# Homework 4

1. Examples of centers of groups.

   (a) (10 points) Show that $Z(\text{Sym}(n)) = \{id\}$ if $n > 2$.

   (b) (10 points) Show that

   $$Z(\text{GL}_2(\mathbb{R})) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R}, a \neq 0 \right\}$$

2. Let

   $$\text{GL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \right\}$$

   be the set of invertible two by two matrices with complex number coefficients. Convince yourself that $\text{GL}_2(\mathbb{C})$ is a group. Let

   $$\text{SL}_2(\mathbb{C}) = \{A \in \text{GL}_2(\mathbb{C}) : \det(A) = 1\}$$

   Convince yourself that $\text{SL}_2(\mathbb{C})$ is a subgroup of $\text{GL}_2(\mathbb{C})$. Let $N$ be the subgroup of $\text{GL}_2(\mathbb{C})$ of diagonal matrices (not a typo this time):

   $$N = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{C}, a \neq 0 \right\}$$

   In this problem, you will show that

   $$\text{SL}_2(\mathbb{C})/\{\pm I_2\} \cong \text{GL}_2(\mathbb{C})/N$$

   where

   $$\{\pm I_2\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \subset \text{SL}_2(\mathbb{C})$$

   (a) (5 points) Show that $\text{GL}_2(\mathbb{C}) = \text{SL}_2(\mathbb{C})N$. That is, show that any invertible $2 \times 2$ matrix can be written as the product of a matrix with determinant 1 and a diagonal matrix.

   (b) (10 points) Show that $N \cap \text{SL}_2(\mathbb{C}) = \{\pm I_2\}$ and apply the second isomorphism theorem to conclude that $\text{SL}_2(\mathbb{C})/\{\pm I_2\} \cong \text{GL}_2(\mathbb{C})/N$.

   (c) (5 points) Where does this proof break down if we replace $\mathbb{C}$ with $\mathbb{R}$ and try to do the same thing?

3. Define an action of $\text{Sym}(3)$ on $\mathbb{R}^3$ as follows: for $\sigma \in \text{Sym}(3)$ and $v = (x_1, x_2, x_3) \in \mathbb{R}^3$, set

   $$\sigma * v = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$$

(a) (10 points) Show that the subspace $V = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$ has the following property: if $v \in V$, then $\sigma * v$ is also in $V$.

(b) (10 points) Find a line (that is a one-dimensional subspace) which has this same property?

4. Sylow theorems.

   (a) (10 points) Show that if $G$ is a group and $|G| = 15$, then $G$ cyclic.

   (b) (10 points) Show that if $G$ is a group and $|G| = 1001$, then $G$ is cyclic.

5. Extra credit (10 points): Let $G$ be a group such that $|G| = 1000$. Show that $G$ has a proper, nontrivial normal subgroup.

6. Extra credit (10 points): Let $G$ be a group such that $|G| = 30$. Prove that $G$ has a normal, nontrivial, proper subgroup.

7. Extra credit (10 points): Let $\mathscr{L}$ denote the set of lines through the origin in $\mathbb{R}^2$, so

$$\mathscr{L} = \{\ell = \text{span}(v) : v \in \mathbb{R}^2, v \neq 0\}$$

Define an action of $\text{GL}_2(\mathbb{R})$ on $\mathscr{L}$ by the rule

$$* : \text{GL}_2(\mathbb{R}) \times \mathscr{L} \longrightarrow \mathscr{L}$$
$$A * \text{span}(v) = \text{span}(Av)$$

Show that this action is well defined, show that it is indeed an action, and show that the action is transitive.