Practice Problems for Final

These are practice problems for the final. Some of them are harder than the problems that will be on the final, but they will all help with the final.

1. How can we define the center of the group without using group actions and how can we define the center of the group using group actions?
2. If G is a group with p elements where p is a prime number, and X is a set with q elements where q is also a prime number, what are the possible number of orbits and sizes of orbits of an action of G on X?
3. Is the set of rotations a subgroup of D_{2n}? What about the set of reflections?
4. Let G be a group and let H be a subgroup of G. Prove that $b \in aH$ if and only if $aH = bH$.
5. Prove that D_6 and $\mathrm{Sym}(3)$ are isomorphic. Prove that D_{2n} and $\mathrm{Sym}(n)$ are not isomorphic for any $n > 3$. Prove D_{24} and $\mathrm{Sym}(4)$ are not isomorphic.
6. Prove that $\mathbb{Z}/49\mathbb{Z}$ and $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$ are not isomorphic.
7. What are the possible group homomorphisms from $\mathbb{Z}/N\mathbb{Z}$ to $\mathbb{Z}/M\mathbb{Z}$ for various $N, M \in \mathbb{Z}$?
8. Let $\mathscr{L}$ denote the set of lines through the origin in $\mathbb{R}^2$, so
$$\mathscr{L} = \{\ell = \mathrm{span}(v) : v \in \mathbb{R}^2, v \neq 0\}$$
Define an action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathscr{L}$ by the rule
$$\begin{aligned} * : \mathrm{GL}_2(\mathbb{R}) \times \mathscr{L} &\longrightarrow \mathscr{L} \\ A * \mathrm{span}(v) &= \mathrm{span}(Av) \end{aligned}$$
Show that this action is well defined, show that it is indeed an action, and show that the action is transitive.
9. Let G be a group such that $|G| = 1000$. Show that G has a proper, nontrivial normal subgroup.
10. Let G be a group such that $|G| = 30$. Prove that G has a normal, nontrivial, proper subgroup.
11. Prove that D_{2n} is solvable. Prove that $\mathrm{Alt}(4)$ is solvable. Prove that $\mathrm{Sym}(4)$ is solvable. Prove that Q_8 is solvable.
12. Know the statement of the first isomorphism theorem. You will not need to remember the statement of the second or third isomorphism theorems.