Solvable Groups

Types of groups we've studied that have nice properties:

abelian groups and p-groups

Note: Every subgroup of an abelian (resp p-group) is an abelian (resp. p-group) and every quotien of an abelian (resp p-group) is an abelian (resp p-group). Further, the cartesian product of two abelian groups (resp p-groups) is an abelian (resp. p-group).

Types of groups with are nice. Solvable groups also have these three properties. We will see that

abelian groups are solvable

and

p-groups are solvable

Def: Let G be a group. A <u>subnormal series</u> of G is a finite sequence of subgroups of the form

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

so every subgroup is normal in the following one. The factor groups G_i/G_{i-1}, $i=1,\dots,n$ are called the <u>factors</u> of the subnormal series.

Note that in a subnormal series it is not required that the G_i be normal in G.

Def: A group G is called <u>solvable</u> if it has a subnormal series with abelian factors.

Remark: 1. If G is an abelian group, then G is solvable because the sequence $\{1_G\} \trianglelefteq G$ is a subnormal series with abelian factor.

2. If G is a p-group, say $|G| = p^n$, then Sylow's First Thm. guarantees a subnormal series

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G$$

where $|G_i| = p^i$. Then $|G_{i+1}/G_i| = p$, so $G_{i+1}/G_i \cong \mathbb{Z}/p\mathbb{Z}$.
Therefore G is solvable.

~~Prop~~

Examples: 1. $G = Sym(3)$ is solvable:

$\{id\} \trianglelefteq \langle(123)\rangle \trianglelefteq Sym(3)$, $\quad |Sym(3)/\langle(123)\rangle| = 2$ so $Sym(3)/\langle(123)\rangle \cong \mathbb{Z}/2\mathbb{Z}$

$\langle(123)\rangle/\{id\} = \langle(123)\rangle \cong \mathbb{Z}/3\mathbb{Z}$

2. $G = Alt(4)$ is solvable: Let

$$V_4 = \{id, (12)(34), (13)(24), (14)(23)\} \subseteq Alt(4)$$

Then $V_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and we have that

$\{id\} \trianglelefteq V_4 \trianglelefteq Alt(4)$

$\left|Alt(4)/V_4\right| = \dfrac{4!/2}{4} = 3$ so $Alt(4)/V_4 \cong \mathbb{Z}/3\mathbb{Z}$

3. $G = Sym(4)$ is solvable:

$$\{id\} \trianglelefteq V_4 \trianglelefteq Alt(4) \trianglelefteq Sym(4)$$

$Sym(4)/Alt(4) \cong \mathbb{Z}/2\mathbb{Z}$

4. D_{2n} is solvable: Let $R \subseteq D_{2n}$ be the subgroup of rotations. Then $[D_{2n}:R] = 2$, so R is normal in D_{2n}. $R \cong \mathbb{Z}/n\mathbb{Z}$, so the subnormal series

$$\{id\} \trianglelefteq R \trianglelefteq D_{2n}, \quad D_{2n}/R \cong \mathbb{Z}/2\mathbb{Z}, \quad R \cong \mathbb{Z}/n\mathbb{Z}$$

shows that D_{2n} is solvable.

5. $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, $Z(Q_8) = \{\pm 1\}$ and $|Q_8/Z(Q_8)| = 4$

Exercise $Q_8/Z(Q_8) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence

$$\{1\} \trianglelefteq \{\pm 1\} \trianglelefteq Q_8$$

is a subnormal series that shows that Q_8 is solvable.

(Note: Q_8 is a 2-gp so we already know Q_8 is solvable.)

6. $Alt(5)$ is not solvable because $Alt(5)$ is simple and non-abelian.

__Theorem__ G-gp. If G is solvable and H is a subgroup of G, then H is solvable.

__proof__: Let

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G$$

be a subnormal series with abelian factors. We claim that

$$\{1\} = G_0 \cap H \trianglelefteq G_1 \cap H \trianglelefteq \cdots \trianglelefteq G_n \cap H = H$$

is a subnormal series with abelian factors. Fix an $i > 0$, and let $g \in G_i \cap H$, $h \in G_{i-1} \cap H$. Then since G_{i-1} is normal in G_i, $ghg^{-1} \in G_{i-1}$. Since H is a subgp, $ghg^{-1} \in H$. Hence $ghg^{-1} \in G_{i-1} \cap H$, so $G_{i-1} \cap H$ is normal in G_i. Now by the 2^{nd} isom. thm

$$\cancel{G_i \cap H / G_{i-1} \cap H \cong G_i \cap H / (G_{i-1} \cap H) \cap \quad (H / H \cap N \cong HN/N)}$$

$$G_i \cap H / G_{i-1} \cap H = G_i \cap H / (G_i \cap H) \cap G_{i-1} \cong (G_i \cap H) G_{i-1} / G_{i-1} \subseteq G_i / G_{i-1}$$

$$(H / H \cap N \cong HN/N)$$

Then since G_i / G_{i-1} is abelian, $G_i \cap H / G_{i-1} \cap H$ is abelian because $G_i \cap H / G_{i-1} \cap H$ is isomorphic to a subgp of G_i / G_{i-1}. $\square$

Thm: G gp. If G is solvable and N is a normal subgp of G, then G/N is solvable.

proof: Let

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G.$$

be a subnormal series of G with abelian factors. We claim that

$$\{1\} = G_0N/N \trianglelefteq G_1N/N \trianglelefteq \cdots \trianglelefteq G_nN/N = G/N$$

is a subnormal series of G/N with abelian factors.

1. G_iN is a subgp of G since N is normal in G.
2. G_iN/N is a subgp of $G_{i+1}N/N$ by the correspondence thm.
3. We show that G_iN/N is normal in $G_{i+1}N/N$. Let $gn_1N \in G_{i+1}N/N$ and $hn_2N \in G_iN/N$. Then we know that
$g \in G_{i+1}, n_1 \in N$ $\quad$ $h \in G_i, n_2 \in N$

$$gn_1N = gN \quad \text{and} \quad hn_2N = hN$$

so $(gn_1N)hn_2N(gn_1N)^{-1} = ghg^{-1}N$.

$ghg^{-1} \in G_i$ since G_i is normal in G_{i+1}. Hence $ghg^{-1}N \in G_iN/N$.

4. By 2nd isom. thm. $(G_iN/N)/(G_{i-1}N/N) \cong G_iN/G_{i-1}N$. Then the inclusion $G_i \hookrightarrow G_iN$ induces a group homomorphism

$$G_i \longrightarrow G_iN/G_{i-1}N.$$

We claim this homomorphism is surjective: Let $g \in G_i$
$gnG_{i-1}N \in G_iN/G_{i-1}N$.

Then $gnG_{i-1}N = gG_{i-1}N$ because $(gn)^{-1}g = n^{-1} \in G_{i-1}N$.

Therefore g maps to $gnG_{i-1}N$.

Now observe the G_{i-1} is in the kernel of the group hom $G_i \to G_iN/G_{i-1}N$ since if

$g \in G_{i-1}$, then g maps to $gG_{i-1}N$ and $g \in G_{i-1}N$.

By the 1st isom. thm $G_i/\ker \cong G_iN/G_{i-1}N$ and $G_i/\ker$ is a factor group of G_i/G_{i-1}. Therefore since G_i/G_{i-1} is abelian, so is $G_i/\ker \cong G_iN/G_{i-1}N$. $\square$

Thm: G - gp, $N \trianglelefteq G$ an normal subgroup of G such that N and G/N are solvable. Then G is solvable.

proof: Let

$$\{1_G\} = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_k = N$$

be a subnormal series of N with abelian quotients.

By the correspondence theorem, we know that a subnormal series of G/N with abelian quotients is given as

$$\{N\} = N/N \trianglelefteq H_1/N \trianglelefteq H_2/N \trianglelefteq \cdots \trianglelefteq H_n/N = G/N$$

where the H_i are subgps of G such that $H_i \trianglelefteq H_{i+1}$.

Then we have that

$$\{1_G\} = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_k = N = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$$

is a subnormal series of G. Further, we have ~~that~~

$$N_i / N_{i-1} \text{ is abelian.}$$

and by the 3rd isom. thm,

$$H_i / H_{i-1} \cong \frac{H_i / N}{H_{i-1} / N}$$

so H_i / H_{i-1} is abelian.