Makeup Class Notes

Symmetry Groups

Def: A symmetry of $\mathbb{R}^n$ (nonstandard definition) is a linear map

$$f: \mathbb{R}^n \to \mathbb{R}^n$$

such that f preserves angles and lengths. That is, f is a symmetry of $\mathbb{R}^n$, if f is a ~~map~~ linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and for all vectors $v, w \in \mathbb{R}^n$,

(i) the angle between v and w is the same as the angle between f(v) and f(w).

(ii) the ~~len~~ distance from v to w is the same as the distance from f(v) to f(w)

Remarks: 1. If you think about your intuitive definition of a symmetry of an object, then this definition seems reasonable.

2. Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and define the Euclidean inner product by

$$\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$$

Then if $f: \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, f is a symmetry iff

$$\langle x, y \rangle = \langle f(x), f(y) \rangle \quad \text{for all } x, y \in \mathbb{R}^n$$

3. Linear algebra: The set of all matrices $A \in GL_n(\mathbb{R})$ such that

$$\langle x, y \rangle = \langle Ax, Ay \rangle \quad \forall x, y \in \mathbb{R}^n$$

is called the set of <u>orthogonal matrices</u>, and denoted $O_n(\mathbb{R})$. In linear algebra, you show that

$$O_n(\mathbb{R}) = \{ A \in GL_n(\mathbb{R}) : AA^t = Id \}$$

$$A^{-1} = A^t$$

Therefore, ~~the~~ $O_n(\mathbb{R})$ is the set of symmetries of $\mathbb{R}^n$. Exercise: Show $O_n(\mathbb{R})$ is a group.

<u>Def</u>: The set of symmetries of $\mathbb{R}^n$ is called the <u>orthogonal group</u>. The orthogonal group is the set

$$O_n(\mathbb{R}) = \{ A \in GL_n(\mathbb{R}) : AA^t = Id \}$$

with group operation given by matrix multiplication.

<u>Remarks</u>: 1. The equation $AA^t = Id$ implies that

$$1 = \det(Id) = \det(AA^t) = \det(A)\det(A^t) = \det(A)^2$$

so $\det(A) = \pm 1$ for $A \in O_n(\mathbb{R})$.

2. Lets consider the case when $n=2$, so considering symmetries of $\mathbb{R}^2$. Then

$$O_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) : \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right\}$$

<u>Fact</u> (Extra Credit exercise on HW)

Every element of $O_2(\mathbb{R})$ is either

(i) a rotation about the origin.

$$\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \text{ — counterclockwise rotation by } \phi$$

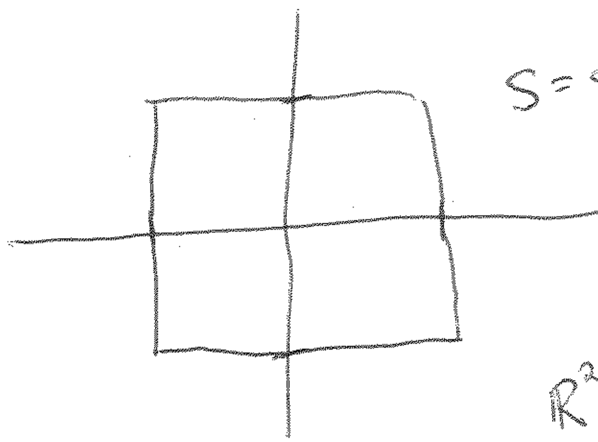
(ii) a reflection about a line through the origin

$A \in O_2(\mathbb{R})$ is a rotation iff $\det(A) = 1$

<u>Def</u>: Let $S \subseteq \mathbb{R}^2$ be a subset of $\mathbb{R}^2$. The <u>symmetry group</u> of S, $\Sigma(S)$, is the set of symmetries of $\mathbb{R}^2$ that sends S to itself.

<u>Note</u>: We can make same definition for a subset $S \subseteq \mathbb{R}^n$.

Example: 1. Consider a square with center at the origin



S = square centered at origin

$\mathbb{R}^2$

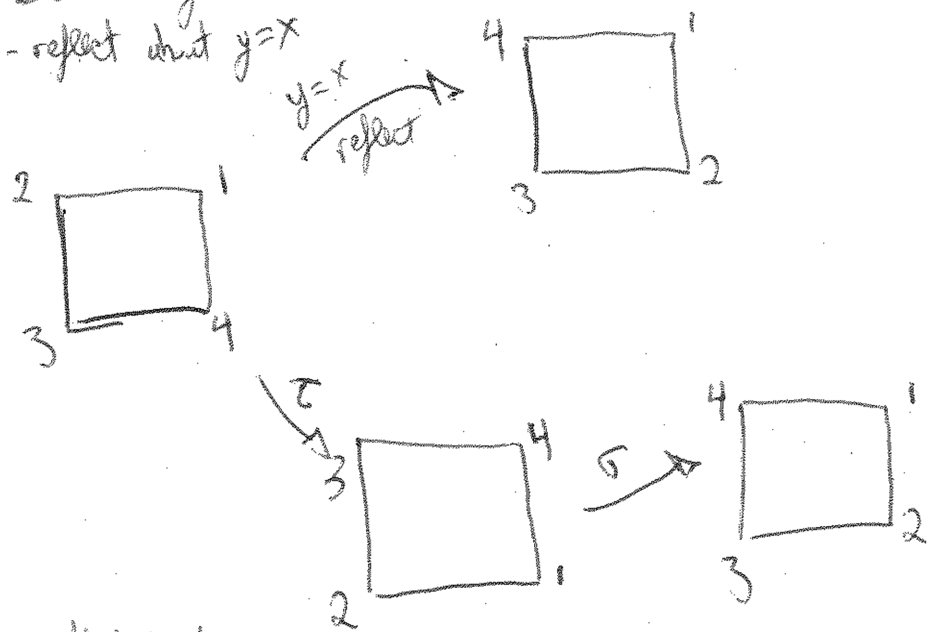
The symmetries of S, $\Sigma(S)$, are the rotations and reflections that send the square to itself.

Rotations: rotate by $0°, 90°, 180°, 270°$

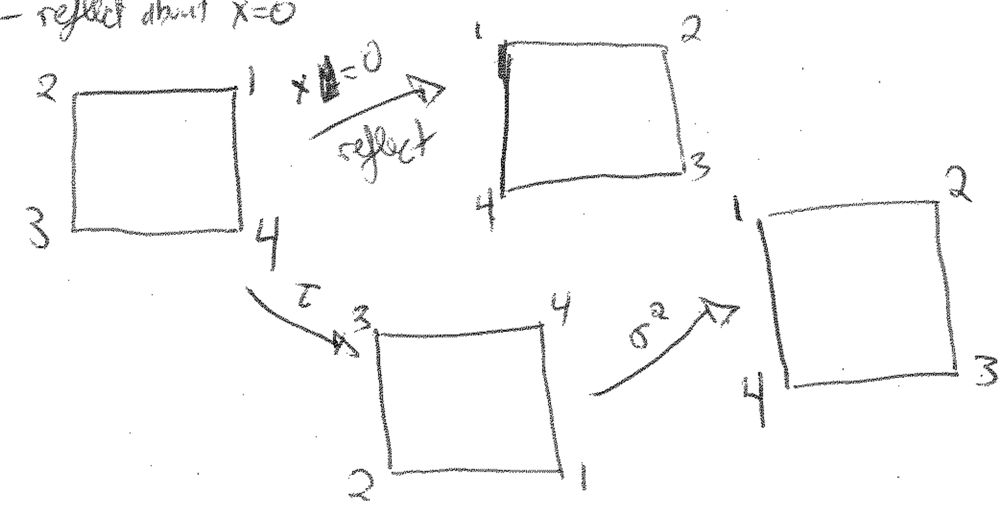
Reflections: reflect about
$y=0$ line
$y=x$ line
$x=0$ line
$y=-x$ line

Let $\sigma \in O_2(\mathbb{R})$ be rotation by $90°$. Then σ^2, σ^3 are rotations by $180°$ and $270°$ respectively.
Let τ be reflection about the $y=0$ line. Then

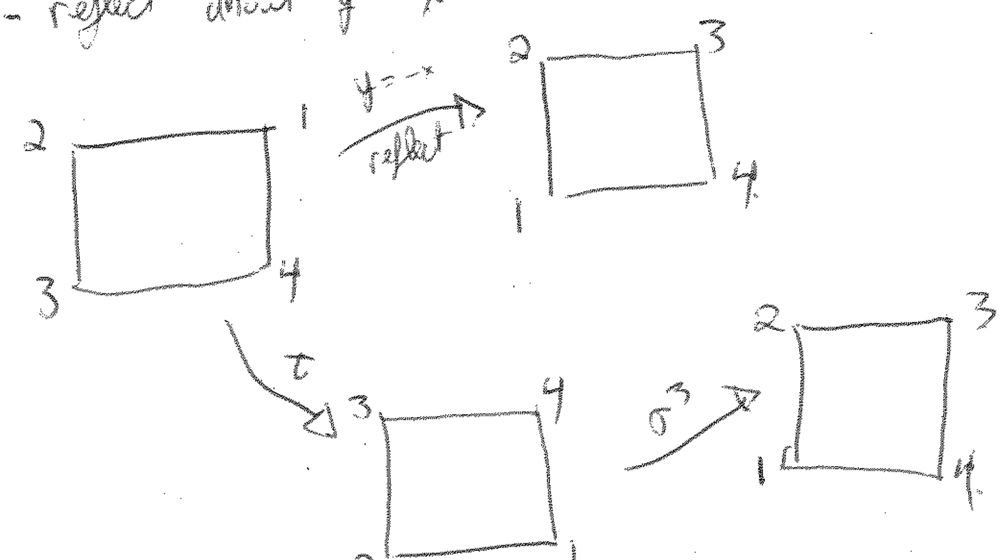
$\sigma\tau$ - reflect about $y=x$



$\sigma^2\tau$ - reflect about $x=0$



$\sigma^3\tau$ - reflect about $y=-x$



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Therefore: $\sum(S) = \{1, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\} \leq O_2(\mathbb{R})$

2. Generalize example 1 to any n-gon P_n in $\mathbb{R}^2$ centered at the origin.

Always n rotations by $360°/i$ for $1 \leq i \leq n$

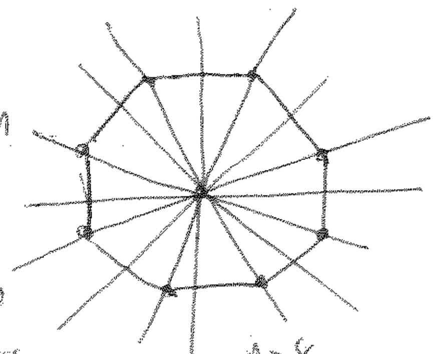
Also n - reflection.

Dichotomy of reflections:

n - even

$\frac{n}{2}$ lines through origin bisecting 2 sides

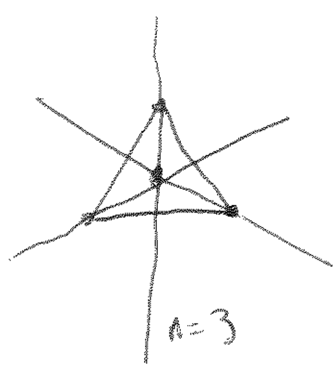
$\frac{n}{2}$ lines through origin connecting 2 vertices



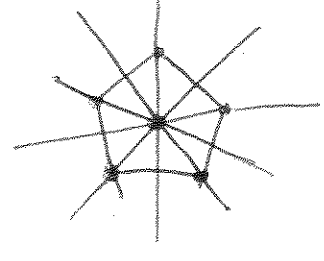
$n=8$

n - odd

n - lines through the origin, a vertex, and bisecting opposite side



$n=3$



$n=5$

$\Sigma(P_n)$ is denoted D_{2n} and called the **dihedral group** of order $2n$.

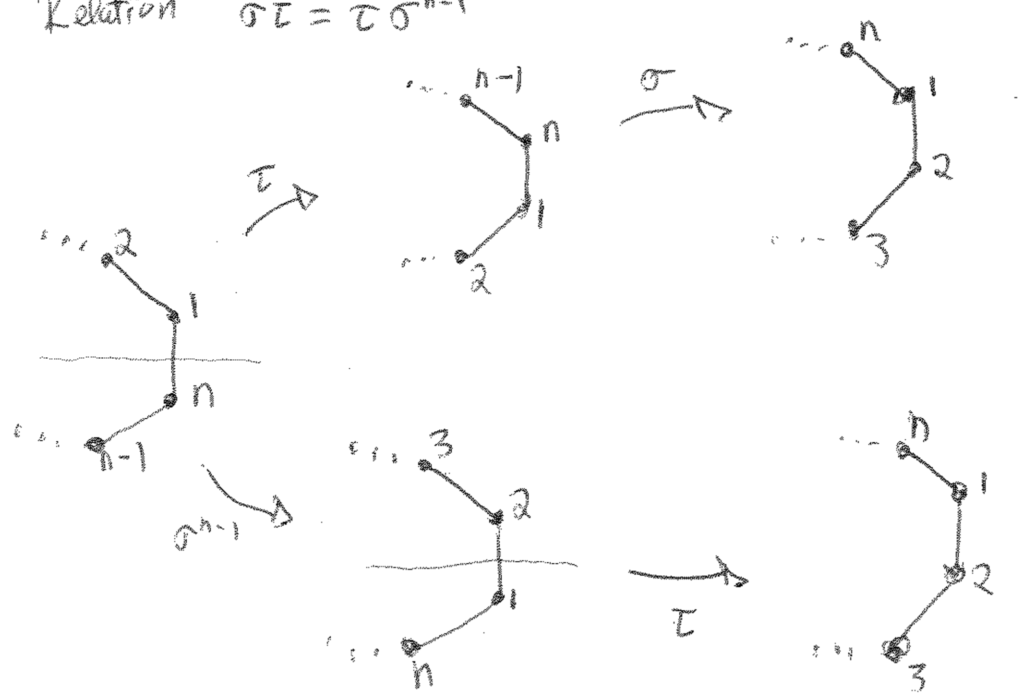
As a set, if σ is rotation by $360/n$, and τ is the reflection about the line that bisects the edge connecting vertices 1 and n, then

$$D_{2n} = \{1, \sigma, \sigma^2, \dots, \sigma^{n-1}, \tau, \sigma\tau, \dots, \sigma^{n-1}\tau\}$$

Note that σ, τ satisfy the relations

$$\sigma^n = 1, \quad \tau^2 = 1, \quad \sigma\tau = \tau\sigma^{n-1}$$

Relation $\sigma\tau = \tau\sigma^{n-1}$



Another Perspective on the dihedral group

Cyclic groups: Groups generated by one element.

$$\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/n\mathbb{Z} \text{ for some } n \in \mathbb{N}$$

Generators and Relations for groups generated by two elements:

$$G = \langle a, b \mid \text{equations with } a, b \rangle$$

$$= \left\{ x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \; ; \; n \in \mathbb{N}, \; \varepsilon_i \in \{1, -1\}, \; x_i = a \text{ or } b \text{ and equations with } a, b \text{ hold} \right\}$$

Examples

(1) $G = \langle a, b \mid ab = ba \rangle$

$$abaaba^{-1}b^{-1}b^{-1}ab = a^3 b$$

$$G = \{ a^n b^m \; ; \; n, m \in \mathbb{Z} \} \cong \mathbb{Z} \times \mathbb{Z}$$

(2) $G = \langle a, b \mid a^n = 1, \; b^m = 1, \; ab = ba \rangle \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$

$$a \longmapsto (1, 0)$$
$$b \longmapsto (0, 1)$$

(3) $G = \langle a, b \mid ab = ba, \; b^n = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$

$$a \longmapsto (1, 0)$$
$$b \longmapsto (0, 1)$$

If $ab = ba$ is one of the equations, then get direct product of 2 cyclic gps.

(4) Non-commutative example is Dihedral gp

$$D_{2n} = \langle \sigma, \tau : \sigma^n = 1, \tau^2 = 1, \sigma\tau = \tau\sigma^{n-1} \rangle$$

(5) Non-commutative example: Quaternions

$$Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, ba = a^3 b \rangle$$