Group homomorphisms and Isomorphism Thms

Recall: 1. A function $f: G \to H$, where G, H are gps is a homomorphism if for all $a, b \in G$ $f(ab) = f(a)f(b)$.

2. We define the kernel of f to be the set

$$\ker(f) = \{a \in G : f(a) = 1_H\}$$

and the image of f to be the set

$$\text{im}(f) = \{b \in H : \exists\, a \in G \text{ such that } f(a) = b\}$$

We have that $\ker(f)$ is a subgp of G and image of f is a subgp of H.

3. $N \subseteq G$ is a normal subgp of G if for all $g \in G$, $gNg^{-1} = N$.

Prop: Let $f: G \to H$ be a group homomorphism. Then f is injective iff $\ker(f) = \{1_G\}$.

proof: $(\Rightarrow)$ If f is injective, then $\ker(f) = \{1_G\}$, for if $\exists\, a \in \ker(f)$, $a \neq 1_G$, then $f(a) = 1_H = f(1_G)$.

$(\Leftarrow)$ Assume $\ker(f) = \{1_G\}$, and let $a, b \in G$ be such that $f(a) = f(b)$. We need to show that $a = b$. Multiplying $f(a) = f(b)$ by $f(b)^{-1}$ gives

$$f(a)f(b)^{-1} = 1_H$$

Since f is a group homomorphism we have.

$$1_H = f(a)f(b)^{-1} = f(a)f(b^{-1}) = f(ab^{-1})$$

Hence $ab^{-1} \in \ker(f)$. By assumption $\ker(f) = \{1_G\}$, so $ab^{-1} = 1_G$. Then $a = b$, so f is injective. □

Prop: Let $f: G \to H$ be a gp hom. Then $\ker(f) \subseteq G$ is a normal subgp.

proof: We will show that $\forall g \in G$, $\forall x \in \ker(f)$, $gxg^{-1} \in \ker(f)$.

Let $g \in G$ and let $x \in \ker(f)$. Then because f is a gp hom we have.

$$f(g^{-1}xg) = f(g^{-1})f(x)f(g) = f(g^{-1})1_H f(g) = f(g)^{-1}f(g) = 1_H$$

because $x \in \ker(f)$

Hence $g^{-1}xg \in \ker(f)$. $\square$

<u>Prop</u>: Let N be a normal subgp of a gp G and let H be a subgroup of G containing N. Then N is ~~norm~~ a normal subgp of H and H/N is a subgp of G/N.

<u>proof</u>: That N is a normal subgp of H follows immediately from the fact that N is a normal subgp of G: Let $h \in H$, $n \in N$, then $hnh^{-1} \in N$ because $h \in G$ and N is a normal subgp of G.

Now we show that H/N is a subgp of G/N. Since.

$$H/N = \{hN : h \in H\}.$$

H/N is a subset of $G/N = \{gN : g \in G\}$. We check the three axioms:

1. (identity) N is the identity in G/N and $N \in H/N$ since $1_G \in H$.
2. (closed under mult): Let $aN, bN \in H/N$ so $a, b \in H$. Then.

   $aN \cdot bN = abN$ and $ab \in H$ because H is a subgp.

   Hence $abN \in H/N$.
3. (inverses): Let $aN \in G/N$. Then $(aN)^{-1} = a^{-1}N$ and $a^{-1} \in H$ because H is a subgp. Therefore $a^{-1}N \in H/N$. $\square$.

Remark: If $N \subseteq H \subseteq G$ is a sequence of subgps such that $N \trianglelefteq H$ and $H \trianglelefteq G$, then it is not necessarily true that N is a normal subgp of G. See the HW.

normal subgp

Thm (Fundamental Theorem of Homomorphisms): Let $f: G \to H$ be a group homomorphism and let N be a normal subgp of G contained in $\ker(f)$. Then

$$\bar{f} : G/N \longrightarrow H$$
$$aN \longmapsto f(a) \qquad \left(\text{means } \bar{f}(aN) = f(a)\right)$$

is a well-defined function that is a group homomorphism.
Furthermore, $\operatorname{im}(\bar{f}) = \operatorname{im}(f)$ and $\ker(\bar{f}) = \ker(f)/N$.

proof: First we show that $\bar{f}$ is well-defined: Let $a, b \in G$ be such that $aN = bN$. ~~Then $\bar{f}(aN) = \bar{f}(a)$~~ Then $a^{-1}b \in N$. Because $N \subseteq \ker(f)$, we then have that $f(a^{-1}b) = 1_H$. Since f is a gp hom, $f(a^{-1}b) = f(a)^{-1}f(b)$. ~~So~~ Therefore we have

$$\begin{aligned} 1_H &= f(a^{-1}b) \\ &= f(a)^{-1}f(b) \end{aligned}$$

which implies

$$\bar{f}(aN) = f(a) = f(b) = \bar{f}(bN)$$

Therefore since $\bar{f}(aN) = \bar{f}(bN)$, $\bar{f}$ is well-defined.

Now we show $\bar{f}$ is a gp hom: Let $aN, bN \in G/N$. Then

$$\bar{f}(aN\,bN) = \bar{f}(abN) = f(ab) = f(a)f(b) = \bar{f}(aN)\bar{f}(bN)$$

because f is a gp hom

Therefore $\bar{f}$ is a gp hom.

Now we show that $\operatorname{im}(\bar{f}) = \operatorname{im}(f)$. Let $h \in H$. We have that

$h \in \mathrm{im}(\bar{f})$ iff $\exists\, aN \in G/N$ such that $\bar{f}(aN) = h$

iff $\exists\, aN \in G/N$ such that $f(a) = h$

iff $\exists\, a \in G$ such that $f(a) = h$

iff $h \in \mathrm{im}(f)$

Therefore $\mathrm{im}(\bar{f}) = \mathrm{im}(f)$.

Finally, we show that $\ker(\bar{f}) = \ker(f)/N$.

We have

$$\begin{aligned}
\ker(\bar{f}) &= \{aN : \bar{f}(aN) = 1_H\} && \text{by def of kernel of } \bar{f} \\
&= \{aN : f(a) = 1_H\} && \text{by def of } \bar{f} \\
&= \{aN : a \in \ker(f)\} && \text{by def of kernel of } f \\
&= \ker(f)/N && \text{by def of } \ker(f)/N. \ \square
\end{aligned}$$

Cor (1st Isomorphism Theorem): Let $f : G \to H$ be a group homomorphism. Then

$$\bar{f} : G/\ker(f) \to \mathrm{im}(f), \quad a\ker(f) \mapsto f(a)$$

is an isomorphism.

Proof: By the previous theorem we have a gp hom

$$\bar{f} : G/\ker(f) \to H, \quad a\ker(f) \mapsto f(a)$$

with $\ker(\bar{f}) = \ker(f)/\ker(f) = 1_{G/\ker(f)}$ and $\mathrm{im}(\bar{f}) = \mathrm{im}(f)$. Hence $\bar{f} : G/\ker(f) \to \mathrm{im}(f)$ is surjective and has trivial kernel. Trivial kernel implies injective, so $\bar{f}$ is an isomorphism $\square$

Examples: 1. $\mathrm{sgn}: \mathrm{Sym}(n) \to \{\pm 1\}$ is a surjective gp hom. with $\ker(\mathrm{sgn}) = \mathrm{Alt}(n)$. Therefore.

$$\mathrm{Sym}(n)/\mathrm{Alt}(n) \cong \{\pm 1\}.$$

2. $\det: GL_n(\mathbb{R}) \to \mathbb{R}^*$ is a surjective gp hom with $\ker(\det) = SL_n(\mathbb{R})$.
Hence $GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong \mathbb{R}^*$.

3. $\pi: \mathbb{R}^2 \to \mathbb{R}$ projection to x-axis
$\pi(x,y) = x$
π is surjective, so $\mathbb{R}^2/\ker(\pi) \cong \mathbb{R}$. Let's calculate

$$\begin{aligned} \ker(\pi) &= \{(x,y) : \pi(x,y) = 0\} \\ &= \{(x,y) : x = 0\} \\ &= \{(0,y) : y \in \mathbb{R}\} \leftarrow \text{y-axis} \end{aligned}$$

Hence $\mathbb{R}^2/\{(0,y) : y \in \mathbb{R}\} \cong \mathbb{R}$.

Lemma: Let N be a normal subgp of a group G, let $M \leq G/N$ be a subgp, and let $\pi: G \to G/N$ be the group homomorphism $\pi(a) = aN$. Then $\pi^{-1}(M) := \{a \in G : \pi(a) \in M\}$ is a subgp of G.

proof: 1. (identity) Since M is a subgp of G/N, $N \in M$. Since π is a gp hom, $\pi(1_G) = N$. Therefore $1_G \in \pi^{-1}(M)$.
2. (closed under mult.) Let $a, b \in G$ be such that $\pi(a), \pi(b) \in M$ (this means $a, b \in \pi^{-1}(M)$). Then since π is a gp hom, $\pi(ab) = \pi(a)\pi(b)$. Since M is a subgp $\pi(a)\pi(b) \in M$. Therefore $ab \in \pi^{-1}(M)$.

3. (closed under inversion) Let $a \in \pi^{-1}(M)$. We need to show that $a^{-1} \in \pi^{-1}(M)$. Since $a \in \pi^{-1}(M)$, $\pi(a) \in M$. Since M is a subgp, $\pi(a)^{-1} \in M$. Since π is a gp hom, $\pi(a)^{-1} = \pi(a^{-1})$. Therefore $a^{-1} \in \pi^{-1}(M)$. $\square$

Remark: Not that in the notation of the previous lemma,

$$\pi^{-1}(N) = \ker(\pi) = N$$

so $N \subseteq M$.

Thm (Third Isom. Thm / Correspondence Thm.): Let N be a normal subgp of a gp G and let $\pi : G \to G/N$ be the group hom. $\pi(a) = aN$. Then there is an order preserving bijection between subgps of G containing N and subgps of G/N given by

$$\alpha : \{\text{subgps of } G \text{ containing } N\} \longrightarrow \{\text{subgps of } G/N\}$$
$$H \longmapsto H/N$$

$$\beta : \{\text{subgps of } G/N\} \longrightarrow \{\text{subgps of } G \text{ containing } N\}.$$
$$M \longmapsto \pi^{-1}(M)$$

Proof: First observe that by the previous two lemmas the functions α and β are well defined.

Thm (Third Isom. Thm/Correspondence Thm): Let N be a normal subgroup of a group G and let $\pi: G \to G/N$ be the group homomorphism $\pi(a) = aN$. Then there is an order preserving bijection between subgps of G containing N and subgps of G/N given by

$$\alpha : \{\text{subgps of } G \text{ containing } N\} \longrightarrow \{\text{subgps of } G/N\}$$
$$H \longmapsto H/N$$

$$\beta : \{\text{subgps of } G/N\} \longrightarrow \{\text{subgps of } G \text{ containing } N\}$$
$$M \longmapsto \pi^{-1}(M)$$

Proof: First observe that by the previous two lemmas, the functions α and β are well-defined. To show that α and β are bijections we show that they are inverse functions to each other. Let H be a subgp of G containing N. Then we need to show that $\beta \circ \alpha(H) = H$. We have

$$\begin{aligned}
\beta \circ \alpha(H) &= \beta(H/N) && \text{by def. of } \alpha \\
&= \pi^{-1}(H/N) && \text{by def of } \beta \\
&= \{g \in G : \pi(g) \in H/N\} && \text{by def of } \pi^{-1}(H/N) \\
&= \{g \in G : gN \in H/N\} && \text{by def of } \pi \\
&= H && \text{because } gN \in H/N \text{ iff } g \in H.
\end{aligned}$$

Therefore $\beta \circ \alpha = \mathrm{id}$.

Now let M be a subgp of G/N. We need to show $\beta\circ\alpha(M) = M$.
We have

$$
\begin{aligned}
\beta\circ\alpha(M) &= \beta(\pi^{-1}(M)) && \text{by def of } \alpha \\
&= \pi^{-1}(M)/N && \text{by def of } \beta \\
&= \{ gN : g \in \pi^{-1}(M) \} && \text{by def of } \pi^{-1}(M)/N \\
&= \{ gN : \pi(g) \in M \} && \text{by def of } \pi^{-1}(M) \\
&= \{ gN : gN \in M \} && \text{by def. of } \pi \\
&= M
\end{aligned}
$$

Hence $\beta\circ\alpha(M) = M$ and so we've shown that α and β are bijections.
To finish, we show the order preserving property of α and β.
First α: Let H, H' be subgps containing N such that $H \subseteq H'$. Then we need to show that $\alpha(H) = H/N$ is a subset of $\alpha(H') = H'/N$.
Let $hN \in H/N$. Then $h \in H$, so $h \in H'$, since $H \subseteq H'$. Therefore $hN \in H'/N$.
Now β: Let M, M' be subgps of G/N ~~we need to show that~~ such that $M \subseteq M'$. We need to show that $\beta(M) = \pi^{-1}(M)$ is a subset of $\beta(M') = \pi^{-1}(M')$. Let $g \in \pi^{-1}(M)$, then $\pi(g) \in M$ so $\pi(g) \in M'$, ~~therefore~~ since $M \subseteq M'$. Therefore $g \in \pi^{-1}(M')$. $\square$

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Group Homomorphisms and Isomorphism Thms Cont.

Def: Let $H, K \leq G$ be two subgps of G. Define the following two subsets of G:

$$HK = \{hk : h \in H, k \in K\}$$
$$KH = \{kh : k \in K, h \in H\}$$

Note that it is not necessarily true that HK or KH are subgps of G or that $HK = KH$.

~~Proposition~~

Prop: The following are equivalent

(i) HK is a subgp of G

(ii) KH is a subgp of G

(iii) $HK = KH$

proof: Assume HK is a subgp of G. We show that $HK = KH$. Let $x \in HK$. ~~so $x = hk$ for some $h \in H$, $k \in K$~~ Then $x^{-1} \in HK$ since HK is a subgp, so $\exists h \in H, k \in K$ s.t. $x^{-1} = hk$. Then $x = (x^{-1})^{-1} = k^{-1}h^{-1}$ is an element of KH since $k^{-1} \in K$, $h^{-1} \in H$.

Let $x \in KH$, so $x = kh$ for some $k \in K, h \in H$. Then $x^{-1} = h^{-1}k^{-1} \in HK$. Since HK is a subgp, $x \in HK$. Hence $HK = KH$, so (i) $\Rightarrow$ (iii).

We also get that (i) $\Rightarrow$ (ii) since HK a subgp and $HK = KH$ implies KH is a subgp. By symmetry (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i). Hence we've shown

(i) $\Leftrightarrow$ (ii), (i) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iii)

It is left to show that (iii) $\Rightarrow$ (i) or (iii) $\Rightarrow$ (ii).

We show (iii) $\Rightarrow$ (i). Assume $HK = KH$. We show HK is a subgp of G.

1. (identity): $1_G = 1_G 1_G$, so $1_G \in HK$.
2. (mult.) Let $x, y \in HK$, so $x = h_1 k_1$, $y = h_2 k_2$ for $h_1, h_2 \in H$, $k_1, k_2 \in K$. Then

$$xy = h_1 k_1 h_2 k_2$$

Since $HK = KH$, $k_1 h_2 = h_3 k_3$ for some $h_3 \in H$, $k_3 \in K$. Then

$$xy = h_1 k_1 h_2 k_2 = h_1 h_3 k_3 k_2 \in HK$$

3. (inv). Let $x \in HK$, so $x = h_1 k_1$, $h_1 \in H$, $k_1 \in K$. Then $x^{-1} = k_1^{-1} h_1^{-1}$. Since $KH = HK$, $\exists\, h_2 \in H$, $k_2 \in K$ s.t. $k_1^{-1} h_1^{-1} = h_2 k_2$. Then $x^{-1} = k_1^{-1} h_1^{-1} = h_2 k_2 \in HK$. $\blacksquare$

<u>Def</u>: Let $H \subseteq G$ be a subgp of a group G. Define the <u>normaliser</u> of H in G to be the set

$$N_G(H) = \{ g \in G : gHg^{-1} = H \}$$

Exercise: Show $N_G(H)$ is a subgp of G.

<u>Remarks</u>: 1. Note that by construction H is a normal subgp of $N_G(H)$. In fact by construction, $N_G(H)$ is the largest subgp N of G for which H is a normal subgp of N.

2. On the homework you show that

$$* \quad G \times \mathrm{Sub}(G) \to \mathrm{Sub}(G)$$
$$(g, H) \mapsto gHg^{-1}$$

defines an action of G on the set of subgps of G.

We have that under this action,

$$\mathrm{stab}(H) = \{ g \in G : gHg^{-1} = H \} = N_G(H)$$

so the normaliser in G of H is the stabiliser of H under this action. This is a proof that $N_G(H)$ is a subgp of G. One can also prove it by hand.

<u>Prop:</u> Let $H, K \subseteq G$ be ~~these~~ two subgps of a gp G. If $K \leq N_G(H)$ or $H \leq N_G(K)$, then $HK = KH$, and so HK is a subgp of G. In particular, if one of the subgroups H or K is normal in G, then $HK = KH$ and HK is a subgroup of G.

<u>proof:</u> By symmetry, we just need to show that if $K \leq N_G(H)$, then $HK = KH$.

Let $x \in HK$, so $x = h_1 k_1$ for some $h_1 \in H$, $k_1 \in K$. Then since $K \subseteq N_G(H)$, $k_1^{-1} h_1 k_1 \in H$. Let $k_1^{-1} h_1 k_1 = h_2$. Then $x = h_1 k_1 = k_1 h_2$, so $x \in KH$.

Let $x \in KH$, so $x = k_1 h_1$ for some $k_1 \in K$, $h_1 \in H$. Then since $K \leq N_G(H)$, $k_1 h_1 k_1^{-1} \in H$. Let $k_1 h_1 k_1^{-1} = h_2$. Then $x = k_1 h_1 = h_2 k_1 \in HK$. $\square$

Thm (2nd Isom. Thm): G gp, $H \subseteq G$ subgp, $N \subseteq G$ normal subgp.
Then N is normal in $HN = NH$, $H \cap N$ is normal in H, and

$$\overline{\varphi}: H/(H\cap N) \longrightarrow HN/N$$
$$\overline{\varphi}(aH\cap N) = aN$$

is an isomorphism.

Proof: By previous prop, $HN = NH$ is a subgp of G since $N \subseteq G$ is a normal subgp. Furthermore, since N is normal in G, N is normal in HN. Consider the composite group homomorphism

$$\varphi: H \xhookrightarrow{\iota} HN \xrightarrow{\pi} HN/N, \quad \varphi = \pi \circ \iota$$
$$a \longmapsto aN$$

where ι is the inclusion map and π is the projection $\pi(a) = aN$.
If $\ker(\varphi) = H \cap N$ and $\operatorname{im}(\varphi) = HN/N$, then by the 1st isom. thm we are done.

We show $\operatorname{im}(\varphi) = HN/N$. Let $aN \in HN/N$. Then $a = hn$ for some $h \in H$, $n \in N$. Observe that

$$hnN = hN \text{ because } (hn)^{-1}h = n^{-1}h^{-1}h = n^{-1} \in N.$$

Then $\varphi(h) = hN = hnN$, so $\operatorname{im}(\varphi) = HN/N$.

Now we show $\ker(\varphi) = H \cap N$.

$a \in \ker(\varphi)$ iff $\varphi(a) = N$ since $N \in HN/N$ is the identity
iff $aN = N$ by def. of φ
iff $a \in N$ how left cosets work

iff $a \in H \cap N$ since $a \in H$ by assumption

Therefore $\ker(\varphi) = H \cap N$.

Hence by the 1^{st} Isom. Thm.

$$\bar{\varphi} : H/_{H\cap N} \to HN/_N$$
$$\bar{\varphi}(aH\cap N) = \varphi(a) = aN$$

is an isomorphism. $\square$

<u>Thm</u> (3rd Isom. Thm) Let G be a group. Let N, H be normal subgps of G such that $N \subseteq H$. Then H/N is a normal subgp of G/N and

$$(G/N)/(H/N) \cong G/H$$

<u>proof</u>: We leave it as an exercise to show that H/N is a normal subgp of G/N. Consider the composition

$$\varphi : G \xrightarrow{\pi_N} G/N \xrightarrow{\pi_{H/N}} (G/N)/(H/N), \quad \varphi = \pi_{H/N} \circ \pi_N.$$

where $\pi_N(g) = gN$ and $\pi_{H/N}(aN) = (aN)(H/N)$. By construction, $\pi_{H/N}$ and π_N are both surjective, so φ is surjective. We have that $a \in \ker(\varphi)$ iff $\varphi(a) = H/N$

iff $(aN)(H/N) = H/N$.

iff $aN \in H/N$

iff $a \in H$

Therefore $\ker(\varphi) = H$, so by the 1^{st} isomorphism thm we have

$$\overline{\varphi}: G/H \longrightarrow (G/N)/(H/N)$$

$$\overline{\varphi}(aH) = \varphi(a) = (aN)(H/N)$$

is an isomorphism. $\square$