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STARK'S CONJECTURES FOR P-ADIC L-FUNCTIONS

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Table of Contents

Abstract

Stark's Conjectures for p-adic L-functions

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We give a new definition of a p-adic L-function for a mixed signature character of a real quadratic field and for a nontrivial ray class character of an imaginary quadratic field. We then state a p-adic Stark conjecture for this p-adic L-function. We prove our conjecture in the case when p is split in the imaginary quadratic field by relating our construction to Katz's p-adic L-function. We also prove our conjecture in the real quadratic setting for one special case and give numerical evidence in one specific example.

Dedicated to my parents.

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Introduction

Let F be a number field and let

$$\chi : G_F \longrightarrow \mathbb{C}^\times$$

be a continuous one dimensional representation of the absolute Galois group of F. Let K be the fixed field of the kernel of χ. Let p be an odd prime number and fix embeddings (for the rest of this thesis) $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ so we may view any algebraic number as a complex or p-adic number.

Via the Artin map, to χ we may associate the complex Hecke L-function, $L(\chi, s)$, defined by the series

$$L(\chi, s) = \sum_{\mathfrak{a} \subset \mathscr{O}_F} \frac{\chi(\mathfrak{a})}{\mathrm{N}\,\mathfrak{a}^s}$$

for $\mathrm{Re}(s) > 1$. The function $L(\chi, s)$ has a meromorphic continuation to the whole complex plane. In the late 1970s, in a series of papers, Stark made precise conjectures concerning the leading term of the Taylor series expansion at $s = 0$ of $L(\chi, s)$ ([28], [29], [30]). Stark's conjectures relate the leading term of $L(\chi, s)$ at $s = 0$ to the determinant of a matrix of linear combinations of logarithms of units in K. His conjectures refine

Dirichlet's class number formula. Stark proved his conjectures when the field F is equal to $\mathbb{Q}$ or to an imaginary quadratic field. In general the conjectures are open.

Around the same time that Stark made his conjectures, p-adic L-functions were constructed interpolating the critical values of complex Hecke L-functions for general number fields. This vastly generalized Kubota and Leopoldt's work on the p-adic Riemann zeta function. When F is a totally real field and $\chi : G_F \to \mathbb{C}^\times$ is a totally even character, Cassou-Nogues, and Deligne–Ribet ([5], [10]) defined a p-adic meromorphic function

$$L_p(\chi, s) : \mathbb{Z}_p \longrightarrow \mathbb{C}_p$$

determined by the following interpolation property: for all $n \in \mathbb{Z}_{\leq 0}$, we have

$$L_p(\chi, n) = \prod_{\mathfrak{p}|p}(1 - \chi\omega^{n-1}(p)\,\mathrm{N}\,\mathfrak{p}^{-n})L(\chi\omega^{n-1}, n) \tag{0.1}$$

where ω is the Teichmüller character. Siegel and Klingen ([26]) showed that the values $L(\chi\omega^{n-1}, n)$ lie in the field obtained by adjoining the values of $\chi\omega^{n-1}$ to $\mathbb{Q}$. The equality (0.1) takes place in $\overline{\mathbb{Q}}$.

Now let F be a CM field with maximal totally real subfield E. A prime p is called ordinary for F if every prime above p in E splits in F. For such primes p, Katz ([17],[18]) defined a p-adic L-function associated to any ray class character $\chi : G_F \to \mathbb{C}^\times$. Katz's p-adic L-function interpolates the values of complex L-functions of algebraic Hecke characters with nonzero infinity type. To specify the interpolation property we specialize to the case that F is imaginary quadratic. Let p be a rational prime that is split in F. Let λ be a Hecke character of infinity type $(1, 0)$. Then Katz constructed a p-adic meromorphic function

$$L_p(\chi, t, s) : \mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow \mathbb{C}_p$$

determined by the following interpolation property: for all $k, j \in \mathbb{Z}$ such that $1 \leq j \leq k-1$, we have

$$\frac{L_p(\chi, k, j)}{\Omega_p^{k-1}} = E_p(\chi, k, j) \frac{L(\chi\lambda^{k-1}, j)}{\Omega_\infty^{k-1}}.$$

In the formula above, $E_p(\chi, k, j)$ is an explicit complex number and $\Omega_p \in \mathbb{C}_p^\times$, $\Omega_\infty \in \mathbb{C}^\times$ are p-adic and complex periods that make both sides of the above equality algebraic.

In the two cases F is totally real and F is imaginary quadratic, p-adic Stark conjectures have been made for $L_p(\chi, s)$ and $L_p(\chi, t, s)$, and some progress has been made on these conjectures. When F is totally real and χ is totally odd Gross ([16]) stated a conjecture for the order of vanishing of $L_p(\chi\omega, s)$ at $s = 0$ and the leading term of the Taylor series of $L_p(\chi\omega, s)$ at $s = 0$. Progress has been made on the order of vanishing, and recently the formula for the leading term was proved in [8] building off of earlier work in [7]. When F is totally real and χ is totally even there is a conjecture for the value $L_p(\chi, 1)$ known as the Serre-Solomon-Stark conjecture ([27], [31]). This conjecture is open except in the cases when $F = \mathbb{Q}$ (when the formula is due to Leopoldt) and when χ is trivial (where Colmez has proven a p-adic class number formula ([6]). When F is imaginary quadratic and p is split in F, Katz stated and proved a p-adic Stark conjecture for the value $L_p(\chi, 1, j)$ known as Katz's p-adic Kronecker's 2nd limit formula ([17] and see section 4.1 of *loc. cit.*).

One of the original motivations for Stark's conjectures is that when the order of vanishing of $L(\chi, s)$ at $s = 0$ is exactly one, then the conjectures shed light on Hilbert's 12th problem about explicit class field theory. More precisely, when the order vanishing is exactly one then Stark's conjectures predict the existence of a unit $u \in \mathscr{O}_K^\times$ such that the relation

$$L'(\psi, 0) = -\frac{1}{e} \sum_{\sigma \in \mathrm{Gal}(K/F)} \psi(\sigma) \log |\sigma(u)| \tag{0.2}$$

holds for all characters of the Galois group $\mathrm{Gal}(K/F)$ and such that $K(u^{1/e})$ is an abelian extension of F. Here e is the number of roots of unity in K and the absolute value is a particular absolute value on K. When F is real quadratic, $\mathrm{ord}_{s=0}(L(\chi, s)) = 1$ if and only if χ is mixed signature. In this case, we choose the absolute value on K to correspond to one of the real places of K. Then by varying ψ and exponentiating (0.2) one can solve for the unit u from the L-values $L'(\psi, 0)$. In this way, Stark's conjectures give a way to construct units in abelian extensions of F.

The goal of this thesis is to define a p-adic L-function and state a p-adic Stark conjecture in the setting when F is a quadratic field and $\mathrm{ord}_{s=0}(L(\chi, s)) = 1$ (the rank one setting). This is the case when

$$\chi : G_F \longrightarrow \mathbb{C}^\times$$

is any nontrivial character if F is imaginary quadratic, and when χ is a mixed signature character when F is real quadratic. When F is imaginary quadratic and p is split in F our p-adic L-function is related to Katz's. In the cases when F is imaginary quadratic and p is inert, as well as when F is real quadratic and χ is mixed signature, our p-adic L-function is new. Our p-adic Stark conjecture has a similar shape to (0.2) above with the complex logarithm replaced with the p-adic logarithm and the same units appearing. One of the main issues with defining the p-adic L-function for χ when F is quadratic and $\mathrm{ord}_{s=0}(L(\chi,s)) = 1$ is that the complex L-function $L(\chi,s)$ has no critical values. Therefore the p-adic L-function of χ will not interpolate any of the special values of $L(\chi,s)$. In order to define the p-adic L-function in spite of the fact that $L(\chi,s)$ has no critical values we p-adically deform χ into a family of p-adic representations where complex L-functions in the family do have critical values to interpolate.

We now explain in more detail our definition, conjectures, and results. Let

$$\rho = \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

be the induction of χ from G_F to $G_{\mathbb{Q}}$. Then the q-expansion

$$f = \sum_{\mathfrak{a} \subset \mathscr{O}_F} \chi(\mathfrak{a}) q^{N\mathfrak{a}}$$

is the q-expansion of a weight one modular form and ρ is the representation associated to f. The modular form f has character $\varepsilon = \det \rho$ and level $N = |d_F| N_{F/\mathbb{Q}}\mathfrak{m}$ where d_F is the discriminant of F and $\mathfrak{m}$ is the conductor of χ. Let

$$x^2 - a_p(f)x + \varepsilon(p) = (x-\alpha)(x-\beta)$$

be the Hecke polynomial of f at p. Then α and β are roots of unity, so f has two (possibly equal) ordinary p-stabilizations. Let $f_\alpha(z) = f(z) - \beta f(pz)$ be a p-stabilization of f. Under the assumption that $\alpha \neq \beta$, Bellaïche and Dmitrov ([2]) have shown that the eigencurve is smooth at the point corresponding to f_α. In order to use Bellaïche and Dmitrov's result we assume $\alpha \neq \beta$ and let V be a neighborhood of f_α on the eigencurve such that the weight map is étale at all points of V except perhaps f_α. Furthermore, let $\mathcal{W}$ be weight space. Using the constructions of [1] there exists a two-variable p-adic rigid analytic function

$$L_p(f_\alpha, z, \sigma) : V \times \mathcal{W} \longrightarrow \mathbb{C}_p$$

such that for all classical points $y \in V$, all finite order characters $\psi \in \mathcal{W}(\mathbb{C}_p)$, and all integers j, $1 \leq j \leq k-1$ where k is the weight of y,

$$\frac{L_p(f_\alpha, y, \psi^{-1}(\cdot)\langle\cdot\rangle^{j-1})}{\Omega_{p,y}^{sgn(\psi)}} = E_p(f_\alpha, y, \psi, j)\frac{L(g_y, \psi\omega^{j-1}, j)}{\Omega_{\infty,y}^{sgn(\psi)}}.$$

Here $E_p(f_\alpha, y, \psi, j)$ is an explicit complex number and $\Omega^\pm_{\infty,y}, \Omega^\pm_{p,y}$ are p-adic and complex periods respectively that make both sides of the equality algebraic. Conceptually, it makes sense to define the p-adic L-function of χ as

$$L_p(\chi, \alpha, s) : \mathbb{Z}_p \longrightarrow \mathbb{C}_p$$

$$L_p(\chi, \alpha, s) = L_p(f_\alpha, x, \langle\cdot\rangle^{s-1})$$

where $x \in V$ is the point corresponding to f_α. The problem with this definition is that while the function $L_p(f_\alpha, z, \sigma)$ is determined by the above interpolation property, the triple of the function $L_p(f_\alpha, z, \sigma)$, the p-adic periods $\Omega^\pm_{p,y}$, and the complex periods $\Omega^\pm_{\infty,y}$ is not canonically defined. The choice of the function $L_p(f_\alpha, z, \sigma)$ may be changed by an invertible function (meaning having no zeros or poles) on V and we would then obtain a new function with new p-adic and complex periods satisfying the same interpolation formula. We would like to state a p-adic Stark conjecture for the function $L_p(\chi, \alpha, s)$, but because the function is not canonically defined it does not make sense to specify its value at any point with a precise conjecture.

To define a function that does not depend on any choices, we fix two finite order Dirichlet characters $\eta, \psi \in \mathcal{W}$ and define the p-adic L-function of χ with the auxiliary characters η and ψ as

$$L_p(\chi, \alpha, \eta, \psi, s) = \frac{L_p(f_\alpha, x, \eta^{-1}(\cdot)\langle\cdot\rangle^{s-1})}{L_p(f_\alpha, x, \psi^{-1}(\cdot)\langle\cdot\rangle^{s-1})}.$$

The function $L_p(\chi, \alpha, \eta, \psi, s)$ does not depend on the choices made to define $L_p(f_\alpha, x, \sigma)$, and so $L_p(\chi, \alpha, \eta, \psi, s)$ is the p-adic L-function we make a Stark conjecture for.

Now assume that η and ψ have orders p^m and p^n respectively. Let M_m and M_n be the fixed fields of the kernels of the representations $\rho \otimes \eta$ and $\rho \otimes \psi$ respectively. We conjecture the existence of units $u_m \in \mathscr{O}_{M_m}^\times$ and $u_n \in \mathscr{O}_{M_n}^\times$ such that

$$L_p(\chi, \alpha, \eta\omega, \psi\omega, 0) = E_p(\chi, \alpha, \eta, \psi, 0) \frac{\displaystyle\sum_{\sigma \in \mathrm{Gal}(M_m/F)} \chi\eta(\sigma) \log_p |\sigma(u_m)|_{1/\alpha}}{\displaystyle\sum_{\sigma \in \mathrm{Gal}(M_n/F)} \chi\psi(\sigma) \log_p |\sigma(u_n)|_{1/\alpha}}$$

where $E_p(\chi, \alpha\eta, \psi, 0)$ is an explicit p-adic number. The absolute value $|\cdot|_{1/\alpha}$ is a projection that depends on the choice of p-stabilization. This projection is a key part of the conjecture and the idea is from Greenberg and Vatsal ([14]). In [14], Greenberg and Vatsal study Iwasawa theory for representations associated to weight one modular forms. When the weight one modular form is the same as the ones we consider in this thesis, the p-adic L-functions they study are very closely related to the ones we study here.

We prove our conjecture in the case when F is imaginary and p is split in F by comparing the p-adic L-function $L_p(\chi, \alpha, \eta\omega, \psi\omega, s)$ to Katz's p-adic L-function. In addition, we use Katz's p-adic L-function to prove our conjecture in the following additional case when F is real quadratic. Assume F real quadratic and χ is a mixed signature character of G_F. If there exists an imaginary quadratic field F' and a ray class character χ' of $G_{F'}$ such that $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \chi = \mathrm{Ind}_{G_{F'}}^{G_{\mathbb{Q}}} \chi'$ then our conjecture holds for χ. We also give numerical evidence for our conjecture in one particular example. Unfortunately the example we give numerical evidence for falls into the setting just described for a real quadratic field. In the future we hope to numerically verify the conjecture in at least one more additional case.

We finish this introduction by giving a brief outline of each of the chapters in this thesis. In chapter 1 we review the rank one abelian Stark conjecture for quadratic fields. In the imaginary quadratic case when Stark's conjecture is a theorem we state the explicit definition of the Stark units. We end chapter 1 with an informal discussion and outline of how we will define the p-adic L-function associated to χ. In chapter 2 we give all the background needed to use the constructions of [1] to define p-adic L-functions on the ordinary locus of the eigencurve. This includes using the language of rigid analytic geometry to discuss overconvergent modular symbols and families of overconvergent modular symbols. We include in chapter 2 the construction of the p-adic L-function of an ordinary weight $k \geq 2$ modular form for comparison to the situation that we consider in weight one.

In chapter 3 we give the definition of our p-adic L-function as well as the statement of our conjectures. We state an integral conjecture at $s = 0$ and at $s = 1$ as well as rational conjectures at $s = 0$ and $s = 1$. In chapter 4 we prove our conjecture when F is imaginary quadratic and p is split in F. In this chapter we give an explanation of Katz's p-adic L-function drawing from many sources ([17],[9],[15],[3]). In chapter 5, we use our result from chapter 4 to prove our conjecture for the case when F is real quadratic described above. In chapter 6 we give numerical evidence for our conjecture in a specific example.

Chapter 7 is an appendix. The section on rigid analytic geometry was included for the reader not familiar with the subject. We do not do any heavy rigid analytic geometry in this thesis, but we do use it as a language to discuss p-adic L-functions. The appendix has all the definitions and theorems used. The section on topological rings, modules, and the completed tensor product is included for completeness to give definitions that do not appear in the body of the thesis. The section on Hecke characters is included to set language and conventions since many different conventions for Hecke characters and their names is used in the literature. Finally the section on Hecke L-functions is included for completeness and to have for comparison to the p-adic L-functions considered in this thesis.

Chapter 1

The rank one abelian Stark conjecture for quadratic fields

1.1 The archimedean conjecture

In this section we state the rank one abelian Stark conjecture for quadratic fields and introduce notation that will be used in later sections. We state the conjecture at $s = 0$ and at $s = 1$ since our p-adic conjecture will be stated at $s = 0$ and $s = 1$. Let F be a quadratic extension of $\mathbb{Q}$ and let K be a nontrivial finite abelian extension of F. If F is real quadratic we assume that one infinite place of F stays real in K and the other becomes complex.

Let S be a finite set of places of F that contains the infinite places and the places that ramify in K, and such that $|S| \geq 2$. Let S_K denote the places of K above those in S. Let v denote an infinite place of K such that $v(K) \subset \mathbb{R}$ if F is real quadratic.

We also let v denote the infinite place of F that is $v|_F$, so $v \in S$. Let $U_{v,S}$ denote the subgroup of $K^\times$ defined by

$$U_{v,S} = \begin{cases} \{u \in K^\times : |u|_{w'} = 1, \forall w' \text{ such that } w'|_F \neq v|_F\} & \text{if } |S| \geq 3 \\ \{u \in K^\times : |u|_{w'} = |u|_{w''}, \forall w', w'' \mid v' \text{ and } |u|_w = 1, \forall w \notin S_K\} & \text{if } S = \{v, v'\}. \end{cases}$$

Let e denote the number of roots of unity in K.

Conjecture 1.1.1. *(Stark [30] at $s = 0$) There exists $u \in U_{v,S}$ such that for all characters χ of Gal(K/F),*

$$L'_S(\chi, 0) = -\frac{1}{e} \sum_{\sigma \in Gal(K/F)} \chi(\sigma) \log |\sigma(u)|_v.$$

Remarks 1.1.2.

1. Stark conjectured the additional conclusion that the $u \in U_{v,S}$ in the above conjecture is such that $K(u^{1/e})$ is an abelian extension of F. For our purposes we will not be considering this part of the conjecture, so we leave it out.

2. Stark proved the above conjecture when F is imaginary quadratic ([30]). The conjecture is open when F is real quadratic.

3. If $|S| \geq 3$, then the element $u \in U_{v,S}$ has its absolute value specified at every infinite place of K, so u if it exists is determined up to multiplication by a root of unity. The element $u \in U_{v,S}$ which the conjecture predicts is called a Stark unit of K/F.

4. In the real quadratic case, we can always take S to be the infinite places of F union the places of F that ramify in K. In this case, the conjectural $u \in U_{v,S}$ is

an actual unit in $\mathscr{O}_K$. Similarly in the imaginary quadratic case if at least two primes of F ramify in K and we take S to be the infinite place of F union the places of F that ramify in K, then the Stark unit $u \in U_{v,S}$ is a unit in K.

5. If there exists a finite prime w in S such that $\chi(w) = 1$, then $L'_S(\chi, 0) = 0$ for all χ and we may take $u = 1$. Therefore the conjecture is somewhat trivial in this case. When we state the conjecture at $s = 1$ will assume that $\chi(w) \neq 1$ for any $w \in S$.

Proposition 1.1.3. *Keeping the notation from above assume F is real quadratic, $|S| \geq 3$, and let χ be a character of Gal(K/F) that is totally even. Then $L'_S(\chi, 0) = 0$ and if $u \in U_{v,S}$, then*

$$\sum_{\sigma \in Gal(K/F)} \chi(\sigma) \log |\sigma(u)|_v = 0.$$

Proof. If χ is totally even, the order vanishing at $s = 0$ of $L_S(\chi, s)$ is at least two, so $L'_S(\chi, 0) = 0$.

Now let $u \in U_{v,S}$ and let w be an infinite place of K such that $w|F \neq v|F$. Then w is above a complex place of K, so $|u|_w = 1$. Let $\{1, \delta\}$ be the decomposition group of w. The condition $|u|_w = 1$ is equivalent to $\overline{w(u)} = w(u)^{-1}$. Since w is a ring homomorphism $w(u)^{-1} = w(u^{-1})$. By definition of δ, $\overline{w(u)} = w \circ \delta(u)$. Then $\delta(u) = u^{-1}$ since w is injective. That is, the condition $|u|_w = 1$ is equivalent to $\delta(u) = u^{-1}$.

On the other hand, since χ is even, $\chi(\delta) = 1$. We then have

$$\begin{aligned}
\sum_{\sigma \in \mathrm{Gal}(K/F)} \chi(\sigma) \log |\sigma(u)|_v &= \sum_{\sigma \in \mathrm{Gal}(K/F)/\{1,\delta\}} \chi(\sigma) \log |\sigma(u)\sigma\delta(u)|_v \\
&= \sum_{\sigma \in \mathrm{Gal}(K/F)/\{1,\delta\}} \chi(\sigma) \log |\sigma(u)\sigma(u^{-1})|_v \\
&= 0.
\end{aligned}$$

□

Remarks 1.1.4. 1. By the proposition, in the real quadratic case Conjecture 1.1.1 is equivalent to the statement that there exists $u \in U_{v,S}$ such that for all mixed signature characters χ of $\mathrm{Gal}(K/F)$

$$L'(\chi, 0) = -\frac{1}{e} \sum_{\sigma \in \mathrm{Gal}(K/F)} \chi(\sigma) \log |\sigma(u)|_v.$$

2. Let χ be a character of $\mathrm{Gal}(K/F)$ such that $\mathrm{ord}_{s=0}(L(\chi, s)) = 1$. Let d_F be the discriminant of F, $\mathfrak{f}$ the conductor of χ, and $N = |d_F| N_{F/\mathbb{Q}}\mathfrak{f}$. In this setting, the functional equation for the primitive L-function, $L(\chi, s)$ is

$$L(1-s)2(2\pi)^{s-1}\Gamma(1-s)N^{(1-s)/2} = W(\chi)L(\overline{\chi}, s)2(2\pi)^{-s}\Gamma(s)N^{s/2}$$

where $W(\chi)$ is the root number of χ.

Using the functional equation we state Stark's conjecture also at $s = 1$. We make the following assumption on S: for all finite $w \in S$ and all characters χ of $\mathrm{Gal}(K/F)$ such that $\mathrm{ord}_{s=0}(L(\chi, s)) = 1$, $\chi(w) \neq 1$.

Conjecture 1.1.5. *(Stark at $s = 1$) There exists a $u \in U_{v,S}$ such that for all characters χ of Gal(K/F) that are of mixed signature if F is real quadratic and are nontrivial if F is imaginary quadratic,*

$$L_S(\chi, 1) = -\frac{2\pi}{e} \frac{W(\chi)}{\sqrt{|d_F| N\mathfrak{f}(\chi)}} E(\chi, S) \sum_{\sigma \in Gal(K/F)} \chi(\sigma^{-1}) \log |\sigma(u_K)|_v$$

where $\mathfrak{f}(\chi)$ is the conductor of χ, d_F is the discriminant of F, e is the number of roots of unity in K, and

$$E(\chi) = \prod_{\substack{v \in S \\ finite}} \left(1 - \frac{\chi(v)}{Nv}\right) (1 - \overline{\chi}(v))^{-1}.$$

Remarks 1.1.6. By Proposition 1.1.3 and the functional equation, when $|S| \geq 3$ and S is as above, Conjectures 1.1.1 and 1.1.5 are equivalent. The term $E(\chi)$ appearing comes from the functional equation for $L_S(\chi, s)$.

Definition 1.1.7. *Let K/F, S and v be as above. An element in $U_{v,S}$ satisfying the above conjecture is called a* ***Stark unit for*** *K/F and is denoted u_K. If $|S| \geq 3$, then u_K is determined up to multiplication by a root of unity. When F is imaginary quadratic the units u_K will be specified in section 1.2.*

Remarks 1.1.8. It is important to note that if S is fixed, then the Stark unit u_K in the above definition depends on the choice of infinite place v of K. In the setting we consider this will be particularly important. In our setting we have a nontrivial ray class character

$$\chi : G_F \longrightarrow \mathbb{C}^\times$$

that is of mixed signature if F is real quadratic. We will then take K to be the fixed field of the kernel of χ. Our conjecture will depend on the two-dimensional representation

$$\rho = \operatorname{Ind}_{G_F}^{G_{\mathbb{Q}}} \chi : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C}).$$

The representation ρ has the property that for any $\tau \in G_{\mathbb{Q}} - G_F$ the character χ_τ defined by

$$\chi_\tau(\sigma) = \chi(\tau^{-1}\sigma\tau)$$

also satisfies $\rho = \operatorname{Ind}_{G_F}^{G_{\mathbb{Q}}} \chi_\tau$. The fixed field of the kernel of χ_τ is $\tau(K)$, and we will need to consider the Stark unit for K/F and for $\tau(K)/F$.

When F is imaginary quadratic, $K\tau(K)$ is the fixed field of the kernel of ρ, and is an abelian extension of F, so there exists a Stark unit for $K\tau(K)/F$.

When F is real quadratic, $\tau(K)K$ is abelian over F, but both infinite places of F become complex in $\tau(K)K$. Therefore the extension $\tau(K)K/F$ does not fall into the setting of the rank one abelian Stark conjecture. For this reason, when F is real quadratic we consider the two extensions K/F and $\tau(K)/F$ separately. If v is an infinite place of K such that $v(K) \subset \mathbb{R}$ if F is real quadratic, then $v^\tau := v \circ \tau^{-1}$ is an infinite place of $\tau(K)$ such that $v^\tau(\tau(K)) \subset \mathbb{R}$ if F is real quadratic. If u_K is the Stark unit for K determined by v, then $\tau(u_K)$ is the Stark unit for $\tau(K)$ determined by v^τ.

One may also fix a character χ of $\mathrm{Gal}(K/F)$ and state a rank one abelian Stark conjecture for the one L-function $L_S(\chi, s)$. This Stark conjecture for $L(\chi, s)$ is still open in the real quadratic case for most χ. We call this Stark conjecture the rational Stark conjecture for χ.

We keep the setting and notation as above for K/F, S, and v. Let χ be a character of $\mathrm{Gal}(K/F)$ such that $\mathrm{ord}_{s=0}(L_S(\chi, s)) = 1$, and let k be the field obtained by adjoining the values of χ to $\mathbb{Q}$. We extend $\log|\cdot|_v$ from $U_{v,S}$ to $k \otimes_{\mathbb{Z}} U_{v,S}$ by k-linearity. Let

$$(k \otimes_{\mathbb{Z}} U_{v,S})^{\chi^{-1}} = \{u \in k \otimes_{\mathbb{Z}} U_{v,S} : \sigma(u) = \chi^{-1}(\sigma)u, \forall \sigma \in \mathrm{Gal}(K/F)\}$$

$$(k \otimes_{\mathbb{Z}} U_{v,S})^{\chi} = \{u \in k \otimes_{\mathbb{Z}} U_{v,S} : \sigma(u) = \chi(\sigma)u, \forall \sigma \in \mathrm{Gal}(K/F)\}$$

be the χ^{-1} and χ isotypic components of $k \otimes_{\mathbb{Z}} U_{v,S}$ where $\mathrm{Gal}(K/F)$ acts via its action on $U_{v,S}$.

Conjecture 1.1.9. *(Stark for χ at $s = 0$). There exists an element $u_\chi^* \in (k \otimes_{\mathbb{Z}} U_{v,S})^{\chi^{-1}}$ such that*

$$L_S'(\chi, 0) = \log|u_\chi^*|_v.$$

Conjecture 1.1.10. *(Stark for χ at $s = 1$). There exists an element $u_\chi \in (k \otimes_{\mathbb{Z}} U_{v,S})^{\chi}$ such that*

$$L_S(\chi, 1) = -\frac{2\pi W(\chi) E(\chi, S)}{\sqrt{|d_F| N\mathfrak{f}(\chi)}} \log|u_\chi|_v.$$

Remarks 1.1.11.

1. Since we are assuming $\mathrm{ord}_{s=0}(L_S(\chi, s)) = 1$, the k-dimension of $(k \otimes_{\mathbb{Z}} U_{v,S})^{\chi^{-1}}$ and $(k \otimes_{\mathbb{Z}} U_{v,S})^{\chi}$ is one.

2. Conjecture 1.1.1 implies Conjecture 1.1.9 by taking

$$u_\chi^* = -\frac{1}{e} \sum_{\sigma \in \mathrm{Gal}(K/F)} \chi(\sigma) \otimes \sigma(u) \in (k \otimes_{\mathbb{Z}} U_{v,S})^{\chi^{-1}}$$

where $u \in U_{v,S}$ is the unit satisfying Conjecture 1.1.1. Conjecture 1.1.5 implies conjecture 1.1.10 by taking

$$u_\chi = -\frac{1}{e} \sum_{\sigma \in \mathrm{Gal}(K/F)} \chi(\sigma^{-1}) \otimes \sigma(u) \in (k \otimes_{\mathbb{Z}} U_{v,S})^\chi$$

where $u \in U_{v,S}$ is the unit satisfying conjecture 1.1.5.

3. The $*$ in the notation for u_χ^* is to indicate that u_χ is in the χ^{-1} component of $U_{v,S}$ and not the χ component. In some references u_χ^* is denoted u_χ and in some references it is $u_{\chi^{-1}}$. Since we will be state the conjectures at $s = 0$ and $s = 1$, we use a $*$ to denote being in the χ^{-1} component and the absence of a $*$ to denote being in the χ component.

1.2 The imaginary quadratic case

In this section we define the Stark units that exist in the imaginary quadratic case of the rank one abelian Stark conjecture. These units will be used in later sections.

We begin by introducing Robert's units. They appear in Kronecker's second limit formula. A note on notation in this section is that depending on what looks better, we may write $\exp(z)$ or e^z for the exponential function.

Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ be a lattice in $\mathbb{C}$ with ordered basis so that $\tau = \omega_1/\omega_2$ is in the upper half plane. Define the sigma and delta functions of a complex number z and lattice L to be

$$\sigma(z, L) = z \prod_{\substack{\omega \in L \\ \omega \neq 0}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2\right)$$

$$\Delta(L) = \left(\frac{2\pi i}{\omega_2}\right)^{12} e^{2\pi i\tau} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})^{24}.$$

Let

$$A(L) = \frac{\omega_1\overline{\omega_2} - \overline{\omega_1}\omega_2}{2\pi i}$$

so $A(L)$ the area of $\mathbb{C}/L$ divided by π. Further let

$$\eta_1(L) = \omega_1 \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}, m\neq 0} \frac{1}{(m\omega_1 + n\omega_2)^2}$$

and

$$\eta_2(L) = \omega_2 \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{(m\omega_1 + n\omega_2)^2}$$

and define

$$\eta(z, L) = \frac{\omega_1\eta_2 - \omega_2\eta_1}{2\pi i A(L)}\overline{z} + \frac{\overline{\omega_2}\eta_1 - \overline{\omega_1}\eta_2}{2\pi i A(L)} z.$$

Then $\sigma(z, L)$ satisfies the following transformation law for all $\omega \in L$

$$\sigma(z + \omega, L) = \pm\sigma(z, L)\exp(\eta(\omega, L)(z + \omega/2)).$$

Define the fundamental theta function by

$$\theta(z, L) = \Delta(L)\exp(-6\eta(z, L)z)\sigma(z, L)^{12}.$$

The function $\theta(z, L)$ satisfies the transformation law $\theta(cz, cL) = \theta(z, L)$ for all $c \in \mathbb{C} - \{0\}$. As a function of z, $\theta(z, L)$ is not holomorphic. If $L = \mathbb{Z}\tau + \mathbb{Z}$ where τ is in the upper half plane, then $\theta(z, L)$ has the product expansion

$$\theta(z, L) = e^{\frac{6}{A(L)} z(z-\overline{z})} e^{2\pi i\tau} (e^{\pi i z} - e^{-\pi i z})^{12} \prod_{n=1}^{\infty} [(1 - e^{2\pi i(n\tau + z)})(1 - e^{2\pi i(n\tau - z)})]^{12}.$$

We now define Robert's units associated to an integral ideal of an imaginary quadratic field. Let F be any imaginary quadratic field, let $\mathfrak{f}$ be a non-trivial integral ideal of F. Let $G_{\mathfrak{f}} = \mathrm{Gal}(F(\mathfrak{f})/F)$ and for a fractional ideal $\mathfrak{a}$ coprime to $\mathfrak{f}$, let $\sigma_{\mathfrak{a}} \in G_{\mathfrak{f}}$ be the image of $\mathfrak{a}$ under the Artin map.

Definition 1.2.1. *Let $\mathfrak{f}$ be a nontrivial integral ideal of F and let f be the least positive integer in $\mathfrak{f} \cap \mathbb{Z}$. Define for $\sigma \in G_{\mathfrak{f}}$, the* ***Robert unit*** *associated to σ by*

$$E(\sigma) = \theta(1, \mathfrak{f}\mathfrak{c}^{-1})^f$$

where $\sigma_{\mathfrak{c}} = \sigma$. The complex number $E(\sigma)$ is well defined and we have the following proposition about its algebraic properties.

Proposition 1.2.2. *(page 55 in [9]) Let $\mathfrak{f}$ be a nontrivial integral ideal of F. Then for all $\sigma \in G_{\mathfrak{f}}$, we have*

(i) $E(\sigma) \in F(\mathfrak{f})$.

(ii) For all $\sigma' \in G_{\mathfrak{f}}$, $\sigma'(E(\sigma)) = E(\sigma'\sigma)$.

(iii) If $\mathfrak{f}$ is divisible by two distinct primes then $E(\sigma)$ is a unit in $F(\mathfrak{f})$. If $\mathfrak{f} = \mathfrak{q}^n$ for a prime $\mathfrak{q}$ of F, then $E(\sigma)$ is a $\mathfrak{q}$-unit.

The following theorem is known as Kronecker's second limit formula. It relates Robert units to special values of L-function associated to the imaginary quadratic field F.

Theorem 1.2.3. *(Kronecker's second limit formula) Let $\mathfrak{f}$ be a nontrivial ideal of F, let f be the least positive integer in $\mathfrak{f} \cap \mathbb{Z}$, and let $w_{\mathfrak{f}}$ be the number of roots of unity in*

F congruent to 1 mod $\mathfrak{f}$. Furthermore, let S be the infinite place of F and the places dividing $\mathfrak{f}$, and let v be the infinite place of $F(\mathfrak{f})$ induced by ι_∞. Then for all characters χ of $G_\mathfrak{f}$,

$$L'_S(\chi, 0) = -\frac{1}{12fw_\mathfrak{f}} \sum_{\sigma \in G_\mathfrak{f}} \log |E(\sigma)|_v.$$

When Stark stated his conjectures, he recast this theorem using the following lemma.

Lemma 1.2.4. *(Lemma 9 on page 225 of [30]) Keeping the notation of the theorem, let $K \subset F(\mathfrak{f})$ be a subfield of $F(\mathfrak{f})$ that is a nontrivial extension of F. Let $J \subset G_\mathfrak{f}$ be the subgroup such that $G_\mathfrak{f}/J = Gal(K/F)$, and define for $\sigma J \in G_\mathfrak{f}/J$*

$$E(\sigma J) = \prod_{\sigma' \in \sigma J} E(\sigma') = N_{F(\mathfrak{f})/K}(E(\sigma)).$$

Let e be the number of roots of unity in K. Then $E(\sigma J)^e$ is a $12fw_\mathfrak{f}$ power in K.

Definition 1.2.5. *Let $\mathfrak{f}$ be a nontrivial ideal of F and let $K \subset F(\mathfrak{f})$ be a nontrivial extension of F such that $Gal(K/F) = G_\mathfrak{f}/J$. Let e be the number of roots of unity in K, f the least positive integer in $\mathfrak{f} \cap \mathbb{Z}$, and $w_\mathfrak{f}$ the number of roots of unity in F congruent to 1 mod $\mathfrak{f}$. Define the **Stark unit** of the extension K/F, denoted u_K to be an element of K such that*

$$u_K^{12fw_\mathfrak{f}} = E(J)^e$$

where $E(J) = \prod_{\sigma \in J} E(\sigma)$. Such an element u_K exists by the previous lemma and is unique up to multiplication by a root of unity in K.

Theorem 1.2.6. *([30]) Keeping the notation as in the previous definition, we have that for all characters χ of Gal(K/F),*

$$L'_S(\chi,0) = -\frac{1}{e}\sum_{\sigma\in Gal(K/F)} \chi(\sigma)\log|\sigma(u_K)|_v$$

and $K(u_K^{1/e})$ is an abelian extension of F.

Remarks 1.2.7.

1. This theorem implies conjecture 1.1.1 when F is imaginary quadratic.

2. We could define a collection of Stark units, one for each $\sigma J \in G_{\mathfrak{f}}/J$ as

$$u(\sigma J) = \prod_{\sigma'\in\sigma J} = N_{F(\mathfrak{f})/K}(u(\sigma))$$

   where $u(\sigma)$ the Robert unit defined in $F(\mathfrak{f})$. Then for the correct choice of the $12fw_{\mathfrak{f}}$th roots we would have the reciprocity law

$$\sigma'(u(\sigma)) = u(\sigma'\sigma)$$

   for all $\sigma,\sigma' \in \mathrm{Gal}(K/F)$. In this case we would define $u_K = u(J)$ and then $\sigma(u_K) = u(\sigma J)$.

1.3 Ideas for a p-adic L-function in this case

Let F be a quadratic field and let χ be a ray class character of F such that $\mathrm{ord}_{s=0}(L(\chi,s)) = 1$. As stated in the introduction the goal of this thesis is to define a p-adic L-function associated to χ and to state a p-adic Stark conjecture. In this section we first explain the philosophy of how to define a p-adic L-function associated to χ. We then outline how we will rigorously carry out the philosophy in chapter 2.

The initial difficulty in defining a p-adic L-function associated to χ is that the complex L-function $L(\chi, s)$ has no critical values. That is, $L(\chi, n) = 0$ for all $n \in \mathbb{Z}_{\leq 0}$ because of the poles in the gamma factors in the functional equation for $L(\chi, s)$. It follows then that any p-adic continuous function that is determined by an interpolation formula involving the values of $L(\chi, s)$ at negative integers would have to be the zero function. Any meaningful p-adic L-function associated to χ would therefore not interpolate any of the values of $L(\chi, s)$.

The idea to define a meaningful p-adic L-function associated to χ is to use the theory of p-adic families of automorphic forms. We will put χ into a continuous family V of a p-adic automorphic forms where there exists a dense subset $X \subset V$ of "algebraic" automorphic forms with p-adic L-functions defined. The p-adic L-function of χ is then defined to be the limit of the p-adic L-functions already defined for the points X in V.

A first attempt to put χ into a p-adic family of automorphic forms would be to deform χ inside of the space of p-adic Hecke characters. When F is imaginary quadratic and p is split in F one can deform χ into a family of p-adic Hecke characters that contains enough algebraic Hecke characters whose complex L-function have critical values in order to define the p-adic L-function of χ. This is what Katz does ([17],[18], and see section 4.1 of *loc. cit.*). When F is real quadratic one cannot deform χ into a family of p-adic Hecke characters where any of the algebraic Hecke characters' complex L-functions have critical values.

A key idea of this thesis that allows us to deform χ into a family where complex L-functions of members of the family have critical values is to not view χ as a Hecke character of F, but to base change from F to $\mathbb{Q}$ and view χ as a modular form. Explicitly, χ becomes the weight one modular form f whose q-expansion is given by

$$f = \sum_{\mathfrak{a} \subset \mathscr{O}_F} \chi(\mathfrak{a}) q^{N\mathfrak{a}}.$$

One can explicitly see that the complex L-function of f is the same as the complex L-function of χ, while philosophically it follows functorially. Using the theory of p-adic families of modular forms we p-adically deform f into a family V of modular forms. If g is a newform of weight $k \in \mathbb{Z}_{\geq 2}$ (an "algebraic" modular form), then the complex L-function of g has critical values at the integers j with $1 \leq j \leq k-1$ and there is a p-adic L-function of g defined interpolating these values. We define the p-adic L-function of f to be the limit of the p-adic L-functions of the newforms g in of weights $k \in \mathbb{Z}_{\geq 2}$ in the family V.

Because the modular form f we are p-adically deforming into a family of modular forms is of weight one, the family of modular forms we consider will be a family of ordinary modular forms, known as a Hida family. Even though it is not strictly necessary to construct the eigencurve to talk about Hida families, we construct the ordinary locus of the eigencurve because we are interested in the weight one point corresponding to f. In [2], Bellaïche and Dmitrov do an analysis of the geometry of the eigencurve at weight one points and give a sufficient condition for the eigencurve to be smooth at a given weight one point. This smoothness condition allows one to define a two-variable

p-adic L-function whose first variable is on the eigencurve parameterizing a p-adic family of modular forms, and whose second variable is the usual cyclotomic variable on weight space. In order to use Bellaïche and Dmitrov's result we must introduce the formalism originally created by Glenn Stevens and then developed by Bellaïche in [1] to construct the the eigencurve using families of overconvergent modular symbols. This uses the language of rigid analytic geometry and the p-adic L-functions we consider are then p-adic rigid analytic functions.

In chapter 2, after giving our conventions for modular forms in section 2.1, we introduce weight space, the rigid analytic space which will be part of the domain of our p-adic L-function. We then introduce general modular symbols, which are used to define the p-adic L-function of an ordinary weight $k \geq 2$ modular form. In order to consider p-adic L-functions of p-adic modular forms of p-adic weights that are not integers $k \in \mathbb{Z}_{\geq 2}$ we introduce overconvergent modular symbols. This in particular will allow use to associate a p-adic L-function to an ordinary weight one modular form. Finally at the end of chapter 2 we introduce families of overconvergent modular symbols which we use to construct the ordinary locus of the eigencurve. This allows us to use the result of Bellaïche and Dmitrov ([2]) and the constructions of Bellaïche in [1] to construct two-variable p-adic L-functions.

Chapter 2

Background for definition of the p-adic L-function

2.1 Conventions for modular forms

In this section, we set some notation and conventions that will be fixed throughout for modular forms. We also state some relevant definitions for later reference.

Fix a positive integer N such that $p \nmid N$. The congruence subgroups we consider will be $\Gamma_1(N)$ and $\Gamma_1(N) \cap \Gamma_0(p)$. Let Γ denote either of these groups. Our Hecke actions will come via double coset algebras following the conventions in [13] and [25]. Let $\Sigma = \mathrm{GL}_2(\mathbb{Q}) \cap M_2(\mathbb{Z})$ and let $D(\Gamma, \Sigma)$ be the double coset algebra of the double cosets of Γ in Σ. Given $\gamma \in \Sigma$, we let $T(\gamma) \in D(\Gamma, \Sigma)$ denote the element corresponding to the double coset $\Gamma\gamma\Gamma$. We define the Hecke operators as the following elements of $D(\Gamma, \Sigma)$:

1. $T_\ell = T\left(\begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}\right)$ for $\ell \nmid Np$.

2. $U_p = T\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right)$ if $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$ and $T_p = T\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right)$ if $\Gamma = \Gamma_1(N)$.

3. $\iota = T\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$

To define the diamond operators, when $\Gamma = \Gamma_1(N)$, for $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, let $x, y \in \mathbb{Z}$ be such that $ax - Ny = 1$ and $\beta_d = \begin{pmatrix} x & y \\ Np^r & a \end{pmatrix} \in \Gamma_0(N)$. Then we define $[a] \in D(\Gamma, \Sigma)$ as $[a] = T(\beta_a)$. When $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$ for $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, let $x, y \in \mathbb{Z}$ be such that $ax - Npy = 1$ and $\beta_a = \begin{pmatrix} x & y \\ Np & a \end{pmatrix} \in \Gamma_0(Np)$. Then we define $[a] \in D(\Gamma, \Sigma)$ as $[a] = T(\beta_a)$.

Let the Hecke algebra be the algebra

$$\mathcal{H} = \begin{cases} \mathbb{Z}[T_\ell, \ell \nmid Np, U_p, [a], a \in (\mathbb{Z}/N\mathbb{Z})^\times] \subset D(\Gamma, \Sigma) & \text{if } \Gamma = \Gamma_1(N) \cap \Gamma_0(p) \\ \mathbb{Z}[T_\ell, \ell \nmid N, [a], a \in (\mathbb{Z}/N\mathbb{Z})^\times] \subset D(\Gamma, \Sigma) & \text{if } \Gamma = \Gamma_1(N). \end{cases}$$

If Σ' is a subsemigroup of Σ containing the matrices needed to define $\mathcal{H}$, then we also consider $\mathcal{H} \subset D(\Gamma, \Sigma')$.

For $k \geq 1$, we let $S_k(\Gamma, \overline{\mathbb{Q}})$ denote the space of holomorphic weight k and level Γ cusp forms with algebraic q-expansions, and let $S_k(N, \varepsilon, \overline{\mathbb{Q}}) \subset S_k(\Gamma_1(N), \overline{\mathbb{Q}})$ be the space of holomorphic cuspforms of level N and nebentypus ε with algebraic q-expansions. Let

$$S_k(\Gamma, \mathbb{C}_p) = S_k(\Gamma, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}_p \text{ and } S_k(\Gamma, \mathbb{C}) = S_k(\Gamma, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$$

and similarly let

$$S_k(N, \varepsilon, \mathbb{C}_p) = S_k(N, \varepsilon, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}_p \text{ and } S_k(N, \varepsilon, \mathbb{C}) = S_k(N, \varepsilon, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}.$$

It is a fact that $S_k(\Gamma, \mathbb{C})$ is the space of holomorphic cusp forms of level Γ.

Let $\mathcal{F}$ be the set of holomorphic functions f on the upper half plane such that for all $c \in \mathbb{P}^1(\mathbb{Q})$

$$\lim_{z \to c} |f(z)| = 0.$$

To make sense of the limit, we view $\mathbb{P}^1(\mathbb{Q})$ and the upper half place as a subsets of $\mathbb{P}^1(\mathbb{C})$. For $k \geq 1$, there is a weight-k action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $\mathcal{F}$ defined as follows: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, $f \in \mathcal{F}$,

$$f|_{\gamma,k}(z) = (cz+d)^k f\left(\frac{az+b}{cz+d}\right).$$

By definition the space of holomorphic cusp forms of weight k and level Γ is the set of invariants of Γ with respect to the weight-k action. Let $\Sigma^+ = \mathrm{GL}_2^+(\mathbb{Q}) \cap M_2(\mathbb{Z}) \subset \Sigma$. Since $S_k(\Gamma, \mathbb{C})$ is the space of holomorphic cusp forms, the action of Σ^+ on $\mathcal{F}$ induces an action of $\mathcal{H}$ on $S_k(\Gamma, \mathbb{C})$. This action leaves the space $S_k(\Gamma, \overline{\mathbb{Q}})$ invariant defining an action of $\mathcal{H}$ on $S_k(\Gamma, \overline{\mathbb{Q}})$ which we extend by linearity to an action on $S_k(\Gamma, \mathbb{C}_p)$. The fact that $\mathcal{H}$ preserves $S_k(\Gamma, \overline{\mathbb{Q}})$ can be seen by the explicit description of the action of $\mathcal{H}$ on q-expansions.

For the rest of this thesis, we adopt the notation $\Gamma_0 = \Gamma_1(N) \cap \Gamma_0(p)$.

Definition 2.1.1. *A **Hecke eigenform** (or just **eigenform**) of level N and character ε is an element $f \in S_k(N, \varepsilon, \mathbb{C}_p)$ which is an eigenvector for all the elements of $\mathcal{H}$. A **normalized eigenform** is a Hecke eigenform $f \in S_k(N, \varepsilon, \mathbb{C}_p)$ such that the leading term of the q-expansion of f is 1. If f is a normalized eigenform, then $f \in S_k(N, \varepsilon, \overline{\mathbb{Q}})$ and so we may also view f as an element of $S_k(N, \varepsilon, \mathbb{C})$. If $f \in S_k(N, \varepsilon, \overline{\mathbb{Q}})$ is a normalized eigenform that is new at level N, we call f a **newform**.*

Definition 2.1.2. *Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(N, \varepsilon, \overline{\mathbb{Q}})$ be a newform. Then the Hecke polynomial of f at p is the polynomial $x^2 - a_p x + \varepsilon(p)p^{k-1}$. Let α and β be the roots of this polynomial, so*

$$x^2 - a_p x + \varepsilon(p)p^{k-1} = (x - \alpha)(x - \beta).$$

*Then $f_\alpha(z) := f(z) - \beta f(pz)$ and $f_\beta(z) := f(z) - \alpha f(pz)$ are called the **p-stabilizations of** f.*

We have that $f_\alpha, f_\beta \in S_k(\Gamma_0, \overline{\mathbb{Q}})$ and are eigenvectors for the action of $\mathcal{H}$. The T_ℓ eigenvalues of f_α (respectively f_β) are the same as for f when $\ell \neq p$, and the U_p-eigenvalue of f_α (respectively f_β) is α (respectively β).

Definition 2.1.3. *Let $S_k^{ord}(N, \varepsilon, \mathbb{C}_p)$ (respectively $S_k^{ord}(\Gamma_0, \mathbb{C}_p)$) denote the maximal invariant subspace of $S_k(N, \varepsilon, \mathbb{C}_p)$ (respectively $S_k(\Gamma_0, \mathbb{C}_p)$) with respect to the action of T_p (respectively U_p) such that the characteristic polynomial of T_p (respectively U_p) restricted to this subspace has roots which are p-adic units. We call the subspace $S_k^{ord}(N, \varepsilon, \mathbb{C}_p)$ (respectively $S_k^{ord}(\Gamma_0, \mathbb{C}_p)$ the **ordinary subspace** of $S_k(N, \varepsilon, \mathbb{C}_p)$ (respectively $S_k(\Gamma_0, \mathbb{C}_p)$.*

*A cuspform $f \in S_k(N, \varepsilon, \mathbb{C}_p)$ (respectively $S_k(\Gamma_0, \mathbb{C}_p)$) is called p-**ordinary** if f is an element of the subspace $S_k^{ord}(N, \varepsilon, \mathbb{C}_p)$ (respectively $S_k^{ord}(\Gamma_0, \mathbb{C}_p)$).*

We remark that if $f \in S_k^{ord}(N, \varepsilon, \mathbb{C}_p)$ is a newform and $k \geq 2$, then there is a unique p-ordinary p-stabilization of f because the p-adic valuation of $\alpha + \beta = a_p(f)$ is 0 while the p-adic valuation of $\alpha\beta = \varepsilon(p)p^{k-1}$ is $k - 1 \geq 1$. On the other hand, if $f \in S_1(N, \varepsilon, \mathbb{C}_p)$ is a weight one newform, then there are two (possibly equal) p-ordinary p-stabilizations of f. This is because when f is a weight one newform, there is an odd irreducible Galois representation

$$\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}})$$

with finite image such that for all primes $\ell \nmid N$, ρ_f is unramified at ℓ and the characteristic polynomial of a Frobenius element at ℓ is the Hecke polynomial of f at ℓ:

$$x^2 - a_\ell(f)x + \varepsilon(\ell).$$

In particular, when $\ell = p$ we see that the roots of the Hecke polynomial, α and β are roots of unity because the image of a Frobenius at p has finite order. Therefore both of the p-stabilizations f_α and f_β have U_p-eigenvalue of p-adic unit.

Definition 2.1.4. *Let $f \in S_k(Np^r, \mathbb{C})$ for $k \geq 1$ have q-expansion*

$$f = \sum_{n=1}^{\infty} a_n q^n$$

*and let $\psi : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character. Then the **complex L-function of f twisted by ψ** is defined by*

$$L(f, \psi, s) = \sum_{n=1}^{\infty} \frac{\psi(n)a_n}{n^s}, \quad \mathrm{Re}(s) > \frac{k+1}{2}.$$

Here $\psi(n)$ is defined to be 0 if $(n, M) > 1$. $L(f, \psi, s)$ has an analytic continuation to all of $\mathbb{C}$ and satisfies a functional equation. If the character ψ is trivial, then we omit ψ from the notation and write $L(f, s)$.

If $f \in S_k(N, \varepsilon, \overline{\mathbb{Q}})$ is a newform, then $L(f, \psi, s)$ has an Euler product representation as

$$L(f, \psi, s) = \prod_{\ell | N} \frac{1}{1 - \psi(\ell) a_\ell \ell^{-s}} \prod_{\ell \nmid N} \frac{1}{1 - \psi(\ell) a_\ell \ell^{-s} + \psi^2 \varepsilon(\ell) \ell^{k-1-2s}}$$

where the two products are over primes $\ell \mid N$ and $\ell \nmid N$ respectively. Furthermore, the L-function of the p-stabilization of f, $f_\alpha(z) = f(z) - \beta f(pz)$ satisfies the relation

$$L(f_\alpha, \psi, s) = (1 - \psi(p)\beta p^{-s}) L(f, \psi, s).$$

2.2 Weight space

In this section we set our conventions for weight space.

Let $\mathcal{W} = \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{G}_m)$ denote weight space. As a rigid analytic space, $\mathcal{W}$ is the disjoint union of $p-1$ open unit disks. For any topological $\mathbb{Z}_p$-algebra R we have $\mathcal{W}(R) = \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, R^\times)$. Let $\mathcal{R}$ denote the $\mathbb{Q}_p$-algebra of rigid analytic functions on $\mathcal{W}$.

Let $D(0, r)$ denote the open unit disk of radius r around 0 in $\mathbb{C}_p$. If we fix a topological generator γ of $1 + p\mathbb{Z}_p$ then we identify $\mathcal{W}(\mathbb{C}_p)$ with $p - 1$ copies of $D(0, 1)$ as follows: First, the decomposition $\mathbb{Z}_p^\times = \mu_{p-1} \times 1 + p\mathbb{Z}_p$ gives a decomposition

$$\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}(\mu_{p-1}, \mathbb{C}_p^\times) \times \mathrm{Hom}_{cont}(1 + p\mathbb{Z}_p, \mathbb{C}_p^\times)$$

$$\kappa \longmapsto (\kappa|_{\mu_{p-1}}, \kappa|_{1+p\mathbb{Z}_p}).$$

Then we have a bijection

$$\begin{aligned} \mathrm{Hom}_{cont}(1+p\mathbb{Z}_p, \mathbb{C}_p^\times) &\longrightarrow D(0,1) \\ \kappa &\longmapsto \kappa(\gamma)-1 \end{aligned}$$

with inverse given by

$$\begin{aligned} D(0,1) &\longrightarrow \mathrm{Hom}_{cont}(1+p\mathbb{Z}_p, \mathbb{C}_p^\times) \\ w &\longmapsto \chi_w : z \mapsto (1+w)^{\log_\gamma \langle z \rangle}. \end{aligned}$$

Here $\log_\gamma(\cdot) := \log_p(\cdot)/\log_p(\gamma)$ and

$$(1+X)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} X^n.$$

The sum converges in our case since $\alpha \in \mathbb{Z}_p$ and $|X| < 1$. Let

$$\omega : (\mathbb{Z}/p\mathbb{Z})^\times \cong \mu_{p-1} \hookrightarrow \mathbb{C}_p$$

be the Teichmüller character. The $p-1$ open unit disks parameterizing $\mathcal{W}(\mathbb{C}_p)$ are given by powers of ω. Let ω^m for $0 \leq m \leq p-2$ be a power of ω. For m with $0 \leq m \leq p-2$, let $\mathcal{W}_m \subset \mathcal{W}$ denote the subset of $\mathcal{W}$ consisting of characters whose restriction to $(\mathbb{Z}/p\mathbb{Z})^\times$ is equal to ω^m. Then we have a parameterization of all the elements of $\mathcal{W}_m(\mathbb{C}_p)$:

$$\begin{aligned} D(0,1) &\longrightarrow \mathcal{W} \\ w &\longmapsto \chi_{m,w} : \begin{array}{rcl} \mathbb{Z}_p^\times & \to & \mathbb{C}_p^\times \\ z & \mapsto & \omega^m(z)(1+w)^{\log_\gamma \langle z \rangle} \end{array} \end{aligned}.$$

This identifies $\mathcal{W}(\mathbb{C}_p)$ with the disjoint union of $p-1$ open unit disks on $\mathbb{C}_p$.

For any topological $\mathbb{Z}_p$-algebra R,

$$\mathcal{W}_m(R) = \{\kappa \in \mathcal{W}(R) : \kappa|_{\mu_{p-1}} = \omega^m\}.$$

Here we use the $\mathbb{Z}_p$-algebra structure on R to view $\mu_{p-1} \subset R$.

We give an explicit description of certain admissible open subsets of the $\mathbb{Q}_p$-points of $\mathcal{W}_m$. For any $\kappa \in \mathcal{W}_m(\mathbb{Q}_p)$ and any $r \geq 1$, let $W(\kappa, 1/p^r)$ denote the closed disk of radius $1/p^r$ in $\mathcal{W}_m$ around κ. Then

$$W(\kappa, 1/p^r)(\mathbb{C}_p) = \{\kappa' \in \mathcal{W}_m(\mathbb{C}_p) : |\kappa'(\gamma) - \kappa(\gamma)| \leq 1/p^r\},$$

and $W(\kappa, 1/p^r)$ is an an admissible open subset of $\mathcal{W}_m$. The ring of $\mathbb{Q}_p$-rigid analytic functions on $W(\kappa, 1/p^r)$ is the $\mathbb{Q}_p$-algebra

$$R = \left\{ \sum_{n=0}^{\infty} a_n (w - (\kappa(\gamma) - 1))^n \in \mathbb{Q}_p[[w - (\kappa(\gamma) - 1)]] : |a_n p^{rn}| \to 0 \text{ as } n \to \infty \right\}$$

and $W(\kappa, 1/p^r) = \operatorname{Sp} R \subset \mathcal{W}$. We remark that R is isomorphic to the Tate algebra

$$\mathbb{Q}_p\langle T \rangle = \left\{ \sum_{n=0}^{\infty} a_n T^n \in \mathbb{Q}_p[[T]] : |a_n| \to 0 \text{ as } n \to \infty \right\}$$

via the map

$$\begin{array}{rcl} \mathbb{Q}_p\langle T \rangle & \longrightarrow & R \\ T & \longmapsto & (x - (\kappa(\gamma) - 1)/p^r \end{array}.$$

The sets $W(\kappa, 1/p^r)$ form a basis of admissible open neighborhoods of κ in $\mathcal{W}_m$.

2.3 Modular symbols

In this section we define modular symbols valued in general modules following [20]. Let $\triangle_0 = \operatorname{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the set of degree zero divisors on $\mathbb{P}^1(\mathbb{Q})$ and we view

$\triangle_0$ as a $\mathrm{GL}_2(\mathbb{Q})$-module via the usual action of linear fractional transformations. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and let V be a right Γ module. We define a right action of Γ on $\mathrm{Hom}(\triangle_0, V)$ via the rule

$$(\varphi|\gamma)(D) = \varphi(\gamma D)|\gamma$$

for $\varphi \in \mathrm{Hom}(\triangle_0, V)$, $\gamma \in \Gamma$, and $D \in \triangle_0$.

Definition 2.3.1. *The set of V**-valued modular symbols on** Γ, denoted $\mathrm{Symb}_\Gamma(V)$, is the set of all $\varphi \in \mathrm{Hom}(\triangle_0, V)$ that are invariant under the action of Γ.*

In the cases we consider, V has an action of a submonoid of $\mathrm{GL}_2(\mathbb{Q})$ which defines an action of $\mathcal{H}$ on $\mathrm{Symb}_\Gamma(V)$ through a double coset algebra. When 2 acts invertible on V and ι acts on $\mathrm{Symb}_\Gamma(V)$, we get a decomposition of $\mathrm{Symb}_\Gamma(V)$ into the sum of the 1 and -1 eigenspaces of ι, denoted

$$\mathrm{Symb}_\Gamma^+(V), \mathrm{Symb}_\Gamma^-(V) \subset \mathrm{Symb}_\Gamma(V).$$

If $\varphi \in \mathrm{Symb}_\Gamma(V)$, then we write $\varphi^\pm$ for the projection of φ onto $\mathrm{Symb}_\Gamma^\pm(V)$.

2.4 The p-adic L-function of an ordinary weight $k \geq 2$ modular form

In this section, we review how to use modular symbols to define the p-adic L-function of a weight $k \geq 2$ p-ordinary newform. For the rest of this thesis we let $\Gamma = \Gamma_1(N)$ and recall that $\Gamma_0 = \Gamma_1(N) \cap \Gamma_0(p)$. Let R be a commutative ring, and for $k \in \mathbb{Z}_{\geq 0}$, let $V_k(R) = \mathrm{Sym}^k(R^2)$ be the R-module of homogeneous polynomials of degree k in two variables X and Y with coefficients in R. We have an action of $\mathrm{GL}_2(R)$ on $V_k(R)$ as follows: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$ and $P \in V_k(R)$, define

$$(P|\gamma)(X,Y) = P((X,Y)\gamma*) = P(dX - cY, -bX + aY)$$

where $\gamma* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Given a modular form $f \in S_{k+2}(N, \epsilon, \mathbb{C})$ (or $S_{k+2}(\Gamma_0, \mathbb{C})$ respectively) we define the standard modular symbol associated to f, denoted ψ_f, to be the function

$$\psi_f : \triangle_0 \longrightarrow V_k(\mathbb{C})$$

$$\psi_f(\{b\} - \{a\}) = 2\pi i \int_a^b f(z)(zX+Y)^k dz.$$

A calculation shows that $\psi_f \in \mathrm{Symb}_\Gamma(V_k(\mathbb{C}))$ (respectively $\mathrm{Symb}_{\Gamma_0}(V_k(\mathbb{C}))$). The action of $\mathrm{GL}_2(\mathbb{C})$ on $V_k(\mathbb{C})$ allows us to define an action of $\mathcal{H}$ on $\mathrm{Symb}_\Gamma(V_k(\mathbb{C}))$ (respectively $\mathrm{Symb}_{\Gamma_0}(V_k(\mathbb{C}))$) and with these actions, the maps

$$\begin{array}{ccc} S_{k+2}(N,\epsilon,\mathbb{C}) & \longrightarrow & \mathrm{Symb}_\Gamma(V_k(\mathbb{C})) \\ f & \longmapsto & \psi_f \end{array} \quad \text{and} \quad \begin{array}{ccc} S_{k+2}(\Gamma_0,\mathbb{C}) & \longrightarrow & \mathrm{Symb}_{\Gamma_0}(V_k(\mathbb{C})) \\ & f \longmapsto \psi_f & \end{array}$$

are Hecke equivariant.

If we now assume that f is a normalized eigenform on Γ (or Γ_0), and let $K(f)$ be the field obtained by adjoining the Hecke eigenvalues of f to $\mathbb{Q}$, then Shimura ([24]) showed that there exist complex periods $\Omega_f^{\pm} \in \mathbb{C}^\times$ such that $\psi_f^{\pm}/\Omega_f^{\pm} \in \mathrm{Symb}_\Gamma(V_k(K(f))$. He showed furthermore, that the Hecke eigenspaces in $\mathrm{Symb}_\Gamma^{\pm}(V_k(K(f))$ (respectively

$\mathrm{Symb}^{\pm}_{\Gamma_0}(V_k(K(f)))$ with the same eigenvalues as f are 1-dimensional over $K(f)$. Therefore the complex periods $\Omega^{\pm}_f$ are well defined up to $K(f)^{\times}$.

This algebraicity result of Shimura allows one to view the modular symbol associated to f p-adically in order to define the p-adic L-function of f. Now assume f is p-ordinary and let f_α be the ordinary p-stabilization of f. Let

$$\varphi^{\pm}_{f_\alpha} = \psi^{\pm}_{f_\alpha}/\Omega^{\pm}_{f_\alpha} \in \mathrm{Symb}_{\Gamma_0}(V_k(K(f_\alpha))).$$

Via ι_p, we may view $\varphi^{\pm}_{f_\alpha}$ as an element of $\mathrm{Symb}_{\Gamma_0}(V_k(\mathbb{C}_p))$.

Mazur-Tate and Teitelbaum ([19]) proved that the function $\mu^{\pm}_{f_\alpha}$ defined by the rule

$$\mu^{\pm}_{f_\alpha}(a + p^m\mathbb{Z}_p) = \alpha^{-m}\varphi^{\pm}_{f_\alpha}\left(\left\{\frac{a}{p^m}\right\} - \{\infty\}\right)|_{X=0,Y=1}$$

is a $\mathbb{C}_p$ valued measure on $\mathbb{Z}_p^{\times}$. Given a finite order character $\psi \in \mathcal{W}(\mathbb{C}_p)$, we then define the p-adic L-function of f_α twisted by ψ to be the analytic function of $s \in \mathbb{Z}_p$ given by

$$L_p(f_\alpha, \psi, s) = \int_{\mathbb{Z}_p^{\times}} \psi^{-1}(t)\langle t\rangle^{s-1} d\mu^{sgn(\psi)}_{f_\alpha}(t).$$

We record here the interpolation property of $L_p(f_\alpha, \psi, s)$ for future reference.

Theorem 2.4.1. *([19]) Let f_α be the ordinary p-stabilization of a p-ordinary newform of level N and weight $k+2 \geq 2$. Let $\psi \in \mathcal{W}(\mathbb{C}_p)$ be a finite order character of conductor p^m. Then $L_p(f_\alpha, \psi, s)$ is a p-adic analytic function on $\mathbb{Z}_p$ with the interpolation property that for all integers j with $0 < j < k+2$,*

$$L_p(f_\alpha, \psi, j) = \frac{1}{\alpha^m}\left(1 - \frac{\psi^{-1}\omega^{1-j}(p)}{\alpha p^{1-j}}\right)\frac{p^{m(j-1)}(j-1)!\tau(\psi^{-1}\omega^{1-j})}{(2\pi i)^{j-1}}\frac{L(f_\alpha, \psi\omega^{j-1}, j)}{\Omega^{sgn(\psi)}_{f_\alpha}}.$$

Here $\tau(\psi^{-1}\omega^{1-j})$ is the Gauss sum associated to $\psi^{-1}\omega^{1-j}$.

Remarks 2.4.2. If f is a non-ordinary Hecke eigenform with Hecke polynomial

$$x^2 - a_p(f)x + \varepsilon(p)p^{k+1} = (x - \alpha)(x - \beta)$$

then one may define the p-adic L-function of either p-stabilization f_α or f_β of f in the same way as above but a little more care is needed because the distribution μ_{f_α} (or μ_{f_β}) is not a measure. For the critical p-stabilization f_β when f is p-ordinary, even more care is needed. See [21] and [1] for more information about these cases.

When f is a weight 1 modular form there is no modular symbol associated to f and so the above constructions do not work. We next introduce overconvergent modular symbols, which allow one to make a definition of a p-adic L-function of either p-stabilization of a weight one modular form.

2.5 Overconvergent modular symbols

In this section we introduce overconvergent modular symbols following the notation and conventions of [1] and [20].

For each $r \in |\mathbb{C}_p^\times| = p^{\mathbb{Q}}$, let

$$B[\mathbb{Z}_p, r] = \{z \in \mathbb{C}_p : \exists a \in \mathbb{Z}_p, |z - a| \leq r\}.$$

$B[\mathbb{Z}_p, r]$ is the set of $\mathbb{C}_p$-points of the $\mathbb{Q}_p$-rigid analytic space which is the union of the closed unit balls of radius r around each point in $\mathbb{Z}_p$. Let $\mathbb{A}[r]$ be the $\mathbb{Q}_p$-algebra of rigid analytic functions on $B[\mathbb{Z}_p, r]$. An element $f \in \mathbb{A}[r]$ is the following data: for each

$a \in \mathbb{Z}_p$, a formal expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(a)(z-a)^n \in \mathbb{Q}_p[[z-a]]$$

which converges on the closed ball centered at a with radius r, such that the collection of power series expansions agree on $B[\mathbb{Z}_p, r]$ as a varies. The sup norm on $\mathbb{A}[r]$ makes $\mathbb{A}[r]$ a $\mathbb{Q}_p$-Banach space. Explicitly the norm is given for $f \in \mathbb{A}[r]$ by

$$\|f\|_r = \sup_{z \in B[\mathbb{Z}_p, r]} |f(z)|.$$

We remark that if $r \geq 1$, then $B[\mathbb{Z}_p, r]$ is the closed disc in $\mathbb{C}_p$ of radius r around 0 and so

$$\mathbb{A}[r] = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{Q}_p[[z]] : |a_n| r^n \to 0 \right\}.$$

Let $\mathbb{D}[r] = \mathrm{Hom}_{\mathbb{Q}_p}(\mathbb{A}[r], \mathbb{Q}_p)$ be the continuous $\mathbb{Q}_p$-dual of $\mathbb{A}[r]$. The space $\mathbb{D}[r]$ is a $\mathbb{Q}_p$-Banach space with norm given by

$$\|\mu\|_r = \sup_{f \in \mathbb{A}[r], f \neq 0} \frac{|\mu(f)|}{\|f\|_r}$$

for $\mu \in \mathbb{D}[r]$. For $r_1 > r_2$, restriction of functions gives a map $\mathbb{A}[r_1] \to \mathbb{A}[r_2]$. This map is injective, has dense image, and is compact. The dual map $\mathbb{D}[r_2] \to \mathbb{D}[r_1]$ is injective and compact. For any real number $r \geq 0$ define

$$\mathbb{A}^\dagger[r] = \varinjlim_{s>r} \mathbb{A}[s] \text{ and } \mathbb{D}^\dagger[r] = \varprojlim_{s>r} \mathbb{D}[s].$$

We give $\mathbb{A}^\dagger[r]$ the inductive limit topology and $\mathbb{D}^\dagger[r]$ the projective limit topology. For the remainder of this thesis, we write $\mathbb{A} = \mathbb{A}^\dagger[0]$ and $\mathbb{D} = \mathbb{D}^\dagger[0]$. We remark that $\mathbb{D}$ is the $\mathbb{Q}_p$-linear dual to $\mathbb{A}$, and that $\mathbb{A}$ may be identified with the set of locally analytic functions on $\mathbb{Z}_p$ and $\mathbb{D}$ the set of locally analytic distributions.

We next explain how to associate a p-adic L-function to a distribution $\mu \in \mathbb{D}$. By integrating, we may view μ as a function on $\mathcal{W}(\mathbb{C}_p)$. We associate $p-1$ power series in $\mathbb{C}_p[[w]]$ to μ. These power series represent the p-adic Mellin transform of μ. Let ω^i be a power of the Teichmüller character. Using the notation from section 2.2, define for $w \in D(0,1)$ and γ a toppological generator of $1 + p\mathbb{Z}_p$,

$$P_\gamma(\mu, \omega^i, w) := \mu(\chi_{i,w}) = \int_{\mathbb{Z}_p^\times} \chi_{i,w}(z) d\mu(z).$$

We have

$$\begin{aligned} P_\gamma(\mu, \omega^i, w) &= \int_{\mathbb{Z}_p^\times} \chi_{i,w}(z) d\mu(z) \\ &= \int_{\mathbb{Z}_p^\times} \omega^i(z)(1+w)^{\log_\gamma \langle z \rangle} d\mu(z) \\ &= \sum_{a=1}^{p-1} \omega^i(a) \int_{1+p\mathbb{Z}_p} (1+w)^{\log_\gamma (z)} d\mu(z) \\ &= \sum_{a=1}^{p-1} \omega^i(a) \int_{1+p\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{\log_\gamma z}{n} w^n d\mu(z) \\ &= \sum_{n=0}^{\infty} \left(\sum_{a=1}^{p-1} \omega^i(a) \int_{1+p\mathbb{Z}_p} \binom{\log_\gamma z}{n} d\mu(z) \right) w^n \end{aligned}$$

so $P_\gamma(\mu, \omega^i, w) \in \mathbb{C}_p[[w]]$. Furthermore, since $\binom{\log_\gamma z}{n} \in \mathbb{Z}_p$ for $z \in 1 + p\mathbb{Z}_p$ and the above series converges for any $w \in B(0,1)$, $P_\gamma(\mu, \omega^i, w)$ is a $\mathbb{Q}_p$-rigid analytic function on $\mathcal{W}_i$.

Recall that $\mathcal{R}$ is the ring of rigid analytic functions on $\mathcal{W}$. Because $P_\gamma(\mu, \omega^i, w)$ is a $\mathbb{Q}_p$-rigid analytic function on $\mathcal{W}_i$ integration defines a map from $\mathbb{D}$ to $\mathcal{R}$. We denote this map by $\mathcal{L}$:

$$\mathcal{L} : \mathbb{D} \longrightarrow \mathcal{R}$$

$$\mathcal{L}(\mu) : \mathcal{W}(\mathbb{C}_p) \longrightarrow \mathbb{C}_p$$

$$\mathcal{L}(\mu)(\chi) = \int_{\mathbb{Z}_p^\times} \chi(z) d\mu(z)$$

for $\mu \in \mathbb{D}$. Note that the element $\mathcal{L}(\mu) \in \mathcal{R}$ does not depend on the choice of γ, but to represent $\mathcal{L}(\mu)$ with the power series $P_\gamma(\mu, \omega^i, w)$ requires the choice of γ.

We now define overconvergent modular symbols. Define

$$\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : p \nmid a, p \mid c \text{ and } ad - bc \neq 0 \right\}.$$

For any integer $k \in \mathbb{Z}$, we define a weight k action of $\Sigma_0(p)$ on $\mathbb{A}[r]$ for $r < p$ as follows. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p),\ f \in \mathbb{A}[r]$, let

$$(\gamma \cdot_k f)(z) = (a + cz)^k f\left(\frac{b + dz}{a + cz}\right).$$

This induces an action of $\Sigma_0(p)$ on $\mathbb{D}[r]$ on the right via

$$(\mu|_k\gamma)(f) = \mu(\gamma \cdot_k f)$$

for $\mu \in \mathbb{D}[r]$. These actions induce actions of $\Sigma_0(p)$ on $\mathbb{A}$ and $\mathbb{D}$. When we consider $\mathbb{A}$ and $\mathbb{D}$ with their weight k actions, we write k in the subscript, $\mathbb{A}_k$, $\mathbb{D}_k$.

Next we define the weight κ action for elements $\kappa \in \mathcal{W}(\mathbb{Q}_p)$. Here we are following the exposition in [11]. Let $W_m \subset \mathcal{W}_m$ be the subset of characters κ such that $|\kappa(\gamma) - 1| \leq 1/p$ for any topological generator γ of $1 + p\mathbb{Z}_p$. That is, W_m is the closed disk of radius $1/p$ around 0 in $\mathcal{W}_m$. We will define a weight κ action for any $\kappa \in W_m(\mathbb{Q}_p)$.

To begin, let $\kappa \in W_0(\mathbb{Q}_p)$ and define

$$F_\kappa(z) = \sum_{n=0}^{\infty} \binom{\log_\gamma(1+z)}{n} (\kappa(\gamma)-1)^n = (1 + (\kappa(\gamma)-1))^{\log_\gamma(1+z)}$$

where as before, γ is any topological generator of $1 + p\mathbb{Z}_p$. Note that the definition of $F_\kappa(z)$ does not depend on the choice of γ. We have ([11]) that $F_\kappa(z)$ converges on $D(0, p^{-1/(p-1)})$ and for all $x \in 1 + p\mathbb{Z}_p$, $F_\kappa(x-1) = \kappa(x)$.

Now for a weight $\kappa \in W_m(\mathbb{Q}_p)$, write $\kappa = \omega^m \kappa_0$ so $\kappa_0 \in W_0(\mathbb{Q}_p)$. Let $\alpha \in \Sigma_0(p)$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Define

$$F_{\kappa,\alpha}(z) := \omega(a)^m F_{\kappa_0}\left(\frac{a+cz}{\omega(a)} - 1\right).$$

Then ([11]) $F_{\kappa,\alpha}(z)$ converges for $z \in B(0, p^{\frac{p-2}{p-1}})$ and for $x \in \mathbb{Z}_p$,

$$F_{\kappa,\alpha}(x) = \kappa(a + cx).$$

Finally, given $\kappa \in W_m(\mathbb{Q}_p)$, $f \in \mathbb{A}[r]$, and $\alpha \in \Sigma_0(p)$ we define the weight-κ action on $\mathbb{A}[r]$ for $r < p^{\frac{p-2}{p-1}}$ as

$$(\alpha \cdot_\kappa f)(z) = F_{\kappa,\alpha}(z) f\left(\frac{b+dz}{a+cz}\right).$$

By the convergence properties of $F_{\kappa,\alpha}(z)$, $\alpha \cdot_\kappa f \in \mathbb{A}[r]$. Dually we get an action on $\mathbb{D}[r]$ where for $\mu \in \mathbb{D}[r]$,

$$(\mu|_\kappa \alpha)(f) = \mu(\alpha \cdot_\kappa f).$$

These actions induce weight κ actions on $\mathbb{D}$ and $\mathbb{A}$. As before, when we want to emphasize the weight κ action we write $\mathbb{A}_\kappa$ or $\mathbb{D}_\kappa$. The spaces of modular symbols of interest are $\mathrm{Symb}_{\Gamma_0}(\mathbb{D}_\kappa)$. These space are Hecke modules via the action of $\Sigma_0(p)$ on $\mathbb{D}_\kappa$.

Definition 2.5.1. *Let $\kappa \in W_m(\mathbb{Q}_p)$. The space of* ***overconvergent modular symbols of weight*** κ *is defined to be* $\mathrm{Symb}_{\Gamma_0}(\mathbb{D}_\kappa)$.

Let $\varphi \in \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_\kappa)$ be an overconvergent modular symbol of weight κ. We define the p-adic L-function of φ by composing the following two maps: first evaluation at $\{0\} - \{\infty\}$ to obtain a locally analytic distribution and then the map $\mathcal{L}$ from before.

Definition 2.5.2. *We call the above composition the* ***Mellin transform of*** φ *and denote it by* Λ_κ*:*

$$\Lambda_\kappa : \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_\kappa) \to \mathcal{R}$$

$$\Lambda_\kappa(\varphi)(\chi) = \int_{\mathbb{Z}_p^\times} \chi(z) d(\varphi(\{0\} - \{\infty\}))(z)$$

for $\varphi \in \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_\kappa)$ *and* $\chi \in \mathcal{W}(\mathbb{C}_p)$*. By definition,* Λ_κ *is a* $\mathbb{Q}_p$*-linear map.*

We end this section by stating the relationship between overconvergent modular symbols of nonnegative integral weight to the spaces of modular symbols introduced in the previous section. Let $k \in \mathbb{Z}_{\geq 0}$ and define the map

$$\rho_k : \mathbb{D}_k \longrightarrow V_k(\mathbb{Q}_p)$$

$$\rho_k(\mu) = \int_{\mathbb{Z}_p} (Y - zX)^k d\mu(z).$$

The integration in the definition of ρ_k takes place coefficient by coefficient. The map ρ_k is $\Sigma_0(p)$-equivarient, so induces a Hecke equivariant map

$$\rho_k^* : \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k) \longrightarrow \mathrm{Symb}_{\Gamma_0}(V_k(\mathbb{Q}_p)).$$

Let $\mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k)^{<k+1}$ and $\mathrm{Symb}_{\Gamma_0}(V_k(\mathbb{Q}_p))^{<k+1}$ denote the subspaces of $\mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k)$ and $\mathrm{Symb}_{\Gamma_0}(V_k(\mathbb{Q}_p))$ where U_p acts with eigenvalue that has p-adic valuation less than

$k+1$. Stevens control theorem ([20]) states that the restriction of ρ_k^* to these subspaces is an isomorphism of Hecke modules.

Theorem 2.5.3. *([20]) For $k \in \mathbb{Z}_{\geq 0}$ the map*

$$\rho_k^* : \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k)^{<k+1} \longrightarrow \mathrm{Symb}_{\Gamma_0}(V_k(\mathbb{Q}_p))^{<k+1}$$

is an isomorphism of Hecke modules.

Remarks 2.5.4. Let $k \in \mathbb{Z}_{\geq 0}$, let f be a p-stabilized newform of weight $k+2$. Let $\varphi_f^{\pm} \in \mathrm{Symb}_{\Gamma_0}^{\pm}(V_k(\mathbb{Q}_p))$ be the modular symbol defined in the section 2.4. By Stevens control theorem there exists unique $\widetilde{\varphi}_f^{\pm} \in \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k)$ such that $\rho_k^*(\widetilde{\varphi}_f^{\pm}) = \varphi_f^{\pm}$. We have the following compatibility:

$$\Lambda_k(\widetilde{\varphi}^{sgn(\psi)})(\psi^{-1}\langle\cdot\rangle^{s-1}) = L_p(f_\alpha, \psi, s)$$

for all $s \in \mathbb{Z}_p$ and finite order characters $\psi \in \mathcal{W}(\mathbb{C}_p)$.

2.6 Families of overconvergent modular symbols

In this section, we introduce families of overconvergent modular symbols over certain open subsets of weight space and use these families to construct the ordinary locus of the eigencurve over these open subsets of weight space following [1] and [11].

Let $\kappa' \in W_m(\mathbb{Q}_p)$ and let $W = W(\kappa', 1/p^r)$ for some $r \in \mathbb{Z}_{\geq 1}$. Let R be the ring of $\mathbb{Q}_p$-affinoid functions on W in the variable w (so we have fixed a topological generator γ of $1 + p\mathbb{Z}_p$ that induces w). Then

$$R = \left\{ \sum_{n=0}^{\infty} a_n(w - (\kappa'(\gamma) - 1))^n \in \mathbb{Q}_p[[w - (\kappa'(\gamma) - 1)]] : |a_n p^{rn}| \to 0 \text{ as } n \to \infty \right\}.$$

Given $\kappa \in W(\mathbb{Q}_p)$ and $F(w) \in R$, we define the evaluation at κ map which we denote by

$$ev_\kappa : R \longrightarrow \mathbb{C}_p$$

$$ev_\kappa(F) = F(\kappa(\gamma) - 1).$$

Define

$$\mathbb{A}[r](R) := \mathbb{A}[r]\widehat{\otimes}_{\mathbb{Q}_p} R$$

for $r \in p^{\mathbb{Q}}$. The evaluation maps induce maps

$$ev_\kappa : \mathbb{A}[r](R) \longrightarrow \mathbb{A}[r]$$

for all r.

We define an action of $\Sigma_0(p)$ on $\mathbb{A}[r](R)$ that is compatible with the evaluation maps and the action defined in the previous section. Note that $\mathbb{A}[1]$ is the Tate algebra. Let z be the variable for $\mathbb{A}[1]$, so $\mathbb{A}[1] = \mathbb{Q}_p\langle z \rangle$. For $r \leq 1$ the injection $\mathbb{A}[1] \hookrightarrow \mathbb{A}[r]$ induces an inclusion

$$\mathbb{A}[1](R) \hookrightarrow \mathbb{A}[r](R).$$

We then define for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p)$, $0 \leq m \leq p-1$ the element $K_{\alpha,m}(z,w) \in \mathbb{Q}_p[[z,w]]$:

$$K_{\alpha,m}(z,w) = \omega(a)^m \sum_{n=0}^{\infty} \binom{\log_\gamma(\frac{a+cz}{\omega(a)})}{n} w^n = \omega(a)^m (1+w)^{\log_\gamma(\frac{a+cz}{\omega(a)})}.$$

We have that ([11]) $K_{\alpha,m}(z,w) \in \mathbb{A}[1](R)$ and for all $\kappa \in W(\mathbb{Q}_p)$,

$$ev_\kappa(K_{\alpha,m}(z,w)) = F_{\kappa,\alpha}(z).$$

We use $K_{\alpha,m}$ to define an action of $\Sigma_0(p)$ on $\mathbb{A}[r](R)$ for $r < p^{\frac{p-2}{p-1}}$. To do this, we view $K_{\alpha,m}$ as an element of $\mathbb{A}[r](R)$ via the inclusion $\mathbb{A}[1](R) \subset \mathbb{A}[r](R)$. We then use the ring structure on $\mathbb{A}[r](R)$ to multiply $K_{\alpha,m}$ with elements of $\mathbb{A}[r](R)$. That is, define for $f \in \mathbb{A}[r]$, $F \in R$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p)$,

$$\alpha \cdot (f(z) \otimes F(w)) = K_{\alpha,m}(z,w)(f\left(\frac{b+dz}{a+cz}\right) \otimes F(w)).$$

on simple tensors and extend this to an action on $\mathbb{A}[r](R)$ by linearity.

Define

$$\mathbb{D}[r](R) := \mathbb{D}[r]\widehat{\otimes}_{\mathbb{Q}_p} R.$$

We remark that $\mathbb{D}[r](R)$ is not the continuous $\mathbb{Q}_p$-dual to $\mathbb{A}[r](R)$. The map

$$\mathbb{D}[r](R) \longrightarrow \mathrm{Hom}_{\mathbb{Q}_p}(\mathbb{A}[r](R), \mathbb{Q}_p)$$

is injective but not surjective (see the appendix of [11]). We then define an action of $\Sigma_0(p)$ on $\mathbb{D}[r](R)$ as follows: Note that $\mathbb{D}[r]$ is an $\mathbb{A}[r]$-module via

$$(g \cdot \mu)(f) = \mu(gf)$$

where $f, g \in \mathbb{A}[r]$, $\mu \in \mathbb{D}[r]$. Then $\mathbb{D}[r](R)$ is an $\mathbb{A}[r](R)$-module. Define for $\mu \otimes F \in \mathbb{D}[r](R)$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p)$,

$$(\mu \otimes F)|\alpha = K_{\alpha,m}(z,w)(\mu|_0\alpha \otimes F)$$

where $\mu|_0\alpha$ is the weight 0 action on $\mathbb{D}[r]$.

Now let $\mathbb{D}(R) = \varprojlim_{r>0} \mathbb{D}[r](R)$. The actions of $\Sigma_0(p)$ on $\mathbb{D}[r](R)$ induce an action on $\mathbb{D}(R)$. We have (lemma 3.2 of [1]) a natural isomorphism

$$\mathbb{D}\widehat{\otimes}_{\mathbb{Q}_p} R \longrightarrow \mathbb{D}(R).$$

The map induced by evaluation at κ from $\mathbb{D}(R)$ to $\mathbb{D}_\kappa$, will be called specialization to weight κ and denoted by sp_κ:

$$\begin{array}{rcl} sp_\kappa : \mathbb{D}(R) & \longrightarrow & \mathbb{D}_\kappa \\ \mu \otimes F & \longmapsto & ev_\kappa(F)\mu \end{array}.$$

The map sp_κ is $\Sigma_0(p)$-equivariant and induces a Hecke equivariant specialization map which we denote by the same name

$$sp_\kappa : \mathrm{Symb}_{\Gamma_0}(\mathbb{D}(R)) \longrightarrow \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_\kappa).$$

To end this section, we rephrase some results of Bellaïche ([1]) about the relation between $\mathrm{Symb}_{\Gamma_0}(\mathbb{D}(R))$ and $\mathrm{Symb}_{\Gamma_0}(\mathbb{D}_\kappa)$ as Hecke modules.

Definition 2.6.1. *Fix a weight* $\kappa \in W(\mathbb{Q}_p)$. *Let* $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)^o \subset \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)$ *(respectively* $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o \subset \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))$*) be the subspace where* U_p *acts with slope bounded by* 0 *in the sense of [1] section 3.2.4. Let* $\mathbb{T}^{\pm}_\kappa$ *(respectively* $\mathbb{T}^{\pm}_W$*) be the* $\mathbb{Q}_p$*-subalgebra of* $\mathrm{End}_{\mathbb{Q}_p}(\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)^o)$ *(respectively the* R*-subalgebra of* $\mathrm{End}_R(\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o)$*) generated by the image of* $\mathcal{H}$*. We call* $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)^o$ *(respectively* $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o$*) the* ***ordinary subspace*** *of* $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)$ *(respectively* $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))$*).*

We have ([1] section 3.2.4) that $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o$ is a finite projective R-module. Since $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o$ is a finite projective R-module, $\mathbb{T}^{\pm}_W$ is a finite R-algebra and so an affinoid algebra (see theorem 7.1.11 of the appendix). Furthermore, $\mathbb{T}_W^{\pm}$ is torsion-free as an R-module and since R is a principal ideal domain, $\mathbb{T}_W^{\pm}$ is flat.

Theorem 2.6.2. *(Bellaïche's specialization theorem (Corollary 3.12 in [1])) Let $\kappa \in W(\mathbb{Q}_p)$. The specialization map restricted to the ordinary subspaces*

$$sp_\kappa : \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o \longrightarrow \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)^o \tag{2.1}$$

is surjective.

Since sp_κ is an $\mathcal{H}$-equivariant surjective map, it induces an $\mathcal{H}$-equivariant map

$$sp_\kappa : \mathbb{T}_W^{\pm} \longrightarrow \mathbb{T}_\kappa^{\pm}$$

which we use in the following definition.

Definition 2.6.3. *Let $x : \mathbb{T}_\kappa^{\pm} \to \mathbb{C}_p$ be a $\mathbb{Q}_p$-algebra homomorphism. The homomorphism x corresponds to a system of $\mathcal{H}$-eigenvalues appearing in $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)^o$. Let $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)_{(x)}$ denote the corresponding generalized eigenspace and let $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)[x]$ denote the eigenspace.*

1. *Let $(\mathbb{T}_\kappa^{\pm})_{(x)}$ be the localization of $\mathbb{T}_\kappa^{\pm} \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ at the kernel of x. We have that*

$$\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)_{(x)} = \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)^o \otimes_{\mathbb{T}_\kappa^{\pm}} (\mathbb{T}_\kappa^{\pm})_{(x)}.$$

2. *Through the specialization map, x induces a $\mathbb{Q}_p$-algebra homomorphism which we also denote by x:*

$$x = x \circ sp_\kappa : \mathbb{T}_W^{\pm} \longrightarrow \mathbb{C}_p.$$

Let $(\mathbb{T}_W^{\pm})_{(x)}$ *be the rigid analytic localization of* $\mathbb{T}_W^{\pm} \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ *at the kernel of* $x \circ sp_\kappa$*, and let*

$$\mathrm{Symb}_{\Gamma_0}^{\pm}(\mathbb{D}(R))_{(x)} = \mathrm{Symb}_{\Gamma_0}^{\pm}(\mathbb{D}(R))^o \otimes_{\mathbb{T}_W^{\pm}} (\mathbb{T}_W^{\pm})_{(x)}.$$

Let $R_{(\kappa)}$ *be the rigid analytic localization of* $R \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ *at the kernel of* ev_κ*. We can then localize the specialization map* sp_κ *to get a map*

$$sp_\kappa : (\mathbb{T}_W^{\pm})_{(x)} \otimes_{R_{(\kappa)},\kappa} \mathbb{C}_p \longrightarrow (\mathbb{T}_\kappa^{\pm})_{(x)}.$$

In ([1]), Bellaïche following Stevens uses these spaces of families of overconvergent modular symbols to construct the eigencurve. Since $\mathbb{T}_W^{\pm}$ is an affinoid algebra, we can let $\mathcal{C}_W^{\pm} = \mathrm{Sp}\,\mathbb{T}_W^{\pm}$. Then $\mathcal{C}_W^{\pm}$ is the ordinary locus of the eigencurve above the open set W of weight space. The weight map

$$\kappa^{\pm} : \mathcal{C}_W^{\pm} \longrightarrow W$$

is the map of rigid analytic spaces induced by the $\mathbb{Q}_p$-algebra homomorphism $R \to \mathbb{T}_W^{\pm}$. Since $\mathbb{T}_W^{\pm}$ is a finite, flat R-module, the map $\kappa^{\pm}$ is finite and flat. Given a point $x \in C_W^{\pm}(\mathbb{C}_p)$, we define the weight of x to be $\kappa^{\pm}(x) \in W(\mathbb{C}_p)$. It is a fact that every point of $C_W^{\pm}(\mathbb{C}_p)$ of weight κ is the pullback of a homomorphism $x : \mathbb{T}_\kappa^{\pm} \to \mathbb{C}_p$ from $\mathbb{T}_\kappa^{\pm}$ to $\mathbb{T}_W^{\pm}$.

Theorem 2.6.4. *([1]) Let* $x \in \mathcal{C}_{W_m}^{\pm}(\mathbb{C}_p)$ *be a smooth point on the eigencurve of weight* $\kappa \in W_m(\mathbb{Q}_p)$*, and let* e *be the ramification index of* $(\mathbb{T}_{W_m}^{\pm})_{(x)}$ *over* $R_{(\kappa)}$*. Then*

1. *The generalized eigenspace* $\mathrm{Symb}_{\Gamma_0}^{\pm}(\mathbb{D}_\kappa)_{(x)}$ *is free of rank one over the algebra* $(\mathbb{T}_\kappa^{\pm})_{(x)}$*, and the eigenspace* $\mathrm{Symb}_{\Gamma_0}^{\pm}(\mathbb{D}_\kappa)[x]$ *is dimension one over* $\mathbb{C}_p$*.*

2. The $(\mathbb{T}^{\pm}_{W_m})_{(x)}$*-module* $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))_{(x)}$ *is free of rank one over* $(\mathbb{T}^{\pm}_{W_m})_{(x)}$*.*

3. The specialization map

$$sp_\kappa : (\mathbb{T}^{\pm}_{W_m})_{(x)} \otimes_{R_{(\kappa)},\kappa} \mathbb{C}_p \longrightarrow (\mathbb{T}^{\pm}_{\kappa})_{(x)} \tag{2.2}$$

is an isomorphism of $\mathbb{C}_p$*-algebras.*

4. There exits a uniformizer $u \in R_{(\kappa)}$ *and an isomorphism of* $R_{(\kappa)}$*-algebras*

$$R_{(\kappa)}[t]/(t^e - u) \longrightarrow (\mathbb{T}^{\pm}_{W_m})_{(x)}$$

sending t *to a uniformizer of* $(\mathbb{T}^{\pm}_{W_m})_{(x)}$*. It then follows that*

$$\mathbb{C}_p[t]/(t^e) \cong (\mathbb{T}^{\pm}_{\kappa})_{(x)}.$$

5. The tensor product of the two specialization maps from (1) and (2):

$$sp_\kappa := sp_\kappa \otimes sp_\kappa : \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))_{(x)} \longrightarrow \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)_{(x)}$$

is surjective.

Proof. Statement (1) is Theorem 4.7 and Corollary 4.8 of [1]. Statement (2) is Proposition 4.5 of [1]. Statement (3) is corollary 4.4 of [1]. Statement (4) is Proposition 4.6 and Theorem 4.7 of [1]. Statement (5) follows from the first four assertions. □

With a little commutative algebra, we can use the theorem above to get the following slightly cleaner statement. We introduce the following notation. For a $\mathbb{Q}_p$-Banach space M and a field extension $K/\mathbb{Q}_p$, we let $M_K := M\widehat{\otimes}_{\mathbb{Q}_p}K$.

Proposition 2.6.5. *Let $x \in C^{\pm}_{W_m}(\mathbb{C}_p)$ be a smooth point of weight $\kappa \in W_m(\mathbb{Q}_p)$. Then there exists a neighborhood, $W = W(\kappa, 1/p^r)$ of κ such that the following hold. Let R be the ring of rigid analytic functions on W. Let T be the direct factor of $\mathbb{T}^{\pm}_W$ corresponding to the connected component of $C^{\pm}_{W_m}$ that x lies in. (Note that T may be defined over a finite extension of $\mathbb{Q}_p$.)*

1. *For all points $y \in C^{\pm}_W$, except perhaps x, the algebra $(\mathbb{T}^{\pm}_W)_{(y)}$ is étale over $R_{(\kappa^{\pm}(y))}$.*

2. *There exists $u \in R_{\mathbb{C}_p}$ such that $ev_\kappa(u) = 0$ and κ is the only 0 of u on W and an element $t \in T$ such that $x(t) \neq 0$ as well as an isomorphism*

$$T_{\mathbb{C}_p} \longrightarrow R_{\mathbb{C}_p}[X]/(X^e - u)$$

   sending t to X.

3. *The $T_{\mathbb{C}_p}$-module $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o \otimes_{\mathbb{T}^{\pm}_W} T_{\mathbb{C}_p}$ is free of rank one.*

4. *For any point $y \in C^{\pm}_W(\mathbb{C}_p)$ of weight $\kappa^{\pm}(y) \in W(\mathbb{Q}_p)$, the $\mathcal{H}$-equivariant map*

$$\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o \otimes_{\mathbb{T}^{\pm}_W} T_{\mathbb{C}_p} \longrightarrow \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_{\kappa^{\pm}(y)})_{(y)}$$

   sends any generator of $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o \otimes_{\mathbb{T}^{\pm}_W} T_{\mathbb{C}_p}$ to a generator of $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_{\kappa^{\pm}(y)})_{(y)}$.

2.7 Two-variable p-adic L-function

We now explain how to use the previous proposition to construct a two-variable p-adic L-function. We give three constructions and explain the interrelations between them, fleshing out what is stated in Bellaïche ([1]).

We begin by defining the Mellin transform of families of overconvergent modular symbols. Let $W = W(\kappa', 1/p^r) = \operatorname{Sp} R$ for some $\kappa' \in W_m(\mathbb{Q}_p)$ and $r \geq 1$. Let $M = \operatorname{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o$. We define the R-linear map

$$\Lambda : M \longrightarrow R \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}$$

to be the composition of evaluation at $\{0\} - \{\infty\}$ and the map $\mathcal{L}$ from before:

$$\Lambda : M \xrightarrow{\{0\}-\{\infty\}} R \widehat{\otimes} \mathbb{D} \xrightarrow{id_R \otimes \mathcal{L}} R \widehat{\otimes} \mathcal{R}.$$

Therefore Λ gives a map from M to rigid analytic functions on $W \times \mathcal{W}$. By construction we then have the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\Lambda} & R \widehat{\otimes} \mathcal{R} \\ \downarrow{\scriptstyle sp_\kappa} & & \downarrow{\scriptstyle ev_\kappa} \\ \operatorname{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)^o & \xrightarrow{\Lambda_\kappa} & \mathcal{R}. \end{array} \tag{2.3}$$

Hence for $\Phi \in M$, the function $\Lambda(\Phi)$ interpolates the functions $\Lambda_\kappa(sp_\kappa(\Phi))$ as κ varies over $W(\mathbb{Q}_p)$.

We now put ourselves in the situation of the previous proposition and we extend scalars to $\mathbb{C}_p$. Let $x \in C^{\pm}_{W_m}(\mathbb{C}_p)$ be of weight $\kappa' \in W_m(\mathbb{Q}_p)$. Let $W = W(\kappa', 1/p^r) = \operatorname{Sp} R$ and T be as in the proposition. Let $\epsilon \in \mathbb{T}^{\pm}_{W,\mathbb{C}_p}$ be such that

$$T_{\mathbb{C}_p} = \epsilon \mathbb{T}^{\pm}_{W,\mathbb{C}_p}.$$

Then

$$\operatorname{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o \otimes_{\mathbb{T}^{\pm}_W} T_{\mathbb{C}_p} = \epsilon \operatorname{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o_{\mathbb{C}_p} \subset \operatorname{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o_{\mathbb{C}_p},$$

so we let

$$M = \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^{o} \otimes_{\mathbb{T}^{\pm}_{W}} T_{\mathbb{C}_p} = \epsilon\, \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^{o}_{\mathbb{C}_p}.$$

We next give our first construction of a two-variable p-adic L-function. We use this construction when the weight map $\kappa^{\pm} : \mathcal{C}^{\pm}_{W} \to W$ is étale.

The module M is a rank one $T_{\mathbb{C}_p}$-module, so let Φ be a generator of M as a $T_{\mathbb{C}_p}$-module. Then let

$$\Lambda(\Phi, \cdot, \cdot) : W \times \mathcal{W} \longrightarrow \mathbb{C}_p$$

be the two-variable rigid analytic function that is the image of Φ in $R\widehat{\otimes}\mathcal{R}$ under Λ. By the commutative diagram (3), we have for all $\sigma \in \mathcal{W}$ and $\kappa \in W(\mathbb{Q}_p)$,

$$\Lambda(\Phi, \kappa, \sigma) = \Lambda_{\kappa}(sp_{\kappa}(\Phi), \sigma).$$

We explain why in the non-étale case this is not the correct p-adic L-function. If the weight map $\kappa^{\pm} : \mathcal{C}^{\pm}_{W} \to W$ has ramification index $e > 1$ at x, then for each weight $\kappa \in W$ with $\kappa \neq \kappa'$ there are e points $y \in \mathcal{C}^{\pm}_{W}$ such that $\kappa^{\pm}(y) = \kappa$. The element $sp_{\kappa}(\Phi) \in \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_{\kappa})^{o}$ is not in the eigenspace corresponding to y. We remark that, while $sp_{\kappa}(\Phi)$ is not in the eigenspace corresponding to y we do have the commutative diagram for each $y \in \mathcal{C}^{\pm}_{W}$ of weight $\kappa \in W(\mathbb{Q}_p)$:

$$\begin{array}{ccccc} M & \xrightarrow{sp_{\kappa}} & \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_{\kappa})^{o}_{\mathbb{C}_p} & \longrightarrow & \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_{\kappa})_{(y)} \\ & \searrow & & \nearrow{\scriptstyle sp_{\kappa}} & \\ & & \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))_{(y)} & & \end{array} \qquad (2.4)$$

which does send the generator Φ of M at a $T_{\mathbb{C}_p}$-module to a generator of $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_{\kappa})_{(y)}$

as a $(\mathbb{T}_{\kappa}^{\pm})_{(y)}$-module. We want the p-adic L-function to interpolate the p-adic L-functions of the image of Φ in $\mathrm{Symb}_{\Gamma_0}^{\pm}(\mathbb{D}_\kappa)_{(y)}$ under the composition of the two maps.

We now give the second construction which works in the non-étale case and clearly gives the same construction as above in the étale case. This construction is due to Bellaïche ([1]). Let

$$N = M \otimes_{R_{\mathbb{C}_p}} T_{\mathbb{C}_p}$$

and let $V = \mathrm{Sp}\, T$. Define

$$\Lambda_T := \Lambda \otimes Id_{T_{\mathbb{C}_p}} : N \longrightarrow (R_{\mathbb{C}_p} \widehat{\otimes} \mathcal{R}_{\mathbb{C}_p}) \otimes_{R_{\mathbb{C}_p}} T_{\mathbb{C}_p} \cong T_{\mathbb{C}_p} \widehat{\otimes} \mathcal{R}_{\mathbb{C}_p}.$$

Then for $\Phi \in N$, the function $\Lambda_T(\Phi) \in T_{\mathbb{C}_p} \widehat{\otimes} \mathcal{R}_{\mathbb{C}_p}$ is a two-variable rigid analytic function on $V_{\mathbb{C}_p} \times \mathcal{W}_{\mathbb{C}_p}$. For each $y \in V(\mathbb{C}_p)$ of weight $\kappa \in W(\mathbb{Q}_p)$, we define a specialization map

$$sp_y : N \longrightarrow \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_\kappa)^o_{\mathbb{C}_p}$$

as the natural map

$$N \longrightarrow N \otimes_{T_{\mathbb{C}_p}, y} \mathbb{C}_p.$$

We view $N \otimes_{T_{\mathbb{C}_p}, y} \mathbb{C}_p$ as a subset of $\mathrm{Symb}_{\Gamma_0}^{\pm}(\mathbb{D}_\kappa)^o_{\mathbb{C}_p}$ via

$$\begin{aligned} N \otimes_{T_{\mathbb{C}_p}, y} \mathbb{C}_p &= (M \otimes_{R_{\mathbb{C}_p}} T_{\mathbb{C}_p}) \otimes_{T_{\mathbb{C}_p}, y} \mathbb{C}_p \\ &= M \otimes_{R_{\mathbb{C}_p}, ev_\kappa} \mathbb{C}_p \hookrightarrow \mathrm{Symb}_{\Gamma_0}^{\pm}(\mathbb{D}_\kappa)^o_{\mathbb{C}_p}. \end{aligned}$$

By construction sp_y is $\mathcal{H}$-equivariant with respect to the action of $\mathcal{H}$ on the first component of N.

Lemma 2.7.1. *([1]) If $\Phi \in N$ and $y \in V(\mathbb{C}_p)$ of weight $\kappa \in W(\mathbb{Q}_p)$, then*

$$\Lambda_T(\Phi)(y, \sigma) = \Lambda_\kappa(sp_y(\Phi))(\sigma).$$

Proof. This is lemma 4.12 of [1] □

We recall that we have an element $t \in T_{\mathbb{C}_p}$ and $u \in R_{\mathbb{C}_p}$ and an isomorphism

$$T_{\mathbb{C}_p} \longrightarrow R_{\mathbb{C}_p}[X]/(X^e - u)$$

sending t to X. Now let ϕ be a generator of M as a $T_{\mathbb{C}_p}$ module and define (following Bellaïche)

$$\Phi = \sum_{i=0}^{e-1} t^i \phi \otimes t^{e-1-i} \in N$$

Lemma 2.7.2. *([1]) Let $T_{\mathbb{C}_p} \otimes_{R_{\mathbb{C}_p}} T_{\mathbb{C}_p}$ act on N with the first factor acting on M and the second factor acting on $T_{\mathbb{C}_p}$. Then*

$$(t \otimes 1 - 1 \otimes t)\Phi = 0.$$

Proof. This is lemma 4.13 of [1]. □

Proposition 2.7.3. *Let $y \in \mathcal{C}_W^{\pm}(\mathbb{C}_p)$ be a point of weight $\kappa \in W(\mathbb{Q}_p)$. Then*

$$sp_y(\Phi) \in \mathrm{Symb}_{\Gamma_0}^{\pm}(\mathbb{D}_\kappa)[y].$$

We note that if $y \neq x$, then $\mathrm{Symb}_{\Gamma_0}^{\pm}(\mathbb{D}_\kappa)[y] = \mathrm{Symb}_{\Gamma_0}^{\pm}(\mathbb{D}_\kappa)_{(y)}$, while if $y = x$ and the ramification index is e, $\mathrm{Symb}_{\Gamma_0}^{\pm}(\mathbb{D}_\kappa)_{(y)}$ is an e-dimensional vector space.

Proof. This is proposition 4.14 of [1]. □

We denote the p-adic L-function determined by the second construction as

$$\Lambda_T(\Phi) : V_{\mathbb{C}_p} \times \mathcal{W}_{\mathbb{C}_p} \longrightarrow \mathbb{C}_p.$$

To compare the second construction with the first construction when the ramification index is 1, we note that $T_{\mathbb{C}_p} \cong R_{\mathbb{C}_p}[X]/(X^e - u) = R_{\mathbb{C}_p}$, so

$$N = M \otimes_{R_{\mathbb{C}_p}} T_{\mathbb{C}_p} \cong M$$

and

$$\Phi = \sum_{i=0}^{e-1} t^i \phi \otimes t^{e-1-i} = \phi \otimes 1$$

so the second construction reduces to the first one when $e = 1$.

We now give the third construction. This construction appears as a remark in [1]. We keep the same notation as before for T, M, and R. Let $M^\vee = \mathrm{Hom}_{R_{\mathbb{C}_p}}(M, R_{\mathbb{C}_p})$ be the $R_{\mathbb{C}_p}$-dual of M. We view $M^\vee$ as a $T_{\mathbb{C}_p}$-module via the dual action. Now, $T_{\mathbb{C}_p}$ is regular at x, and for all $y \in \mathcal{C}_W^{\pm}(\mathbb{C}_p)$ with $y \neq x$, $T_{\mathbb{C}_p}$ is étale over $R_{\mathbb{C}_p}$, so $T_{\mathbb{C}_p}$ is a regular ring. Therefore $T_{\mathbb{C}_p}$ is Gorenstein, so M being a rank one $T_{\mathbb{C}_p}$-module implies that $M^\vee$ is also a rank one $T_{\mathbb{C}_p}$-module. Fix an isomorphism

$$M^\vee \longrightarrow T_{\mathbb{C}_p}$$

and note that this amounts to choosing a generator of $M^\vee$ as a $T_{\mathbb{C}_p}$-module. We have the R-linear map Λ from before

$$\Lambda : M \longrightarrow R_{\mathbb{C}_p} \widehat{\otimes} \mathcal{R}_{\mathbb{C}_p}$$

which we view as an element of $\mathrm{Hom}_{R_{\mathbb{C}_p}}(M, R_{\mathbb{C}_p} \widehat{\otimes} \mathcal{R}_{\mathbb{C}_p})$. We have

$$\begin{aligned} \mathrm{Hom}_{R_{\mathbb{C}_p}}(M, R_{\mathbb{C}_p} \widehat{\otimes} \mathcal{R}_{\mathbb{C}_p}) &= \mathrm{Hom}_{R_{\mathbb{C}_p}}(M, R_{\mathbb{C}_p}) \otimes_{R_{\mathbb{C}_p}} R_{\mathbb{C}_p} \widehat{\otimes} \mathcal{R}_{\mathbb{C}_p} \\ &= M^\vee \widehat{\otimes} \mathcal{R}_{\mathbb{C}_p} \\ &\cong T_{\mathbb{C}_p} \widehat{\otimes} \mathcal{R}_{\mathbb{C}_p} \end{aligned}$$

allowing us to view Λ as an element of $T_{\mathbb{C}_p}\widehat{\otimes}\mathcal{R}_{\mathbb{C}_p}$. Then this is exactly a two-variable rigid analytic function on $V_{\mathbb{C}_p} \times \mathcal{W}_{\mathbb{C}_p}$:

$$\Lambda : V_{\mathbb{C}_p} \times \mathcal{W}_{\mathbb{C}_p} \longrightarrow \mathbb{C}_p$$

where for $(y, \sigma) \in V(\mathbb{C}_p) \times \mathcal{W}(\mathbb{C}_p)$, $\Lambda(y, \sigma)$ is the image of Λ under the maps

$$T_{\mathbb{C}_p}\widehat{\otimes}\mathcal{R}_{\mathbb{C}_p} \xrightarrow{(y,\sigma)} T_{\mathbb{C}_p}/(\ker(y)) \otimes_{\mathbb{C}_p} \mathcal{R}_{\mathbb{C}_p}/(\ker(\sigma)) \longrightarrow \mathbb{C}_p.$$

We show how evaluation of Λ at (y, σ) is compatible with specialization. Let $y \in V(\mathbb{C}_p)$ be a point of weight $\kappa \in W(\mathbb{Q}_p)$. Let $M_\kappa = M \otimes_{R_{\mathbb{C}_p},\kappa} \mathbb{C}_p$. By Bellaïche's specialization theorem, we have that

$$M_\kappa \subset \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)^{o}_{\mathbb{C}_p},$$

and since the map $M \to M_\kappa$ is $\mathcal{H}$-equivariant, we know that $\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)^{o}_{\mathbb{C}_p}[y] \subset M_\kappa$. We let $M_\kappa[y]$ denote the one-dimensional eigenspace corresponding to y in M_κ. Let $M_\kappa^\vee = \mathrm{Hom}_{\mathbb{C}_p}(M_\kappa, \mathbb{C}_p)$, $\mathfrak{p}_\kappa = \ker(\kappa)$, and $\mathfrak{p}_y = \ker(y) = (t - y(t))$. We have a map

$$\begin{aligned} M^\vee &\longrightarrow M_\kappa^\vee \\ \varphi &\longmapsto \overline{\varphi} \end{aligned}$$

which is

$$\begin{aligned} M^\vee &\to M^\vee \otimes_{R_{\mathbb{C}_p}} R_{\mathbb{C}_p}/\mathfrak{p}_\kappa \\ &\cong \mathrm{Hom}_{R_{\mathbb{C}_p}}(M, R_{\mathbb{C}_p}/\mathfrak{p}_\kappa) \\ &\cong \mathrm{Hom}_{R_{\mathbb{C}_p}}(M \otimes_{R_{\mathbb{C}_p}} R_{\mathbb{C}_p}/\mathfrak{p}_\kappa, R_{\mathbb{C}_p}/\mathfrak{p}_\kappa) \\ &= \mathrm{Hom}_{\mathbb{C}_p}(M_\kappa, \mathbb{C}_p) \\ &= M_\kappa^\vee \end{aligned}$$

since M is a finite, flat $R_{\mathbb{C}_p}$-module. Explicitly, if $m \otimes \overline{f} \in M_\kappa = M \otimes_{R_{\mathbb{C}_p}} R_{\mathbb{C}_p}/\mathfrak{p}_\kappa$ and $\varphi \in M^\vee$, then

$$\overline{\varphi}(m \otimes \overline{f}) = ev_\kappa(\varphi(m)) ev_\kappa(f) \in \mathbb{C}_p. \tag{2.5}$$

Now $M^\vee \otimes_{R_{\mathbb{C}_p}} R_{\mathbb{C}_p}/\mathfrak{p}_\kappa = M^\vee/\mathfrak{p}_\kappa M^\vee$ so the above maps induce an isomorphism

$$M^\vee/\mathfrak{p}_\kappa M^\vee \cong M_\kappa^\vee.$$

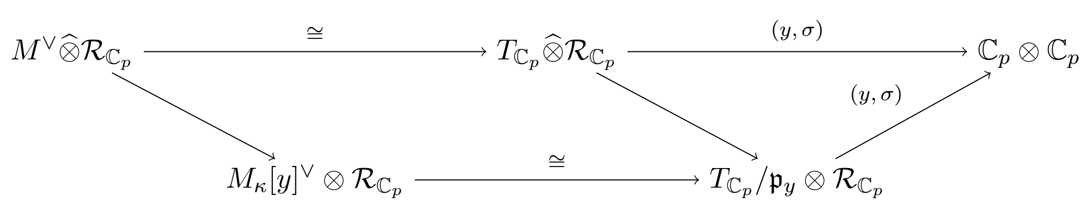
We note that

$$\Lambda_\kappa : \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_\kappa)^o_{\mathbb{C}_p} \longrightarrow \mathcal{R}_{\mathbb{C}_p}$$

and we may restrict Λ_κ to M_κ to view $\Lambda_\kappa \in M_\kappa^\vee \otimes \mathcal{R}_{\mathbb{C}_p}$. From (5), it is clear that $\Lambda \in M^\vee \widehat{\otimes} \mathcal{R}_{\mathbb{C}_p}$ maps to $\Lambda_\kappa \in M_\kappa^\vee \otimes \mathcal{R}_{\mathbb{C}_p}$. If we further quotient out by $\mathfrak{p}_y$, then we have

$$M^\vee/\mathfrak{p}_y M^\vee = M_\kappa^\vee/(t - y(t))M^\vee = M_\kappa[y]^\vee$$

since quotienting $M_\kappa^\vee$ by $t - y(t)$ is the same as taking the dual of the kernel of $t - y(t)$. Then the image of Λ in $M_\kappa[y]^\vee \otimes \mathcal{R}_{\mathbb{C}_p}$ is Λ_κ restricted to $M_\kappa[y]$. To see the connection with specialization, we have the following commutative diagram



and since Λ maps to Λ_κ in the above diagram, we get that for all σ,

$$\Lambda(y, \sigma) = \Lambda_\kappa(\sigma).$$

In the following section we use the first and second constructions given here. The third construction is included for completeness.

Chapter 3

Definition of a p-adic L-function and p-adic Stark Conjectures

We begin this chapter by introducing the objects we are working with and setting notation that will be fixed throughout. We remind the reader that we have fixed embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let F be a quadratic field of discriminant d_F, and let

$$\chi : G_F \longrightarrow \overline{\mathbb{Q}}^\times$$

be a nontrivial ray class character of F that is of mixed signature if F is real quadratic. Let K be the fixed field of the kernel of χ and let $\mathfrak{f}$ be the conductor of χ. Assume that $\iota_\infty(K) \subset \mathbb{R}$ if F is real quadratic. Let

$$\rho = \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \chi : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}})$$

be the induction of χ and let M be the fixed field of the kernel of ρ. Let f be the weight one modular form associated to ρ, so f has level $N = N_{F/\mathbb{Q}}(\mathfrak{f}) \cdot |d_F|$ and character $\varepsilon = \det \rho$. The q-expansion of f is

$$f = \sum_{\substack{\mathfrak{a} \subset \mathscr{O}_F \\ (\mathfrak{a}, \mathfrak{f}) = 1}} \chi(\mathfrak{a}) q^{N\mathfrak{a}}$$

and we have that

$$L(f, s) = L(\chi, s).$$

Let

$$x^2 - a_p(f)x + \varepsilon(p) = (x - \alpha)(x - \beta)$$

be the pth Hecke polynomial of f. We note that when p splits in F, say $p\mathscr{O}_F = \mathfrak{p}\overline{\mathfrak{p}}$, then $\alpha = \chi(\mathfrak{p})$ and $\beta = \chi(\overline{\mathfrak{p}})$, and if p is inert, then $\alpha = \sqrt{\chi(p\mathscr{O}_F)}$ and $\beta = -\sqrt{\chi(p\mathscr{O}_F)}$. Let k be the field obtained by adjoining the values of χ along with α and β to $\mathbb{Q}$.

We now make some assumptions that will be fixed throughout. First we assume that $p \nmid N$, which implies in particular that p does not ramify in M. We further assume that $p \nmid [M : \mathbb{Q}]$, and we assume that $\alpha \neq \beta$. With these assumptions, we let $f_\alpha(z) = f(z) - \beta f(pz)$ be a fixed p-stabilization of f.

3.1 Definition of one and two-variable p-adic L-functions

We use the constructions from the previous section to define our p-adic L-function. In order to do that, we start with the following result of Bellaïche and Dmitrov about the eigencurve at weight one points.

Theorem 3.1.1. *([2]) Let g be a classical weight one newform of level M, whose Hecke polynomial at p has distinct roots. Then the eigencurve is smooth at either p-stabilization of g. Moreover, the eigencurve is smooth but not étale over weight space if and only if the representation associated to g is obtained by induction from a mixed signature character of a real quadratic field in which p splits.*

By our assumption that $\alpha \neq \beta$ the above theorem implies that the eigencurve is smooth at the point corresponding to f_α. We may break our situation into four cases, the cases when F is either imaginary or real quadratic and when p is either inert or split in F. In the case when F is real quadratic and p is split the eigencurve is smooth but not étale at f_α. In the other three cases the eigencurve is étale at f_α. For this reason, we use the second construction of the two-variable p-adic L-function in a neighborhood of f_α. In the étale cases this construction simplifies to the conceptually simplest first construction. We adopt the notation from the previous section except that we base change everything to $\mathbb{C}_p$ so that we can drop all the subscripts. Therefore, let $T = T_{\mathbb{C}_p}$, $M \subset \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^{o}_{\mathbb{C}_p}$, N, and $R = R_{\mathbb{C}_p}$ be as in section 2.7 where the point of interest x is the point on the eigencurve corresponding to f_α. Let $\phi^{\pm}$ be a generator of M as a T-module and let

$$\Phi^{\pm} = \sum_{i=0}^{e-1} t^i \phi^{\pm} \otimes t^{e-1-i} \in N.$$

We note that in the étale cases, $\Phi^{\pm} = \phi^{\pm}$. Let $V^{\pm} = \mathrm{Sp}\, T$, $\mathcal{W} = \mathcal{W}_{\mathbb{C}_p}$, $W = \mathrm{Sp}(R)$, and let $\Lambda(\Phi^{\pm}) = \Lambda_T(\Phi^{\pm})$ to make all the notation uniform. We then have our two-variable rigid analytic function

$$\Lambda(\Phi^{\pm}) : V^{\pm} \times \mathcal{W} \longrightarrow \mathbb{C}_p.$$

We precisely state the interpolation formula of $\Lambda(\Phi^{\pm})$. For each $y \in V$ of weight $k \in \mathbb{Z}_{\geq 2}$, let g_y be the p-stabilized newform corresponding to y. Let $\Omega^{\pm}_{\infty,g_y} \in \mathbb{C}^{\times}$ be the complex period used to define the p-adic L-function associated to g_y as in section 2.4. Let

$$\varphi^{\pm}_{g_y} \in \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_{k-2})_{(y)}$$

be the unique (by Theorem 2.5.3) modular symbol specializing under ρ^*_k to

$$\psi^{\pm}_{g_y}/\Omega^{\pm}_{\infty,g_y} \in \mathrm{Symb}^{\pm}_{\Gamma_0}(V_{k-2}(\overline{\mathbb{Q}})).$$

Let

$$\Omega^{\pm}_{p,g_y} \in \mathbb{C}_p^{\times}$$

be the p-adic period such that

$$sp_y(\Phi^{\pm})/\Omega^{\pm}_{p,g_y} = \varphi^{\pm}_{g_y}.$$

Remarks 3.1.2. The complex periods $\Omega^{\pm}_{\infty,g_y}$ may be defined as a complex number such that

$$\psi^{\pm}_{g_y}/\Omega^{\pm}_{\infty,g_y} \in \mathrm{Symb}^{\pm}_{\Gamma_0}(V_{k-2}(\overline{\mathbb{Q}})) \subset \mathrm{Symb}^{\pm}_{\Gamma_0}(V_{k-2}(\mathbb{C})).$$

Then $\Omega^{\pm}_{\infty,g_y}$ is determined up to multiplication by an element of $\overline{\mathbb{Q}}^{\times}$. Once $\Phi^{\pm}$ is fixed, $\Omega^{\pm}_{p,g_y}$ is determined by the choice of $\omega^{\pm}_{\infty,g_y}$. On the other hand, we could choose $\Phi^{\pm}$ and then choose the $\Omega^{\pm}_{p,g_y}$ such that

$$sp_y(\Phi^{\pm})/\Omega^{\pm}_{p,g_y} \in \rho^{*-1}_k(\mathrm{Symb}^{\pm}_{\Gamma_0}(V_{k-2}(\overline{\mathbb{Q}}))).$$

This would then determine the $\Omega^{\pm}_{\infty,g_y} \in \mathbb{C}^{\times}$ that satisfy the relation

$$\rho_k^*\left(\frac{sp_y(\Phi^{\pm})}{\Omega^{\pm}_{p,g_y}}\right) = \frac{\psi^{\pm}_{g_y}}{\Omega^{\pm}_{\infty,g_y}}$$

in $\mathrm{Symb}^{\pm}_{\Gamma_0}(V_{k-2}(\overline{\mathbb{Q}}))$. Therefore, once $\Phi^{\pm}$ is chosen the pair of periods $(\Omega^{\pm}_{p,g_y}, \Omega^{\pm}_{\infty,g_y}) \in \mathbb{C}_p^{\times} \times \mathbb{C}^{\times}$ may be viewed as an element of $\mathbb{C}_p^{\times} \times \mathbb{C}^{\times}/\overline{\mathbb{Q}}^{\times}$ where we embed $\overline{\mathbb{Q}}^{\times}$ into $\mathbb{C}_p^{\times} \times \mathbb{C}^{\times}$ diagonally. To summarize, once $\Phi^{\pm}$ is chosen, for each $y \in V$ corresponding to a p-stabilized newform g_y of weight $k \in \mathbb{Z}_{\geq 2}$, there is an element $(\Omega^{\pm}_{p,g_y}, \Omega^{\pm}_{\infty,g_y}) \in \mathbb{C}_p^{\times} \times \mathbb{C}^{\times}/\overline{\mathbb{Q}}^{\times}$ such that

$$\rho_k^*\left(\frac{sp_y(\Phi^{\pm})}{\Omega^{\pm}_{p,g_y}}\right) = \frac{\psi^{\pm}_{g_y}}{\Omega^{\pm}_{\infty,g_y}}$$

holds in $\mathrm{Symb}^{\pm}_{\Gamma_0}(V_{k-2}(\overline{\mathbb{Q}}))$ and the equation does not depend on the choice of representative in $\mathbb{C}_p^{\times} \times \mathbb{C}^{\times}/\overline{\mathbb{Q}}^{\times}$ for $(\Omega^{\pm}_{p,g_y}, \Omega^{\pm}_{\infty,g_y})$.

Theorem 3.1.3. *The two-variable rigid analytic functions $\Lambda(\Phi^{\pm})$ on $V \times \mathcal{W}$ are determined by the following interpolation property. For all $y \in V$ corresponding to a p-stabilized newform g_y of weight $k \in \mathbb{Z}_{\geq 2}$, and all characters $\psi\langle\cdot\rangle^{j-1} \in \mathcal{W}(\mathbb{C}_p)$ where ψ is a finite order character of conductor p^m and $1 \leq j \leq k-1$,*

$$\begin{aligned}\frac{\Lambda(\Phi^{sgn(\psi)}, y, \psi\langle\cdot\rangle^{j-1})}{\Omega^{sgn(\psi)}_{p,g_y}} = \; & \frac{1}{a_p(g_y)^m}\left(1 - \frac{\psi\omega^{1-j}(p)}{a_p(g_y)p^{1-j}}\right)\frac{p^{m(j-1)}(j-1)!\tau(\psi\omega^{1-j})}{(2\pi i)^{j-1}} \times \\ & \times \frac{L(g_y, \psi^{-1}\omega^{j-1}, j)}{\Omega^{sgn(\psi)}_{\infty,g_y}}.\end{aligned} \tag{3.1}$$

This equality takes place in $\overline{\mathbb{Q}}$. Here $\tau(\psi\omega^{1-j})$ is the Gauss sum associated to $\psi\omega^{1-j}$.

Proof. With the way everything is set up, the interpolation property follows from the fact that

$$\frac{\Lambda(\Phi^{sgn(\psi)}, y, \sigma)}{\Omega_{p,g_y}^{sgn(\psi)}} = \frac{\Lambda_k(sp_y(\Phi^{sgn(\psi)}), \sigma)}{\Omega_{p,g_y}^{sgn(\psi)}} = L_p(g_y, \psi, j)$$

where $L_p(g_y, \psi, s)$ is defined using that complex periods $\Omega^{\pm}_{\infty,g_y}$. The fact that the interpolation property determines $\Lambda(\Phi^{\pm})$ follows from the Weierstrass preparation theorem. By the Weierstrass preparation theorem, for each y corresponding to a p-stabilized newform g_y of weight $k \in \mathbb{Z}_{\geq 2}$, $\Lambda(y, \cdot) : \mathcal{W} \to \mathbb{C}_p$ is determined by its values of infinitely many points on each connected component of $\mathcal{W}$. By the interpolation property we see that $\psi\langle\cdot\rangle^{j-1}$ varies through infinitely many points on each connected component of weight space. Similarly by the Weierstrass preparation theorem in the other variable, if we fix $\sigma = \psi\langle\cdot\rangle^{j-1}$ and consider $\Lambda(\Phi^{sgn(\psi)}, \cdot, \sigma) : V \to \mathbb{C}_p$, then V is connected and there are infinitely many points $y \in V$ corresponding to p-stabilized newforms of weight $k \in \mathbb{Z}_{\geq 2}$. Therefore, $\Lambda(\Phi^{\pm})$ is determined by its values on these interpolation points. □

At this point, we would like to define the two-variable p-adic L-function associated to χ as

$$\begin{aligned} L_p(\chi, \alpha, \cdot, \cdot) &: V \times \mathbb{Z}_p \longrightarrow \mathbb{C}_p \\ L_p(\chi, \alpha, y, s) &= \Lambda(\Phi^{+}, y, \langle\cdot\rangle^{s-1}). \end{aligned} \tag{3.2}$$

The p-adic L-function $L_p(\chi, \alpha, y, s)$ is determined by the above interpolation formula. The first variable is on the eigencurve varying through the p-adic family of modular forms passing through f_α and second variable is the usual cyclotomic variable. To get the one variable p-adic L-function associated to χ we would plug the point $x \in V$ that corresponds to f_α. It is then natural to make conjectures for the values $L_p(\chi, \alpha, x, 0)$

and $L_p(\chi, \alpha, x, 1)$ that are analogous to Conjectures 1.1.1 and 1.1.5 (or Conjectures 1.1.9 and 1.1.10), replacing the complex logarithm with the p-adic logarithm.

For the purposes of this discussion we focus on the value $L_p(\chi, \alpha, x, 1)$. Let S be the set of places of F containing the infinite places of F and the places of F that ramify in K. Let $u_K \in K^\times$ be the (conjectural if F is real quadratic) Stark unit for the extension K/F with respect to S and the place of K induced by the embedding ι_∞. The p-adic Stark conjecture to make for the value $L_p(\chi, \alpha, x, 1)$ would be that

$$L_p(\chi, \alpha, x, 1) = E_p(\alpha, x, 1) \sum_{\sigma \in \mathrm{Gal}(K/F)} \chi(\sigma^{-1}) \log_p |\sigma(u_K)|_\beta \tag{3.3}$$

where $E_p(\alpha, x, 1)$ is an explicit p-adic number consisting of Euler like factors and a Gauss sum, and $|\cdot|_\beta$ is a projection that depends on the choice of p-stabilization and takes the place of the complex absolute value. Compare this formula with the formula in Conjecture 1.1.5.

The issue with making the conjecture this way is that the p-adic number $L_p(\chi, \alpha, x, 1)$ is not canonically defined because we made a choice for ϕ^+. The condition on the choice of ϕ^+ is that ϕ^+ is a generator of M as a T-module. If we choose a different generator of M as a T-module (changing ϕ^+ by an element of $T^\times$) that would change the value $L_p(\chi, \alpha, x, 1)$. Therefore as it stands now, we cannot precisely conjecture the value $L_p(\chi, \alpha, x, 1)$ (or $L_p(\chi, \alpha, x, 0)$).

This issue of the value $L_p(\chi, \alpha, x, 1)$ not being canonically defined is a central question of this thesis. One way to approach the question is to ask whether or not there is a way to canonically choose the periods $(\Omega^+_{p,g_y}, \Omega^+_{\infty,g_y})$ so that they determine a two-variable modular symbol ϕ^+ which would in turn define the function $L_p(\chi, \alpha, x, s)$ canonically. It is possible to do this in the case when F is imaginary quadratic and p is split in F (see section 4.7). In this case when F is imaginary quadratic and p is split in F the two-variable p-adic L-function $L_p(\chi, \alpha, y, s)$ does not turn out to be canonically defined (it depends on the choice of canonical periods), but the one-variable p-adic L-function $L_p(\chi, \alpha, x, s)$ is. A goal for future research is to determine a way to choose the periods canonically in the other cases when F is real quadratic and when F is imaginary quadratic with p inert in F, so that the function $L_p(\chi, \alpha, x, s)$ is uniquely defined independent of any choices.

To get around these issues and make a precise conjecture we exploit the fact that in (3.1) the function $\Lambda(\Phi^{\pm}, y, \sigma)$ interpolates the values of the complex L-function of g_y twisted by p-power conductor Dirichlet characters. Let $\psi \in \mathcal{W}(\mathbb{C}_p)$ be a finite order character of conductor p^{m+1}. We assume ψ has order p^m, so in particular ψ is even. We could then define generalizing (3.2) the p-adic L-function of χ twisted by ψ to be

$$L_p(\chi, \alpha, \psi, y, s) = \Lambda(\Phi^+, y, \psi^{-1}\langle\cdot\rangle^{s-1}),$$

and state a p-adic Stark conjecture for the value $L_p(\chi, \alpha, \psi, x, 1)$. To determine what the p-adic Stark conjecture should be in this case we make the following observation. The value $L_p(\chi, \alpha, \psi, x, 1)$ is outside the range of interpolation for the function $\Lambda(\Phi^+, y, \sigma)$, but if it was in the range of interpolation it would be related to complex L-value $L(f_\alpha, \psi, 1)$. Recalling that we have the relation of complex L-functions $L(f, s) = L(\chi, s)$, then viewing ψ as a Galois character we have the relation $L(f, \psi, s) = L(\chi\psi, s)$. Therefore a conjecture for the value $L_p(\chi, \alpha, x, \psi, 1)$ should have the same shape as the conjecture for the value $L(\chi\psi, 1)$ with the complex logarithm replaced with the p-adic logarithm. Since ψ is a Dirichlet character and $\mathrm{ord}_{s=0}(L(\chi, s)) = 1$ we also have that $\mathrm{ord}_{s=0}(L(\chi\psi, s)) = 1$, so we are also in the setting of the rank one abelian Stark conjecture for the character $\chi\psi$ of G_F.

Let K_ψ be the fixed field of the kernel of $\chi\psi$, let S be the set of places of F containing the infinite places of F and the places of F that ramify in K_ψ, let v be the infinite place of K_ψ induced by ι_∞, and let u_{K_ψ} be the Stark unit corresponding to this data. Then the p-adic Stark conjecture to make for $L_p(\chi, \alpha, \psi, x, 1)$ following (3.3) would be

$$L_p(\chi, \alpha, \psi, x, 1) = E_p(\alpha, x, \psi, 1) \sum_{\sigma \in \mathrm{Gal}(K_\psi/F)} \chi\psi(\sigma^{-1}) \log_p |\sigma(u_{K_\psi})|_\beta \qquad (3.4)$$

where $E_p(\alpha, x, \psi, 1)$ is an explicit p-adic number. Of course, the value $L_p(\chi, \alpha, \psi, x, 1)$ has the same issue of not being canonically defined as $L_p(\chi, \alpha, 1)$, but now that we have the flexibility of using finite order characters $\psi \in \mathcal{W}(\mathbb{C}_p)$ we can make a function that is canonically defined.

To make a function that is canonically defined independent of any choices we fix two finite order characters $\eta, \psi \in \mathcal{W}(\mathbb{C}_p)$ and consider the ratio of the functions $L_p(\chi, \alpha, \eta, y, s)$ and $L_p(\chi, \alpha, \psi, y, s)$. Let $\eta, \psi \in \mathcal{W}(\mathbb{C}_p)$ be two finite order characters of the same sign $\pm$ and define the function

$$L_p(\chi, \alpha, \eta, \psi, y, s) = \frac{\Lambda(\Phi^\pm, y, \eta^{-1}\langle\cdot\rangle^{s-1})}{\Lambda(\Phi^\pm, y, \psi^{-1}\langle\cdot\rangle^{s-1})}.$$

Then $L_p(\chi, \alpha, \eta, \psi, y, s)$ does not depend on the choice of $\phi^{\pm}$ because the indeterminacy of the periods in the interpolation formula (3.1) cancel out. The value $L_p(\chi, \alpha, \eta, \psi, x, 1)$ is then canonically defined independent of any choices, and we conjecture this values by taking the ratio of the right hand side of (3.4) for η and ψ. We now make this discussion formally precise with the following definitions and conjectures in the following section.

Definition 3.1.4. *Let $\eta, \psi \in \mathcal{W}(\mathbb{C}_p)$ be two finite order characters with the same sign $\pm$. Define the two-variable p-adic L-function of f_α with the auxiliary characters η and ψ as*

$$L_p(f_\alpha, \eta, \psi, \cdot, \cdot) : V \times \mathbb{Z}_p \longrightarrow \mathbb{C}_p \cup \{\infty\}$$

$$L_p(f_\alpha, \eta, \psi, y, s) = \frac{\Lambda(\Phi^{\pm}, y, \eta^{-1}\langle\cdot\rangle^{s-1})}{\Lambda(\Phi^{\pm}, y, \psi^{-1}\langle\cdot\rangle^{s-1})}.$$

The function $L_p(f_\alpha, \eta, \psi, y, s)$ does not depend on the choice of $\Phi^{\pm}$.

Definition 3.1.5. *Let $\eta, \psi \in \mathcal{W}(\mathbb{C}_p)$ be two finite order characters that have the same sign. Define the p-adic L-function of χ with the auxiliary characters η and ψ as*

$$L_p(\chi, \alpha, \eta, \psi, \cdot) : \mathbb{Z}_p \longrightarrow \mathbb{C}_p \cup \{\infty\}$$

$$L_p(\chi, \alpha, \eta, \psi, s) = L_p(f_\alpha, \eta, \psi, x, s).$$

We note here that we can give the definition of $L_p(\chi, \alpha, \eta, \psi, s)$ without making reference to the two-variable p-adic L-function. The two-variable p-adic L-function is introduced for two reasons. The first is that it satisfies an interpolation property, while the one-variable function $L_p(\chi, \alpha, \eta, \psi, s)$ does not. The second is that we will use the two-variable p-adic L-function to prove our conjectures when F is imaginary quadratic and p is split in F.

To define $L_p(\chi, \alpha, \eta, \psi, s)$ without referencing the two-variable p-adic L-function, we consider the space

$$\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_{-1})^o$$

of weight negative one overconvergent modular symbols. Since the eigencurve is smooth at the point x corresponding to f_α the eigenspace

$$\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_{-1})[x]$$

with the same eigenvalues as f_α is one-dimensional. If $\varphi^{\pm}_{f_\alpha}$ is a generator of this eigenspace, then $L_p(\chi, \alpha, \eta, \psi, s)$ may be defined as

$$L_p(\chi, \alpha, \eta, \psi, s) = \frac{\Lambda_{-1}(\varphi^{sgn(\eta)}_{f_\alpha}, \eta^{-1}\langle\cdot\rangle^{s-1})}{\Lambda_{-1}(\varphi^{sgn(\psi)}_{f_\alpha}, \psi^{-1}\langle\cdot\rangle^{s-1})}.$$

Since $\Lambda(\Phi^+, x, \sigma) = \Lambda_{-1}(sp_x(\Phi^+), \sigma)$ and

$$0 \neq sp_x(\Phi^+) \in \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}_{-1})[x]$$

it is clear that this second definition is the same as the first definition.

3.2 p-adic Conjectures

For each $n \in \mathbb{Z}_{\geq 0}$, let $\mathbb{Q}_n$ be the nth layer of the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$, so

$$\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) = 1 + p\mathbb{Z}_p/1 + p^{n+1}\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}.$$

Let $\Gamma_n = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. Let K_n (respectively M_n) be the compositum of K (respectively M) and $\mathbb{Q}_n$. Let $\Delta = \mathrm{Gal}(M/\mathbb{Q})$, $G = \mathrm{Gal}(K/F)$, and $H = \mathrm{Gal}(M/F)$. Further, for $n \geq 0$ let $\Delta_n = \mathrm{Gal}(M_n/\mathbb{Q})$, $G_n = \mathrm{Gal}(K_n/F)$, and $H_n = \mathrm{Gal}(M_n/F)$. By our assumption that p does not ramify in M and $p \nmid [M : \mathbb{Q}]$ restriction gives isomorphisms

$$\Delta_n = \Delta \times \Gamma_n, \quad G_n = G \times \Gamma_n, \quad H_n = H \times \Gamma_n$$

Fix an element $\tau \in \Delta - H$. By the isomorphisms $\Delta_n = \Delta \times \Gamma_n$ we view τ as the element of Δ_n that acts trivially on $\mathbb{Q}_n$ and as τ on M.

It may be the case that $H_n = G_n$ for all n. This happens if and only if K is Galois over $\mathbb{Q}$. When F is real quadratic, since χ is mixed signature K is never Galois over $\mathbb{Q}$. When F is imaginary quadratic a necessary condition for K to be Galois over $\mathbb{Q}$ is that if $\mathfrak{f}$ is the conductor of χ, then $\mathfrak{f} = \tau(\mathfrak{f}) = \bar{\mathfrak{f}}$. (When F is imaginary quadratic τ induces complex conjugation on F.) When K is not Galois over $\mathbb{Q}$, let $\overline{K}_n = \tau(K_n)$. The field $\overline{K}_n$ does not depend on the choice of τ, and we have that M_n is the compositum of K_n and $\overline{K}_n$. Furthermore, since K_n and $\overline{K}_n$ are abelian over F, M_n/F is an abelian extension.

When F is imaginary quadratic, M_n/F is abelian so we are in the rank one abelian Stark conjecture setting for the extension M_n/F. In this case, let S be the set of infinite places of F, the places above p, and the set of places of F that ramify in M. When F is real quadratic, since M_n is totally complex we are not in the setting of the rank one abelian Stark conjecture for M_n/F, but since $\mathbb{Q}_n$ is totally real the extension K_n/F is such that exactly one infinite place of F splits completely in K_n so we are in the rank one abelian Stark conjecture for K_n/F. In this case let S be the set of infinite places of F, the places above p, and the places of F that ramify in K. Since only p ramifies in $\mathbb{Q}_n$, in both cases, S does not depend on n. Further, we have that $|S| \geq 2$. When F is real quadratic we are assuming that $\iota_\infty(K) \subset \mathbb{R}$. For any $n \geq 0$, we let v denote the infinite place of M_n and K_n induced by ι_∞.

Definition 3.2.1. *When F is imaginary quadratic, let*

$$u_n = u_{M_n}$$

where u_{M_n} is the Stark unit associated to the extension M_n/F, the set S, and the place v. When F is real quadratic, let

$$u_n = u_{K_n}\tau(u_{K_n})$$

where u_{K_n} is the conjectural Stark unit associated to the extension K_n/F, the set S, and place v. In this case, while u_n depends on the choice of τ, the values in our conjectures (the right hand sides of (3.5) and (3.6)) that depend on u_n do not depend on the choice of τ (see Proposition 3.2.5). Therefore we leave τ out of the notation for u_n. We also remark that the unit $\tau(u_{K_n})$ is the Stark unit for $\overline{K}_n/F$, the set $\tau(S)$, and the embedding v^τ.

Let η be a character of Γ_n, and let $\rho\eta$ denote the representation $\rho \otimes \eta$ of Δ_n and $(\rho\eta)^*$ denote the representation $\mathrm{Ind}_{G_F}^{G_\mathbb{Q}} \chi^{-1} \otimes \eta^{-1}$ of Δ_n. Given a Δ_n module A, we let $\pi_{\rho\eta}$ and $\pi^*_{\rho\eta}$ denote maps from A to the $\rho\eta$ and $(\rho\eta)^*$ isotypic components of $\overline{\mathbb{Q}} \otimes A$ given by

$$\pi_{\rho\eta} : A \longrightarrow (\overline{\mathbb{Q}} \otimes A)^{\rho\eta}$$

$$\pi_{\rho\eta}(a) = \sum_{\sigma\in\Delta_n} Tr(\rho\eta(\sigma^{-1})) \otimes \sigma(a)$$

$$\pi^*_{\rho\eta} : A \longrightarrow (\overline{\mathbb{Q}} \otimes A)^{(\rho\eta)^*}$$

$$\pi^*_{\rho\eta}(a) = \sum_{\sigma\in\Delta_n} Tr((\rho\eta)^*(\sigma)) \otimes \sigma(a)$$

where Tr denotes the trace. Because G_F is an index 2 subgroup in $G_{\mathbb{Q}}$ and ρ is the induction of a character from G_F the formulas simplify to

$$\pi_{\rho\eta}(a) = \sum_{\sigma\in H_n} (\chi\eta(\sigma^{-1}) + \chi_\tau\eta(\sigma^{-1})) \otimes \sigma(a)$$

$$\pi^*_{\rho\eta}(m) = \sum_{\sigma\in H_n} (\chi\eta(\sigma) + \chi_\tau\eta(a)) \otimes \sigma(a).$$

We remark that the character χ_τ does not depend on the choice of τ.

If ψ is a character of a subgroup H of Δ_n, then we also consider the projections to the ψ and ψ^{-1} components of $\overline{\mathbb{Q}} \otimes A$:

$$\pi_\psi : A \longrightarrow (\overline{\mathbb{Q}} \otimes A)^\psi$$

$$\pi_\psi(a) = \sum_{\sigma\in H} \psi(\sigma^{-1}) \otimes \sigma(a)$$

$$\pi^*_\psi : A \longrightarrow (\overline{\mathbb{Q}} \otimes A)^{\psi^{-1}}$$

$$\pi^*_\psi(a) = \sum_{\sigma\in H} \psi(\sigma) \otimes \sigma(a).$$

We note that with this notation, $\pi^*_\psi = \pi_{\psi^{-1}}$. We also note that in our situation, $\pi_{\rho\eta} = \pi_{\chi\eta} + \pi_{\chi_\tau\eta}$ and $\pi^*_{\rho\eta} = \pi^*_{\chi\eta} + \pi^*_{\chi_\tau\eta}$.

The following local projection is how α is incorporated into our conjectures. It is an idea of Greenberg and Vatsal ([14]), and is a key aspect to the conjecture. Let $D_p \subset \Delta$ be the decomposition group at p determined by ι_p and let δ_p be the arithmetic Frobenius. For a D_p-module A, a root of unity ζ, and an element $a \in A$, let $|m|_\zeta \in \mathbb{C}_p \otimes M$ be the projection of a to the subspace of $\mathbb{C}_p \otimes M$ on which δ_p acts by scaling by ζ. That is, if

$$\varepsilon_p : D_p \longrightarrow \overline{\mathbb{Q}}^\times$$

is the character $\varepsilon_p(\delta_p) = \zeta$, then

$$|a|_\zeta = \frac{1}{|D_p|}\pi_\varepsilon^*(a) = \frac{1}{|D_p|}\sum_{\delta \in D_p} \varepsilon_p(\delta) \otimes \delta(a).$$

We will use the projection when ζ is either $\alpha, \beta, 1/\alpha$, or $1/\beta$. Via the isomorphism $\Delta_n = \Delta \times \Gamma_n$, we view D_p as a subgroup of Δ_n for any n. Then any Δ_n-modules are also D_p-modules.

Let

$$\log_p : \mathbb{C}_p^\times \longrightarrow \mathbb{C}_p$$

denote Iwasawa's p-adic logarithm. On $U = \{u \in \mathbb{C}_p^\times : |1-u| < 1\}$, $\log_p$ is given by the usual power series, and then $\log_p$ is extended to all of $\mathbb{C}_p^\times$ by making $\log_p(p) = 0$ and $\log_p(\zeta_n) = 0$ for any root of unity of order prime to p. We extend $\log_p$ to $\mathbb{C}_p \otimes_\mathbb{Z} \mathbb{C}_p^\times$ by $\mathbb{C}_p$-linearity, and we may view $\log_p$ as a function on $\mathbb{C}_p \otimes \overline{\mathbb{Q}}^\times$ via ι_p.

We can now state our conjectures. We first state integral conjectures at $s = 0$ and $s = 1$ using the units u_n. Compare these conjectures with Conjectures 1.1.1 and 1.1.5.

Conjecture 3.2.2. *Let $\psi, \eta \in \mathcal{W}(\mathbb{C}_p)$ be of orders p^n and p^m, respectively. Then*

$$L_p(\chi, \alpha, \psi\omega, \eta\omega, 0) = \frac{(1-\beta\psi(p))\left(1 - \frac{\psi^{-1}(p)}{\alpha p}\right)\frac{\tau(\psi^{-1})}{p^{n+1}}}{(1-\beta\eta(p))\left(1 - \frac{\eta^{-1}(p)}{\alpha p}\right)\frac{\tau(\eta^{-1})}{p^{m+1}}}\frac{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}}{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}} \tag{3.5}$$

Conjecture 3.2.3. *Let $\psi, \eta \in \mathcal{W}(\mathbb{C}_p)$ be of orders p^n and p^m, respectively. Then*

$$L_p(\chi, \alpha, \psi, \eta, 1) = \frac{\left(1 - \frac{\psi^{-1}(p)}{\alpha}\right)\left(1 - \frac{\psi(p)\beta}{p}\right) \frac{\tau(\psi)}{\psi(N)p^{n+1}}}{\left(1 - \frac{\eta^{-1}(p)}{\alpha}\right)\left(1 - \frac{\eta(p)\beta}{p}\right) \frac{\tau(\eta)}{\eta(N)p^{m+1}}} \frac{\log_p |\pi_{\rho\psi}(u_n)|_\beta}{\log_p |\pi_{\rho\eta}(u_m)|_\beta}. \tag{3.6}$$

Remarks 3.2.4. There should be a functional equation relating the p-adic L-functions $L_p(\chi, \alpha, \psi, \eta, s)$ and $L_p(\chi, \alpha, \omega\psi, \omega\eta, 1-s)$ which makes these two conjectures equivalent. In the future we hope to prove the existence of such a functional equation by relating the construction of the two-variable p-adic L-function given in section 2.7 to the two-variable p-adic L-function associated to a Hida family given in [13]. The two-variable p-adic L-function defined in [13] does satisfy the necessary functional equation, so relating the construction given here to the ones in [13] would give us the desired functional equation.

Proposition 3.2.5. *When F is real quadratic, the quantities*

$$\frac{\log_p |\pi_{\rho\eta}(u_m)|_\beta}{\log_p |\pi_{\rho\psi}(u_n)|_\beta} \text{ and } \frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}}$$

do not depend on the choice of τ.

Proof. We do the proof for $\frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}}$. The proof for $\frac{\log_p |\pi_{\rho\eta}(u_m)|_\beta}{\log_p |\pi_{\rho\psi}(u_n)|_\beta}$ is similar. Let

$$\varepsilon : D_p \longrightarrow \overline{\mathbb{Q}}$$

$$\varepsilon(\delta_p) = \alpha.$$

We break it into two cases, when p is split in F and when p is inert.

Assume p is split in F as $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ where $\mathfrak{p}$ is picked out by ι_p. By definition we have

$$\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha} = \sum_{\sigma\in H_m} (\chi\eta(\sigma) + \chi_\tau\eta(\sigma)) \log_p \left(\frac{1}{|D_p|} \sum_{\delta\in D_p} \varepsilon(\delta) \otimes \delta(u_m) \right)$$

$$= \left(\frac{1}{|D_p|} \sum_{\delta\in D_p} \varepsilon(\delta)\delta \right) \left(\sum_{\sigma\in H_m} (\chi\eta(\sigma) + \chi_\tau\eta(\sigma)) \log_p(\sigma(u_m)) \right)$$

Working with the sum over H_m we have by definition of u_m

$$\sum_{\sigma\in H_m} \chi\eta(\sigma) \log_p(\sigma(u_m)) = \sum_{\sigma\in H_m} \chi\eta(\sigma) \log_p(\sigma(u_{K_m})) + \sum_{\sigma\in H_m} \chi\eta(\sigma) \log_p(\sigma\tau(u_{K_m}))$$

and

$$\sum_{\sigma\in H_m} \chi_\tau\eta(\sigma) \log_p(\sigma(u_m)) = \sum_{\sigma\in H_m} \chi_\tau\eta(\sigma) \log_p(\sigma(u_{K_m})) + \sum_{\sigma\in H_m} \chi_\tau\eta(\sigma) \log_p(\sigma\tau(u_{K_m}))$$

Since $\mathrm{Gal}(M_m/K_n) = \ker(\chi\eta)$ and $K_m \neq \overline{K}_m$, $\chi\eta$ restricted to $\mathrm{Gal}(M_m/\overline{K}_m)$ is non-trivial. Therefore

$$\sum_{\delta\in \mathrm{Gal}(M_m/K_m)} \chi\eta(\delta) = 0.$$

Then since $\tau(u_{K_m}) \in \overline{K}_m$, by summing over the cosets of $\mathrm{Gal}(M_m/\overline{K}_m)$ in H_m we see that

$$\sum_{\sigma\in H_m} \chi\eta(\sigma) \log_p(\sigma(\tau(u_{K_m}))) = 0.$$

Similarly,

$$\sum_{\sigma\in H_m} \chi_\tau\eta(\sigma) \log_p(\sigma(u_{K_m})) = 0.$$

Therefore

$$\sum_{\sigma\in H_m}(\chi\eta(\sigma)+\chi_\tau\eta(\sigma))\log_p(\sigma(u_m)) = \sum_{\sigma\in H_m}\chi\eta(\sigma)\log_p(\sigma(u_{K_m}))+$$

$$+\sum_{\sigma\in H_m}\chi_\tau\eta(\sigma)\log_p(\sigma(\tau(u_{K_m}))),$$

so

$$\log_p|\pi^*_{\rho\eta}(u_m)|_{1/\alpha} = \left(\frac{1}{|D_p|}\sum_{\delta\in D_p}\varepsilon(\delta)\chi^{-1}(\delta)\right)\left(\sum_{\sigma\in H_m}\chi\eta(\sigma)\log_p(\sigma(u_{K_m}))\right)+$$

$$+\left(\frac{1}{|D_p|}\sum_{\delta\in D_p}\varepsilon(\delta)\chi_\tau^{-1}(\delta)\right)\left(\sum_{\sigma\in H_m}\chi_\tau\eta(\sigma)\log_p(\sigma(u_{K_m}))\right)$$

The choice of α matters. Since p is split in F either $\alpha=\chi(\mathfrak{p})$ or $\alpha=\chi(\overline{\mathfrak{p}})$.

Since we are assuming $\alpha\neq\beta$ this means $\chi(\mathfrak{p})\neq\chi(\overline{\mathfrak{p}})$. Then since τ is a nontrivial automorphism of F and ι_p picks out $\mathfrak{p}$

$$\chi(\delta_p)=\chi(\mathfrak{p})\neq\chi(\overline{\mathfrak{p}})=\chi_\tau(\delta_p),$$

so $\chi|_{D_p}\neq\chi_\tau|_{D_p}$.

If $\alpha=\chi(\mathfrak{p})$, then $\varepsilon=\chi|_{D_p}$ so

$$\sum_{\delta\in D_p}\varepsilon(\delta)\chi_\tau^{-1}(\delta)=\sum_{\delta\in D_p}\chi\chi_\tau^{-1}(\delta)=0$$

because $\chi|_{D_p}\neq\chi_\tau|_{D_p}$. Furthermore,

$$\sum_{\delta\in D_p}\varepsilon(\delta)\chi^{-1}(\delta)=|D_p|$$

because $\varepsilon = \chi|_{D_p}$. Therefore

$$\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha} = \sum_{\sigma \in H_m} \chi\eta(\sigma) \log_p(\sigma(u_{K_m})),$$

so $\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}$ does not depend on τ. Hence the ratio

$$\frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}}$$

does not depend on τ.

Now assume that $\alpha = \chi(\overline{\mathfrak{p}}) = \chi_\tau(\delta_p)$. By similar reasoning in this case we get that

$$\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha} = \sum_{\sigma \in H_m} \chi_\tau\eta(\sigma) \log_p(\sigma\tau(u_{K_m}))$$

which does depend on τ. Let $\tau' \in \Delta - H$ be another choice of τ. Since H has index 2 in Δ, $\tau' = h\tau$ for some $h \in H$. Note that $\chi_\tau = \chi_{\tau'}$. We claim that changing τ to τ' scales the sum by $\chi_\tau^{-1}(h)$. Indeed

$$\sum_{\sigma \in H_m} \chi_\tau\eta(\sigma) \log_p(\sigma\tau'(u_{K_m})) = \sum_{\sigma \in H_m} \chi_\tau\eta(\sigma) \log_p(\sigma h\tau(u_{K_m}))$$

$$= \chi_\tau^{-1}(h) \sum_{\sigma \in H_m} \chi_\tau\eta(\sigma) \log_p(\sigma\tau(u_{K_m})).$$

The scalar $\chi_\tau^{-1}(h)$ is independent of m and η so in the ratio

$$\frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}}$$

it will cancel out. Hence the ratio does not depend on the choice of τ.

Now assume p is inert in F. By definition

$$\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha} = \sum_{\sigma\in H_m} \chi\eta(\sigma)\log_p|\sigma(u_m)|_{1/\alpha} + \sum_{\sigma\in H_m} \chi_\tau\eta(\sigma)\log_p|\sigma(u_m)|_{1/\alpha}.$$

Just as in the p-split case these sums will simplify as

$$\sum_{\sigma\in H_m} \chi\eta(\sigma)\log_p|\sigma(u_n)|_{1/\alpha} = \sum_{\sigma\in H_m} \chi\eta(\sigma)\log_p|\sigma(u_{K_m})|_{1/\alpha}$$

and

$$\sum_{\sigma\in H_m} \chi_\tau\eta(\sigma)\log_p|\sigma(u_m)|_{1/\alpha} = \sum_{\sigma\in H_m} \chi_\tau\eta(\sigma)\log|\sigma\tau(u_{K_m})|_{1/\alpha}.$$

Rearranging the second we have

$$\begin{aligned}\sum_{\sigma\in H_m} \chi_\tau\eta(\sigma)\log_p|\sigma\tau(u_{K_m})|_{1/\alpha} &= \sum_{\sigma\in H_m} \chi\eta(\tau^{-1}\sigma\tau)\log_p\left(\sum_{\delta\in D_p}\varepsilon(\delta)\otimes\delta\sigma\tau(u_{K_m})\right)\\ &= \sum_{\sigma\in H_m} \chi\eta(\sigma)\log_p\left(\sum_{\delta\in D_p}\varepsilon(\delta)\otimes\delta\tau\sigma(u_{K_m})\right)\end{aligned}$$

Since p is inert in F, $\delta_p \in \Delta - H$. Therefore there exists $h \in H$ such that $\tau = \delta_p h$.

Then

$$\sum_{\sigma\in H_m} \chi\eta(\sigma)\log_p\left(\sum_{\delta\in D_p}\varepsilon(\delta)\otimes\delta\tau\sigma(u_{K_m})\right)=$$

$$=\sum_{\sigma\in H_m} \chi\eta(\sigma)\log_p\left(\sum_{\delta\in D_p}\varepsilon(\delta)\otimes\delta\delta_p h\sigma(u_{K_m})\right)=$$

$$=\chi^{-1}(h)\varepsilon^{-1}(\delta_p)\sum_{\sigma\in H_m} \chi\eta(\sigma)\log_p\left(\sum_{\delta\in D_p}\varepsilon(\delta)\otimes\delta\sigma(u_{K_m})\right)=$$

$$=\chi^{-1}(h)\varepsilon^{-1}(\delta_p)\sum_{\sigma\in H_m} \chi\eta(\sigma)\log_p|\sigma(u_{K_m})|_{1/\alpha}.$$

Hence

$$\log_p|\pi^*_{\rho\eta}(u_m)|_{1/\alpha}=(1+\chi^{-1}(h)\varepsilon^{-1}(\delta_p))\sum_{\sigma\in H_m}\chi\eta(\sigma)\log_p|\sigma(u_{K_m})|_{1/\alpha}.$$

Since $1+\chi^{-1}(h)\varepsilon^{-1}(\delta_p)$ does not depend on m or η, in the ratio

$$\frac{\log_p|\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p|\pi^*_{\rho\psi}(u_n)|_{1/\alpha}}$$

it will cancel out. Therefore the ratio does not depend on τ. □

In the proof of the previous proposition we simplified the ratio

$$\frac{\log_p|\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p|\pi^*_{\rho\psi}(u_n)|_{1/\alpha}}$$

so that only the Stark unit u_{K_n} appears. We record these simplifications in the following corollary. This corollary shows that Conjectures 3.2.2 and 3.2.3 do have the shape described by equation 3.4 in the discussion preceding the definition of $L_p(\chi,\alpha,\eta,\psi,s)$.

Corollary 3.2.6. *Let η and ψ be p-power order Dirichlet characters of conductors p^{m+1} and p^{n+1}.*

Assume F is imaginary quadratic and $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ in F with ι_p picking out the prime $\mathfrak{p}$. If $\alpha = \chi(\mathfrak{p})$, then

$$\frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}} = \frac{\sum_{\sigma \in H_m} \chi\eta(\sigma) \log_p(\sigma(u_{M_m}))}{\sum_{\sigma \in H_n} \chi\psi(\sigma) \log_p(\sigma(u_{M_n}))}.$$

If $\alpha = \chi(\overline{\mathfrak{p}})$, then

$$\frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}} = \frac{\sum_{\sigma \in H_m} \chi_\tau\eta(\sigma) \log_p(\sigma(u_{M_m}))}{\sum_{\sigma \in H_n} \chi_\tau\psi(\sigma) \log_p(\sigma(u_{M_n}))}.$$

If p is inert in F, then

$$\frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}} = \frac{\sum_{\sigma \in H_m} (\chi\eta(\sigma) + \chi_\tau\eta(\sigma)) \log_p |\sigma(u_{M_m})|_{1/\alpha}}{\sum_{\sigma \in H_n} (\chi\psi(\sigma) + \chi_\tau\psi(\sigma)) \log_p |\sigma(u_{M_n})|_{1/\alpha}}.$$

Assume F is real quadratic and $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ in F with ι_p picking out the prime $\mathfrak{p}$. If $\alpha = \chi(\mathfrak{p})$, then

$$\frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}} = \frac{\sum_{\sigma \in H_m} \chi\eta(\sigma) \log_p(\sigma(u_{K_m}))}{\sum_{\sigma \in H_n} \chi\psi(\sigma) \log_p(\sigma(u_{K_n}))}.$$

If $\alpha = \chi(\overline{\mathfrak{p}})$, then

$$\frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}} = \frac{\sum_{\sigma \in H_m} \chi_\tau\eta(\sigma) \log_p(\sigma(\tau(u_{K_m})))}{\sum_{\sigma \in H_n} \chi_\tau\psi(\sigma) \log_p(\sigma(\tau(u_{K_n})))}.$$

and the ratio does not depend on τ.

If p is inert in F, then

$$\frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}} = \frac{\sum\limits_{\sigma \in H_m} \chi\eta(\sigma) \log_p |\sigma(u_{K_m})|_{1/\alpha}}{\sum\limits_{\sigma \in H_n} \chi\psi(\sigma) \log_p |\sigma(u_{K_n})|_{1/\alpha}}.$$

Proof. When F is real quadratic these formulas follow from the proof of the previous proposition. When F is imaginary quadratic, the formulas follow from the definitions of $\pi^*_{\rho\eta}$ and $|\cdot|_{1/\alpha}$. □

We next state rational conjectures at $s = 0$ and $s = 1$ using the units that appear in Conjectures 1.1.9 and 1.1.10. Compare these two conjectures with Conjectures 1.1.9 and 1.1.10.

Conjecture 3.2.7. *Let $\psi, \eta \in \mathcal{W}(\mathbb{C}_p)$ be of order p^n and p^m respectively. Let k be a finite extension of $\mathbb{Q}$ containing the values of χ, ψ, and η. Let $u^*_{\chi\psi}, u^*_{\chi_\tau\psi}, u^*_{\chi\eta}$ and $u^*_{\chi_\tau\eta}$ be the Stark units from Conjecture 1.1.9. Then*

$$L_p(\chi, \alpha, \psi\omega, \eta\omega, 0) = \frac{(1-\beta\psi(p))\left(1-\frac{\psi^{-1}(p)}{\alpha p}\right)\frac{\tau(\psi^{-1})}{p^{n+1}}}{(1-\beta\eta(p))\left(1-\frac{\eta^{-1}(p)}{\alpha p}\right)\frac{\tau(\eta^{-1})}{p^{m+1}}} \frac{\log_p |u^*_{\chi\psi} + u^*_{\chi_\tau\psi}|_{1/\alpha}}{\log_p |u^*_{\chi\eta} + u^*_{\chi_\tau\eta}|_{1/\alpha}} \tag{3.7}$$

where $u^*_{\chi\psi} + u^*_{\chi\tau\psi}$ and $u^*_{\chi\eta} + u^*_{\chi_\tau\eta}$ are viewed as elements of $(k \otimes U_{M_n})^{(\rho\psi)*}$ and $(k \otimes U_{M_n})^{(\rho\eta)^*}$ respectively.

Conjecture 3.2.8. *Let $\psi, \eta \in \mathcal{W}(\mathbb{C}_p)$ be of order p^n and p^m respectively. Let k be a finite extension of $\mathbb{Q}$ containing the values of χ, ψ, and η. Let $u_{\chi\psi}, u_{\chi_\tau\psi}, u_{\chi\eta}$, and $u_{\chi_\tau\eta}$ be the Stark units from Conjecture 1.1.10. Then*

$$L_p(\chi, \alpha, \psi, \eta, 1) = \frac{\left(1-\frac{\psi^{-1}(p)}{\alpha}\right)\left(1-\frac{\psi(p)\beta}{p}\right)\frac{\tau(\psi)}{\psi(N)p^{n+1}}}{\left(1-\frac{\eta^{-1}(p)}{\alpha}\right)\left(1-\frac{\eta(p)\beta}{p}\right)\frac{\tau(\eta)}{\eta(N)p^{m+1}}} \frac{\log_p |u_{\chi\psi} + u_{\chi_\tau\psi}|_\beta}{\log_p |u_{\chi\eta} + u_{\chi_\tau\eta}|_\beta} \tag{3.8}$$

where $u_{\chi\psi}+u_{\chi_\tau\psi}$ *and* $u_{\chi\eta}+u_{\chi_\tau\eta}$ *are viewed as elements of* $(k\otimes U_{M_n})^{\rho\psi}$ *and* $(k\otimes U_{M_m})^{\rho\eta}$ *respectively.*

If we assume the archimedean rank one abelian Stark conjecture is true, then Conjecture 3.2.2 implies Conjecture 3.2.7 and Conjecture 3.2.3 implies Conjecture 3.2.8. We explain why Conjecture 3.2.2 implies Conjecture 3.2.7. The explanation for why Conjecture 3.2.3 implies 3.2.8 is similar.

Assume Conjecture 1.1.1 for the fields K_n/F and $\overline{K}_n/F$ for $n \geq 0$ and assume Conjecture 3.2.2. In point (2) of Remarks 1.1.11, $u^*_{\chi\eta}$ and $u^*_{\chi_\tau\eta}$ are defined assuming Conjecture 1.1.1. Then by our choice of u_m, we have that

$$u^*_{\chi\eta} = \pi^*_{\chi\eta}(u_m) \text{ and } u^*_{\chi_\tau\eta} = \pi^*_{\chi_\tau\eta}(u_m).$$

By the fact that

$$\pi^*_{\rho\eta} = \pi^*_{\chi\eta} + \pi^*_{\chi_\tau\eta}$$

and the following proposition about the ratios

$$\frac{\log_p |u^*_{\chi\eta} + u^*_{\chi_\tau\eta}|_{1/\alpha}}{\log_p |u^*_{\chi\psi} + u^*_{\chi_\tau\psi}|_{1/\alpha}}$$

in the real quadratic case, we have that

$$\frac{\log_p |u^*_{\chi\eta} + u^*_{\chi_\tau\eta}|_{1/\alpha}}{\log_p |u^*_{\chi\psi} + u^*_{\chi_\tau\psi}|_{1/\alpha}} = \frac{\log_p |\pi^*_{\chi\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\chi\psi}(u_n)|_{1/\alpha}}$$

so Conjecture 3.2.7 is true.

Proposition 3.2.9. *Let* η *and* ψ *be* p*-power order Dirichlet characters of conductors* p^{m+1} *and* p^{n+1}*. Let* $u_{\chi\eta} \in (k \otimes_{\mathbb{Z}} \mathscr{O}_{M_m})^{(\chi\eta)^{-1}}$ *and* $u_{\chi\psi} \in (k \otimes_{\mathbb{Z}} \mathscr{O}_{M_n})^{(\chi\psi)^{-1}}$*. When* F *is* *real quadratic the ratio*

$$\frac{\log_p |u^*_{\chi\eta} + u^*_{\chi_\tau\eta}|_{1/\alpha}}{\log_p |u^*_{\chi\psi} + u^*_{\chi_\tau\psi}|_{1/\alpha}}$$

does not depend on τ*. Assume* $p = \mathfrak{p}\overline{\mathfrak{p}}$ *in* F *with* ι_p *picking our the prime* $\mathfrak{p}$*. If* $\alpha = \chi(\mathfrak{p})$*, then*

$$\frac{\log_p |u_{\chi\eta} + u_{\chi_\tau\eta}|_{1/\alpha}}{\log_p |u_{\chi\psi} + u_{\chi_\tau\psi}|_{1/\alpha}} = \frac{\log_p(u_{\chi\eta})}{\log_p(u_{\chi_\tau\eta})}.$$

If $\alpha = \chi(\overline{\mathfrak{p}})$*, then*

$$\frac{\log_p |u_{\chi\eta} + u_{\chi_\tau\eta}|_{1/\alpha}}{\log_p |u_{\chi\psi} + u_{\chi_\tau\psi}|_{1/\alpha}} = \frac{\log_p(u_{\chi_\tau\eta})}{\log_p(u_{\chi_\tau\psi})}.$$

If F *is real quadratic and* p *is inert in* F*, then*

$$\frac{\log_p |u_{\chi\eta} + u_{\chi_\tau\eta}|_{1/\alpha}}{\log_p |u_{\chi\psi} + u_{\chi_\tau\psi}|_{1/\alpha}} = \frac{\log_p |u_{\chi\eta}|_{1/\alpha}}{\log_p |u_{\chi\psi}|_{1/\alpha}}.$$

If F *is imaginary quadratic and* p *is inert in* F*, then there is no major simplification.*

Proof. The proof is similar to and simpler than the proof of Proposition 3.2.5. □

Chapter 4

Proof of the conjecture when F is imaginary quadratic and p splits in F

4.1 Katz's p-adic L-function

In this section we state relevant facts that are needed about Katz's two variable p-adic L-function. Let F be an imaginary quadratic field of discriminant d_F, and let $p \geq 5$ be a prime that splits in F. Let p factor in F as $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ where $\mathfrak{p}$ is the prime induced by the embedding ι_p. Let $\mathscr{O}_p = \{x \in \mathbb{C}_p : |x| \leq 1\}$ be the closed unit ball in $\mathbb{C}_p$. Let $\mathfrak{f}$ be an integral ideal of F such that $(\mathfrak{f}, p) = 1$.

The domain of Katz's p-adic L-function is the set of all p-adic Hecke characters of modulus $\mathfrak{f}$. Let

$$G(\mathfrak{f}p^\infty) = \mathbb{A}_F^\times / F^\times U_{\mathfrak{f},p}$$

where

$$U_{\mathfrak{f},p} = \left\{ (x_v)_v \in \mathbb{A}_F^\times : \begin{matrix} x_v > 0 \text{ if } v \text{ is real} \\ x_v \equiv 1 \bmod \mathfrak{f}_v \text{ if } v \mid \mathfrak{f} \\ x_v \in \mathscr{O}_{F_v}^\times \text{ if } v \nmid \mathfrak{f}p \text{ and is finite} \\ x_v = 1 \text{ if } v \mid p \end{matrix} \right\}.$$

See the appendix for our conventions on Hecke characters. The set of all p-adic Hecke characters of modulus $\mathfrak{f}$ is

$$\mathrm{Hom}_{cont}(G(\mathfrak{f}p^\infty), \mathbb{C}_p^\times).$$

Let

$$F(\mathfrak{f}p^\infty) = \bigcup_{n=1}^{\infty} F(\mathfrak{f}p^n)$$

where $F(\mathfrak{f}p^n)$ is the ray class field of conductor $\mathfrak{f}p^n$. By class field theory $G(\mathfrak{f}p^\infty)$ is isomorphic to the Galois group of $F(\mathfrak{f}p^\infty)$ over F. We normalize the Artin map so that for a finite place v such that $(v, \mathfrak{f}p) = 1$, a uniformizer π_v at v is sent to an arithmetic Frobenius.

Order the two embeddings, σ_1, σ_2 of F into $\overline{\mathbb{Q}}$ so the first one is how we view F as a subfield of $\overline{\mathbb{Q}}$. If ψ is an algebraic Hecke character of infinity type $T = a\sigma_1 + b\sigma_2$ (see the appendix for the definition of an infinity type) then we say that ψ is of infinity type (a, b).

Let ψ be an algebraic Hecke character of infinity type (a, b) and conductor $\mathfrak{f}'\mathfrak{p}^{a_\mathfrak{p}}\overline{\mathfrak{p}}^{a_{\overline{\mathfrak{p}}}}$ where $\mathfrak{f}'$ divides $\mathfrak{f}$. We may view ψ as a p-adic Hecke character of modulus $\mathfrak{f}$ or as a complex Hecke character

$$\psi_p : G(\mathfrak{f}p^\infty) \longrightarrow \mathbb{C}_p^\times$$

$$\psi_\infty : \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times.$$

For ease of notation we drop the subscripts p and ∞ on ψ_p and ψ_∞. It will be clear from context when we are referring to ψ_p or ψ_∞.

Define the p-adic local root number associated to ψ to be the complex number

$$\begin{aligned} W_p(\psi) &= \frac{|\psi_\mathfrak{p}(\pi_\mathfrak{p}^{a_\mathfrak{p}})|}{N\mathfrak{p}^{a_\mathfrak{p}/2}} W(\psi_\mathfrak{p}^{-1}) \\ &= \frac{\psi_\mathfrak{p}(\pi_\mathfrak{p}^{-a_\mathfrak{p}})}{p^{a_\mathfrak{p}}} \sum_{u \in (\mathscr{O}_{F_\mathfrak{p}}/\mathfrak{p}^{a_\mathfrak{p}})^\times} \psi_\mathfrak{p}(u) \exp(-2\pi i (Tr_{F_\mathfrak{p}/\mathbb{Q}_p}(u/\pi_\mathfrak{p}^{a_\mathfrak{p}})) \end{aligned} \tag{4.1}$$

where $W(\psi_\mathfrak{p}^{-1})$ is the local root number at $\mathfrak{p}$ (as defined in the appendix) and $\psi_\mathfrak{p}$ denotes ψ restricted to $F_\mathfrak{p}^\times$ and $\pi_\mathfrak{p}$ is a uniformizer for $F_\mathfrak{p}$. We note that $F_\mathfrak{p} = \mathbb{Q}_p$ and so we could take $\pi_p = p$.

Let S be the set of places containing the infinite places of F and the places of F dividing $\mathfrak{f}$.

Theorem 4.1.1. *([17], [9]) There exists an $\mathscr{O}_p$-valued measure $\mu = \mu_\mathfrak{f}$ on $G(\mathfrak{f}p^\infty)$ and complex and p-adic periods $\Omega_\infty \in \mathbb{C}^\times$, $\Omega_p \in \mathbb{C}_p^\times$ such that for any algebraic Hecke character ψ of conductor $\mathfrak{f}'\mathfrak{p}^{a_\mathfrak{p}}\overline{\mathfrak{p}}^{a_{\overline{\mathfrak{p}}}}$ where $\mathfrak{f}'$ divides $\mathfrak{f}$ and infinity type (a,b) with $a > 0$ and $b \leq 0$ we have*

$$\frac{\int_{G(\mathfrak{f}p^\infty)} \psi(x) d\mu(x)}{\Omega_p^{a-b}} = \frac{\sqrt{d_F}^b (a-1)!}{(2\pi)^b} W_p(\psi^{-1}) \left(1 - \frac{\psi(\mathfrak{p})}{p}\right) (1 - \psi^{-1}(\overline{\mathfrak{p}})) \frac{L_S(\psi^{-1}, 0)}{\Omega_\infty^{a-b}} \tag{4.2}$$

Once the p-adic and complex periods Ω_p and Ω_∞ are chosen, μ is uniquely determined by the above interpolation property.

Remarks 4.1.2. 1. Let $\mathbb{Q}_p^{ur}$ denote the maximal unramified extension of $\mathbb{Q}_p$ and let

$\widehat{\mathbb{Q}}_p^{ur}$ denote its completion. Let $\mathscr{O}_p^{ur} = \{x \in \widehat{\mathbb{Q}}_p^{ur} : |x| \leq 1\}$. The measure μ in the theorem is valued in $\mathscr{O}_p^{ur}$.

2. Katz originally proved this theorem in [17] for imaginary quadratic fields and then a similar theorem in [18] for CM fields. The above statement is taken from [9] with the correction from [3] and with a slight modifications in order to state everything adelically.

Definition 4.1.3. *Define* ***Katz's*** p***-adic*** L***-function*** *with respect to* F*,* p*, and* $\mathfrak{f}$ *to be the function*

$$L_p = L_{p,Katz} : \mathrm{Hom}_{cont}(G(\mathfrak{f}p^\infty), \mathbb{C}_p^\times) \longrightarrow \mathbb{C}_p$$

$$L_p(\psi) = \int_{G(\mathfrak{f}p^\infty)} \psi^{-1}(x) d\mu(x)$$

where μ *is the measure from the preceding theorem. Depending on the context we may or may not have the subscript* $Katz$*. For the rest of this section we drop the subscript. In following sections when we compare different* p*-adic* L*-functions we will have the subscript.*

We record the rephrasing of (4.2) for L_p in the following theorem.

Theorem 4.1.4. *The function* L_p *is uniquely determined by the interpolation property that for all algebraic Hecke characters* ψ *of conductor* $\mathfrak{f}'\mathfrak{p}^{a_\mathfrak{p}}\overline{\mathfrak{p}}^{a_{\overline{\mathfrak{p}}}}$ *where* $\mathfrak{f}'$ *divides* $\mathfrak{f}$ *and infinity type* (a, b) *with* $a < 0$ *and* $b \geq 0$*, we have*

$$\frac{L_p(\psi)}{\Omega_p^{b-a}} = \frac{(-a-1)!(2\pi)^b}{\sqrt{d_F}^b} W_p(\psi) \left(1 - \frac{\psi^{-1}(\mathfrak{p})}{p}\right) (1 - \psi(\overline{\mathfrak{p}})) \frac{L_S(\psi, 0)}{\Omega_\infty^{b-a}}. \tag{4.3}$$

We now state Katz's p-adic Kronecker's second limit theorem. Fix a nontrivial integral ideal of F and let L_p be Katz's p-adic L-function associated to F, p, and $\mathfrak{f}$. Let $\zeta_n = \iota_\infty^{-1}(e^{2\pi i/n}) \in \overline{\mathbb{Q}}$ be a collection of primitive nth roots of unity in $\overline{\mathbb{Q}}$.

For our purposes we will consider Katz's theorem for algebraic Hecke characters of the form $\chi\psi$ where χ has conductor $\mathfrak{f}$ and trivial infinity type, and ψ is a p-power conductor Dirichlet character. We recall how to view a Dirichlet character adelically. Let

$$\psi : (\mathbb{Z}/p^n\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

be a Dirichlet character of conductor p^n. Then ψ as an algebraic Hecke character is the unique character

$$\psi : \mathbb{A}_{\mathbb{Q}}^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

such that for all primes $\ell \neq p$, $\psi|_{\mathbb{Z}_\ell^\times} = 1$ and $\psi(\pi_\ell) = \psi(\ell)$ where π_ℓ is a uniformizer in $\mathbb{Z}_\ell$, and $\psi(\mathbb{Q}^\times) = 1$. Define the Gauss sum of ψ as the element of $\overline{\mathbb{Q}}$:

$$\tau(\psi) = \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \psi(a)\zeta_{p^n}^a.$$

Then viewing $\tau(\psi^{-1})$ as an element of $\mathbb{C}$ via ι_∞, $\tau(\psi^{-1})$ satisfies the following relation to $W_p(\psi)$ from (4.1):

$$W_p(\psi) = \psi(-1)\tau(\psi^{-1}).$$

Theorem 4.1.5. *([17], [9]) Let χ be an algebraic Hecke character of conductor $\mathfrak{f}$ and trivial infinity type and let ψ be a Dirichlet character of conductor p^n. Let K be the fixed field of the kernel of $\chi\psi$ when $\chi\psi$ is viewed as a Galois character via the Artin* *isomorphism* $G(\mathfrak{f}p^\infty) \cong Gal(F(\mathfrak{f}p^\infty)/F)$. *Let* u_K *be the Stark unit for* K/F, $G = Gal(K/F)$, *and* e *be the number of roots of unity in* K. *Then*

$$L_p(\chi\psi) = -\frac{1}{e}\frac{\psi(-1)\tau(\psi^{-1})}{\chi(\mathfrak{p}^n)p^n}\left(1 - \frac{(\chi\psi)^{-1}(\mathfrak{p})}{p}\right)(1 - \chi\psi(\overline{\mathfrak{p}}))\sum_{\sigma\in G}\chi\psi(\sigma)\log_p(\sigma(u_K))$$

Remarks 4.1.6. A version of this was proved in Katz's original paper. The formulas for this theorem are taken from [9] with a minor correction so the $1 - \chi\psi(\overline{\mathfrak{p}})$ factor is correct (see [15]).

4.2 Definition of the period Ω_∞

Let E be an elliptic curve with CM by $\mathscr{O}_F$ defined over K, where K is a finite extension of F (K necessarily contains the Hilbert class field of F). Let $\omega \in \Omega^1(E/K)$ be a nonzero element. The period lattice of E is by definition

$$\mathscr{L} = \left\{\frac{1}{2\pi i}\int_\gamma \omega : \gamma \in H_1(E(\mathbb{C}), \mathbb{Z})\right\}.$$

Let ω_1, ω_2 be a $\mathbb{Z}$-basis of $\mathscr{L}$. Since E has CM by $\mathscr{O}_F$, $\sqrt{d_F}\mathscr{L} \subset \mathscr{L}$ so in particular

$$\sqrt{d_F}\omega_2 = a\omega_1 + b\omega_2$$

for some $a, b \in \mathbb{Z}$. Hence

$$\frac{\omega_1}{\omega_2} = \frac{\sqrt{d_F} - b}{a} \in F.$$

It follows then that $\mathbb{Z} + \mathbb{Z}\frac{\omega_1}{\omega_2} \subset F \subset \mathbb{C}$ is a fractional ideal of F. Let $\mathfrak{f} = \mathbb{Z} + \mathbb{Z}\frac{\omega_1}{\omega_2}$ and $\Omega_\infty = \omega_2$. Then

$$\mathscr{L} = \Omega_\infty\mathfrak{f}$$

and Ω_∞ is defined to be "the" complex period associated to E. The reason that the word, the, is in quotation marks is because obviously Ω_∞ depends on some choices. The choice of ω_2 can be any element of $\mathscr{L}$ that may be extended to a $\mathbb{Z}$-basis of $\mathscr{L}$. A different choice of ω_2 would scale Ω_∞ by an element of F. We could also change the choice of ω. A different choice of ω would be a nonzero K-scaler of ω because $\Omega^1(E/K)$ is a rank one K-vector space. Changing ω would then scale $\mathscr{L}$ by that element of K which in turn scales Ω_∞. The different choices all scale Ω_∞ by elements of $\overline{\mathbb{Q}}^\times$, so Ω_∞ is a well defined element of $\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$.

If we tensor everything with $\mathbb{Q}$ then that simplifies the choice of ω_2 for then if we let

$$H_1(E(\mathbb{C}), \mathbb{Q}) = H_1(E(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

we have that

$$\left\{ \int_\gamma \omega : \gamma \in H_1(E(\mathbb{C}), \mathbb{Q}) \right\} = \Omega_\infty F$$

where $\Omega_\infty = \frac{1}{2\pi i} \int_{\gamma_0} \omega$ for any $\gamma_0 \in H_1(E(\mathbb{C}), \mathbb{Q})$.

The complex period Ω_∞ has the property that for all algebraic Hecke characters ψ of F with infinity type (a, b) where $a < 0$ and $b \geq 0$ the ratio

$$\frac{(2\pi i)^b L(\psi, 0)}{\Omega_\infty^{b-a}}$$

is algebraic. This ratio being algebraic obviously does not depend on the choice of Ω_∞ since different choices scale Ω_∞ by an algebraic number. Furthermore, after taking into account the functional equation of $L(\psi, s)$, these L-values account for all critical L-values of algebraic Hecke characters of F.

4.3 Definition of the period pair $(\Omega_\infty, \Omega_p)$

Let K be a finite extension of F that contains the Hilbert class field of F. Let $\mathfrak{P}$ be the prime of K determined by ι_p. Let E be an elliptic curve with CM by $\mathscr{O}_F$ defined over K and with good reduction at $\mathfrak{P}$. Let $\omega \in \Omega^1(E/K)$ be an invariant differential of E defined over K. Attached to the pair (E, ω), we let x and y be coordinates on E such that

$$\begin{aligned} \iota : E &\longrightarrow \mathbb{P}^2 \\ P &\longmapsto (x, y, 1) \end{aligned}$$

is an embedding defined over K, which embeds E as the zero set of $y^2 = 4x^3 - g_2 x + g_3$ and such that $\iota^*(\frac{dx}{y}) = \omega$. Let E_ω denote the image of E under ι. Let $E_\omega(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$ denote the complex manifold which consists of the complex points of E_ω. Let $\gamma \in H_1(E_\omega(\mathbb{C}), \mathbb{Q})$ and define the complex period

$$\Omega_\infty = \frac{1}{2\pi i} \int_\gamma \omega.$$

In the previous section we defined Ω_∞ directly from E without the consideration of E_ω. The reason for considering E_ω is that in defining Ω_p we consider the formal group and E and it will be important to keep track of the coordinates on E used to define the formal group.

We now explain how to define Ω_p from Ω_∞. Let

$$\mathscr{L} = \left\{ \frac{1}{2\pi i} \int_\eta \omega : \eta \in H_1(E_\omega(\mathbb{C}), \mathbb{Z}) \right\}$$

be the period lattice of E_ω. Then we have the complex uniformization

$$\begin{aligned}\Phi : \mathbb{C}/\mathscr{L} &\longrightarrow E_\omega(\mathbb{C})\\ z &\longmapsto (\mathscr{P}(\mathscr{L},z), \mathscr{P}'(\mathscr{L},z), 1)\end{aligned}$$

were $\mathscr{P}$ is the Weierstrass function. We consider the element

$$(p^{-n}\Omega_\infty)_{n=1}^\infty \in \varprojlim_n (p^{-n}\Omega_\infty F/\Omega_\infty F) = (\varprojlim_n p^{-n}\mathscr{L}/\mathscr{L}) \otimes \mathbb{Q}_p$$

which is in the Tate module of $\mathbb{C}/\mathscr{L}$ tensored with $\mathbb{Q}_p$. Let $V_pE_\omega = T_pE_\omega \otimes \mathbb{Q}_p$, and let $\xi = (\xi_n)_{n=1}^\infty$ be the image of $(p^{-n}\Omega_\infty)_{n=1}^\infty$ under the composition

$$\varprojlim_n p^{-n}\Omega_\infty F/\Omega_\infty F \xrightarrow{\Phi_p} V_pE_\omega \longrightarrow V_{\mathfrak{p}}E_\omega$$

where the second map is the projection corresponding to $T_pE_\omega = T_{\mathfrak{p}}E_\omega \times T_{\bar{\mathfrak{p}}}E_\omega$.

The coordinates x and y on E_ω determine a formal group of E, $\widehat{E}_\omega$, which we view as a formal group over $K_{\mathfrak{P}}$. Let $V_p\widehat{E}_\omega = T_p\widehat{E}_\omega \otimes \mathbb{Q}_p$. Since p splits in F and $\mathfrak{p}$ is the prime of F determined by ι_p, we have that $T_p\widehat{E}_\omega = T_{\mathfrak{p}}E_\omega$. Let ξ now denote the corresponding elemet of $V_p\widehat{E}_\omega$. Since $V_p\widehat{E}$ is a rank one $\mathbb{Q}_p$-module, ξ is a basis element. Let

$$\zeta = (\zeta_{p^n})_{n=1}^\infty = (\iota_p^{-1}(\exp(2\pi i/p^n)))_{n=1}^\infty$$

so ζ is a basis element of $V_p\widehat{\mathbb{G}}_m := T_p\widehat{\mathbb{G}}_m \otimes \mathbb{Q}_p$. Define

$$\varphi_p : V_p\widehat{E}_\omega \longrightarrow V_p\widehat{\mathbb{G}}_m$$

by $\varphi_p(\xi) = \zeta$. It is a result of Tate in his p-divisible groups paper that the map

$$\mathrm{Hom}_{\mathscr{O}_{\mathbb{C}_p}}(\widehat{E}_\omega, \widehat{\mathbb{G}}_m) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p\widehat{E}_\omega, T_p\widehat{\mathbb{G}}_m)$$

is a bijection. We note that

$$\mathrm{Hom}_{\mathbb{Q}_p}(V_p\widehat{E}_\omega, V_p\widehat{\mathbb{G}}_m) = \mathrm{Hom}_{\mathbb{Z}_p}(T_p\widehat{E}_\omega, T_p\widehat{\mathbb{G}}_m) \otimes \mathbb{Q}_p$$

and let $\varphi \in \mathrm{Hom}_{\mathscr{O}_{\mathbb{C}_p}}(\widehat{E}, \widehat{\mathbb{G}}_m) \otimes \mathbb{Q}_p$ be the element corresponding to φ_p. Define Ω_p by the rule

$$\omega = \Omega_p \varphi^*(dT/(1+T)).$$

The definition of the pair $(\Omega_\infty, \Omega_p)$ depends on the choice of E, ω, and γ. We will examine the dependence on this choice and show that as an element of $\mathbb{C}^\times \times \mathbb{C}_p^\times / \overline{\mathbb{Q}}^\times$, $(\Omega_\infty, \Omega_p)$ does not depend on E, ω or γ but only on the imaginary quadratic field F.

Let the pair (E, ω) be given and suppose we have two choices $\gamma, \gamma' \in H_1(E_\omega(\mathbb{C}), \mathbb{Q})$. Since

$$\left\{ \int_\eta \omega : \eta \in H_1(E_\omega(\mathbb{C}), \mathbb{Q}) \right\} = \Omega_\infty F$$

we have that

$$\Omega'_\infty = \frac{1}{2\pi i} \int_{\gamma'} \omega = \lambda \Omega_\infty$$

for some $\lambda \in F$. Define ξ' and φ'_p corresponding to Ω'_∞ as we did for Ω_∞. Note that $F_\mathfrak{p}$ acts on $V_p\widehat{E}_\omega$ via the action of $\mathscr{O}_{F_\mathfrak{p}}$ on $\widehat{E}_\omega$ as a Lubin-Tate formal group. We denote the action of $\alpha \in \mathscr{O}_{F_\mathfrak{p}}$ on $\widehat{E}_\omega$ as $[\alpha]$ as in the previous secion. With respect to this action, we have the relation $\xi' = [\lambda]_p \xi$. Then

$$(\varphi_p \circ [\lambda^{-1}]_p)(\xi') = \varphi_p(\xi) = \zeta$$

so $\varphi' = \varphi \circ [\lambda^{-1}]$. Then

$$\begin{aligned}\lambda\Omega_p\varphi'^*(dT/(1+T)) &= \lambda\Omega_p[\lambda^{-1}]^*\varphi^*(dT/(1+T))\\ &= \lambda\Omega_p\Omega_p^{-1}[\lambda^{-1}]^*(\omega)\\ &= \omega\end{aligned}$$

by definition of Ω_p and since $[\lambda^{-1}]^*(\omega) = \lambda^{-1}\omega$. Therefore $\Omega'_p = \lambda\Omega_p$, so $(\Omega_\infty, \Omega_p) \equiv (\Omega'_\infty, \Omega'_p) \bmod \overline{\mathbb{Q}}^\times$.

Now suppose we have two pairs (E, ω) and (E, ω'). Then $\omega' = \lambda\omega$ for some $\lambda \in K^\times$. Let

$$\mathscr{L} = \left\{\frac{1}{2\pi i}\int_\eta \omega : \eta \in H_1(E_\omega(\mathbb{C}), \mathbb{Z})\right\}$$

and

$$\mathscr{L}' = \left\{\frac{1}{2\pi i}\int_\eta \omega' : \eta \in H_1(E_{\omega'}(\mathbb{C}), \mathbb{Z})\right\}$$

be the respective period lattices of E_ω and $E_{\omega'}$. We have the $\mathscr{L}' = \lambda\mathscr{L}$ and so multiplication by λ defines an isomorphism from $\mathbb{C}/\mathscr{L}$ to $\mathbb{C}/\mathscr{L}'$. Let λ_{alg} be the corresponding isogeny from E_ω to $E_{\omega'}$ such that the diagram

$$\begin{array}{ccc}\mathbb{C}/\mathscr{L} & \xrightarrow{\Phi} & E_\omega(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})\\ \times\lambda\downarrow & & \downarrow\lambda_{alg}\\ \mathbb{C}/\mathscr{L}' & \xrightarrow{\Phi'} & E_{\omega'}(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})\end{array}$$

commutes. With respect to the coordinates on $\mathbb{P}^2$, λ_{alg} sends $(x, y, 1)$ to $(\lambda^2 x, \lambda^3 y, 1)$. Therefore λ_{alg} is defined over K. Let $\gamma \in H_1(E_\omega(\mathbb{C}), \mathbb{Q})$ and define

$$\Omega_\infty = \frac{1}{2\pi i}\int_\gamma \omega.$$

Since we've already checked to dependence of the period pair on changing γ, we may use any $\gamma' \in H_1(E_{\omega'}(\mathbb{C}), \mathbb{Q})$ to define Ω'_∞. Let γ' then be the image of γ under λ_{alg} and define

$$\Omega'_\infty = \frac{1}{2\pi i}\int_{\gamma'} \omega'.$$

By the change of variables formula we have that

$$\Omega'_\infty = \frac{1}{2\pi i}\int_{\gamma'} \omega' = \frac{1}{2\pi i}\int_{\lambda_{alg}\circ\gamma} \omega' = \frac{1}{2\pi i}\int_{\gamma} \lambda^*_{alg}(\omega') = \frac{1}{2\pi i}\int_{\gamma} \lambda\omega = \lambda\Omega_\infty.$$

Let ξ, φ_p and ξ', φ'_p be used to define Ω_p and Ω'_p respectively. Then we have the following commutative diagram

$$\begin{array}{ccc} V_p\widehat{E}_\omega & \xrightarrow{\varphi_p} & V_p\widehat{\mathbb{G}}_m \\ \lambda_{alg,p}\big\downarrow & & \big\downarrow = \\ V_p\widehat{E}_{\omega'} & \xrightarrow{\varphi'_p} & V_p\widehat{\mathbb{G}}_m \end{array}$$

because $\lambda_{alg,p}(\xi) = \xi'$ and φ_p, φ'_p are defined by sending ξ to ζ and ξ' to ζ respectively. The commutative diegram then gives the relation

$$\begin{aligned} \omega &= \Omega_p\varphi^*\left(\tfrac{dT}{1+T}\right) \\ &= \Omega_p(\varphi' \circ \lambda_{alg})^*\left(\tfrac{dT}{1+T}\right) \\ &= \Omega_p\lambda^*_{alg}\varphi'^*\left(\tfrac{dT}{1+T}\right) \\ &= \Omega_p\Omega'^{-1}_p\lambda^*_{alg}(\omega') \\ &= \Omega_p\Omega'^{-1}_p\lambda\omega \end{aligned}$$

so $\Omega'_p = \lambda\Omega_p$. Hence $(\Omega_\infty, \Omega_p) \equiv (\Omega'_\infty, \Omega'_p) \bmod \overline{\mathbb{Q}}^\times$.

Finally we see what happens if we change E to E' where both E and E' have CM by $\mathscr{O}_F$. By enlarging K, we may assume K is the same for both E and E'. Let ω

and γ be choices for E that define Ω_∞ and Ω_p. Let $\omega' \in \Omega^1(E'/K)$ be any choice. Let $\mathscr{L}$ and $\mathscr{L}'$ be the period lattices for E_ω and $E'_{\omega'}$. Since E and E' both have CM by $\mathscr{O}_F$, there exists $\alpha \in F^\times$ such that $\alpha\mathscr{L} \subset \mathscr{L}'$. We then get a commutative diagram

$$\begin{array}{ccc}
\mathbb{C}/\mathscr{L} & \xrightarrow{\Phi} & E_\omega(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C}) \\
{\scriptstyle \times\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_{alg}} \\
\mathbb{C}/\mathscr{L}' & \xrightarrow{\Phi'} & E'_{\omega'}(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})
\end{array}$$

which is similar to the one we have before. We define $\gamma' = \alpha_{alg}(\gamma)$ and

$$\Omega'_\infty = \frac{1}{2\pi i}\int_{\gamma'} \omega'.$$

Then the same calculations as before show that $\Omega'_\infty = \alpha\Omega_\infty$ and $\Omega'_p = \alpha\Omega_p$ so $(\Omega_\infty, \Omega_p) \equiv (\Omega'_\infty, \Omega'_p)$.

4.4 The CM Hida family

For the remainder of chapter 4, fix a nontrivial ray class character χ of conductor $\mathfrak{f}$ such that $(\mathfrak{f}, p) = 1$, and let

$$f = \sum_{\mathfrak{a} \subset \mathscr{O}_F} \chi(\mathfrak{a}) q^{N\mathfrak{a}}$$

be the weight one modular form associated to χ. The goal of this section is to explicitly describe the rigid analytic function T_ℓ for $\ell \nmid Np$ and U_p on a neighborhood of the point corresponding to f_α on the eigencurve.

We recall that the level of f is $N = |d_F| N_{K/\mathbb{Q}}\mathfrak{f}$ and the character is ε :

$(\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}$ determined by the rule

$$\varepsilon(\ell) = \chi(\ell\mathscr{O}_F)$$

for primes $\ell \nmid Np$. Let f_α be a p-stabilization of f, so α is either $\chi(\mathfrak{p})$ of $\chi(\overline{\mathfrak{p}})$.

We embed $\mathbb{Z}$ into weight space as

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathcal{W}(\mathbb{Q}_p) \\ k &\longmapsto \nu_k : t \mapsto t^{k-2}. \end{aligned}$$

By Bellaíche and Dmitrov's theorem about the eigencurve at weight one points (Theorem 3.1.1), the eigencurve is étale at the point corresponding to f_α. Let $w = \nu_1 \in \mathcal{W}(\mathbb{Q}_p)$ and let $W = W(w, 1/p^r) = \operatorname{Sp} R$ be a neighborhood of w such that the weight map $C^\pm_W \to W$ is étale at all points in the connected component containing the point corresponding to f_α. Let $x \in C^\pm_W(\mathbb{C}_p)$ be the point corresponding to f_α and let $V_{\mathbb{C}_p} = \operatorname{Sp} T_{\mathbb{C}_p} \subset C^\pm_{W,\mathbb{C}_p}$ be the connected component of $C^\pm_{W,\mathbb{C}_p}$ containing x. Then $V_{\mathbb{C}_p} \to W_{\mathbb{C}_p}$ is étale, and we take W to be as in Proposition 2.6.5. Then the weight map on the level of rings

$$R_{\mathbb{C}_p} \longrightarrow T_{\mathbb{C}_p}$$

is an isomorphism, which we use this map to identify $T_{\mathbb{C}_p}$ with $R_{\mathbb{C}_p}$.

We have that

$$W(\mathbb{C}_p) = \{\kappa : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times : \kappa|_{\mu_{p-1}} = \omega^{-1}, |\kappa(\gamma) - w(\gamma)| \leq 1/p^r\}$$

where γ is an topological generator of $1 + p\mathbb{Z}_p$. Fix a choice of topological generator γ of $1 + p\mathbb{Z}_p$. Then

$$R = \left\{ \sum a_n (t - (w(\gamma) - 1))^n \in \mathbb{Q}_p[[t - (w(\gamma) - 1)]] : |a_n p^{rn}| \to 0 \text{ as } n \to \infty \right\}.$$

Let $z = t - (w(\gamma) - 1)$ so R is the set of all $F(z) \in \mathbb{Q}_p[[z]]$ that converge on the closed around 0 disk of radius $1/p^r$ in $\mathbb{C}_p$. Recall that for $F(z) \in R$ and $\kappa \in W$

$$ev_\kappa(F(z)) = F(\kappa(\gamma) - w(\gamma)).$$

By the Weierstrass preparation theorem any $F(z) \in R$ is determined by its values

$$ev_{\nu_k}(F(z)) = F(\nu_k(\gamma) - w(\gamma)) = F(\gamma^{k-2} - \gamma^{-1})$$

at the integers $k \in \mathbb{Z}$ such that $\nu_k \in W$. We record here that for and integer k, $\nu_k \in W = W(w, 1/p^r)$ if and only if $k \equiv 1 \bmod p^{r-1}(p-1)$.

Since $C_W^{\pm}$ is the eigencurve the Hecke operators T_ℓ for $\ell \nmid Np$, U_p and $[a]$ for $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ that generate $R_{\mathbb{C}_p}$ are are the unique elements of $R_{\mathbb{C}_p}$ such that

1. At the weight w we have

$$ev_w(T_\ell) = a_\ell(f_\alpha) = \begin{cases} \chi(\mathfrak{q}) + \chi(\overline{\mathfrak{q}}) & \text{if } \ell\mathscr{O}_F = \mathfrak{q}\overline{\mathfrak{q}} \\ 0 & \text{if } \ell \text{ is inert in } F \end{cases}$$

   $ev_w(U_p) = \alpha$, and $ev_w([a]) = \varepsilon(a)$ for all $a \in (\mathbb{Z}/N\mathbb{Z})^\times$.

2. For all $k \in \mathbb{Z}_{\geq 2}$ such that $\nu_k \in W$, $ev_{\nu_k}(T_\ell), ev_{\nu_k}(U_p)$ are the T_ℓ and U_p Hecke eigenvalues of an eigenform g of weight k, level Γ_0, and character ε which is new at level N. This condition implies that the rigid analytic functions $[a] \in R_{\mathbb{C}_p}$ are the constant functions $[a] = \varepsilon(a)$.

We now exhibit explicit elements of $R_{\mathbb{C}_p}$ with the above two properties which must then be T_ℓ for $\ell \nmid Np$ and U_p.

To begin we define an algebraic Hecke character of F. Since $p \geq 5$, the only root of unity congruent to 1 mod $\mathfrak{p}$ or 1 mod $\overline{\mathfrak{p}}$ in F is 1. Therefore the groups $P_1(\mathfrak{p})$ and $P_1(\overline{\mathfrak{p}})$ are subgroups of $F^\times$:

$$P_1(\mathfrak{p}) = \{\alpha \in F^\times : ((\alpha), \mathfrak{p}) = 1, \alpha \equiv 1 \bmod \mathfrak{p}\}$$

$$P_1(\overline{\mathfrak{p}}) = \{\alpha \in F^\times : ((\alpha), \overline{\mathfrak{p}}) = 1, \alpha \equiv 1 \bmod \overline{\mathfrak{p}}\}.$$

We must consider the two cases of when $\alpha = \chi(\overline{\mathfrak{p}})$ and when $\alpha = \chi(\mathfrak{p})$ somewhat separately. If $\alpha = \chi(\overline{\mathfrak{p}})$, define λ_0 (viewing $P_1(\mathfrak{p})$ as a subgroup of $F^\times$) as

$$\lambda_0 : P_1(\mathfrak{p}) \longrightarrow F^\times \subset \overline{\mathbb{Q}}^\times$$

$$\lambda_0(\alpha) = \alpha.$$

If $\alpha = \chi(\mathfrak{p})$ define λ_0 (viewing $P_1(\overline{\mathfrak{p}})$ as a subgroup of $F^\times$) as

$$\lambda_0 : P_1(\overline{\mathfrak{p}}) \longrightarrow F^\times \subset \overline{\mathbb{Q}}^\times$$

$$\lambda_0(\alpha) = \overline{\alpha}.$$

We may extend λ_0 to $I(\mathfrak{p})$ or $I(\overline{\mathfrak{p}})$ to define an algebraic Hecke character λ of infinity type $(1, 0)$ and modulus $\mathfrak{p}$ when $\alpha = \chi(\overline{\mathfrak{p}})$ (respectively infinity type $(0, 1)$ and modulus $\overline{\mathfrak{p}}$ when $\alpha = \chi(\mathfrak{p})$). (Remarks 7.3.5 of the appendix on Hecke characters explains how to extend λ_0 to λ.) After extending λ_0 to λ we may change λ by any character of $I(\mathfrak{p})/P_1(\mathfrak{p})$ (respectively $I(\overline{\mathfrak{p}})/P_1(\overline{\mathfrak{p}})$) and get another extension of λ_0. We impose a condition on the extension λ we choose. Recall that $\mathbb{C}_p^\times$ may be written as

$$\mathbb{C}_p^\times = p^{\mathbb{Q}} \times W \times U$$

where W is the group of roots of unity of order prime to p and

$$U = \{u \in \mathbb{C}_p^\times : |1 - u| < 1\}.$$

By construction, after composing with ι_p the image of λ_0 is contained in U. Since U is a divisible group, we may choose our extension λ so that the image of λ after composing with ι_p is also contained in U. We assume that we have done this. Since the only torsion in U is the p-power roots of unity, any two extensions λ and λ' of λ_0 that have image in U differ by a character of $I(\mathfrak{p})/P_1(\mathfrak{p})[p^\infty]$ (respectively $I(\overline{\mathfrak{p}})/P_1(\overline{\mathfrak{p}})[p^\infty]$) where the $[p^\infty]$ denotes the maximal quotient of $I(\mathfrak{p})/P_1(\mathfrak{p})$ (respectively $I(\overline{\mathfrak{p}})/P_1(\overline{\mathfrak{p}})$) with p-power order. This quotient is isomorphic to the subgroup of p-power torsion.

Let $p^n = |I(\mathfrak{p})/P_1(\mathfrak{p})[p^\infty]|$ (respectively $|I(\overline{\mathfrak{p}})/P_1(\overline{\mathfrak{p}})[p^\infty]|$). If $p^r \leq p^n$, then we shrink W so that $W = W(w, \frac{1}{p^{n+1}})$. We may do this without changing anything we have assumed previously, and the reason for doing this will become clear momentarily.

Now we begin defining elements of $R_{\mathbb{C}_p}$ that are related to the Hecke operators. Let $M = |I(\mathfrak{p})/P_1(\mathfrak{p})|$ (respectively $|I(\overline{\mathfrak{p}})/P_1(\overline{\mathfrak{p}})|$) and note that $|M| = 1/p^n$. For each prime $\mathfrak{q}$ of F such that $\mathfrak{q} \neq \mathfrak{p}$ if $\alpha = \chi(\overline{\mathfrak{p}})$ (respectively $\mathfrak{q} \neq \overline{\mathfrak{p}}$ if $\alpha = \chi(\mathfrak{p})$) define the power series

$$G_{\mathfrak{q}}(z) = \exp_p(z \log_p(\lambda(\mathfrak{q}))) = \sum_{n=0}^{\infty} \frac{z^n \log_p(\lambda(\mathfrak{q}))^n}{n!}$$

as an element of $\mathbb{C}_p[[z]]$. The power series $G_{\mathfrak{q}}(z)$ converges if

$$|z| < \frac{1}{p^{1/(p-1)} |\log_p(\lambda(\mathfrak{q}))|}.$$

We bound $|\log_p(\lambda(\mathfrak{q}))|$ independent of $\mathfrak{q}$. We have that $\mathfrak{q}^M = (q)$ for some $q \in \mathscr{O}_F$ such that $q \equiv 1 \mod \mathfrak{p}$ (respectively $1 \mod \overline{\mathfrak{p}}$). Hence by definition of λ_0

$$\lambda(\mathfrak{q})^M = \lambda((q)) \equiv 1 \bmod \mathfrak{p}$$

so $|1-\lambda(\mathfrak{q})^M| < p^{-1/(p-1)}$. Then by properties of the p-adic logarithm we get the string of inequalities

$$\frac{1}{p^{1/(p-1)}} > |1-\lambda(\mathfrak{q})^M| = |\log_p(\lambda(\mathfrak{q})^M)| = |M||\log_p(\lambda(\mathfrak{q}))| = \frac{|\log_p(\lambda(\mathfrak{q}))|}{p^n}$$

so

$$\frac{1}{p^n} < \frac{1}{p^{1/(p-1)}|\log_p(\lambda(\mathfrak{q}))|}.$$

Therefore $G_{\mathfrak{q}}(z)$ converges for $|z| \leq \frac{1}{p^n}$ which is independent of $\mathfrak{q}$.

Now recall that

$$\log_\gamma(z) := \frac{\log_p(z)}{\log_p(\gamma)}$$

and define

$$F_{\mathfrak{q}}(z) = G_{\mathfrak{q}} \circ \log_\gamma(1+\gamma z).$$

We claim that if $|z| \leq \frac{1}{p^{n+1}}$ the $F_{\mathfrak{q}}(z)$ converges. Indeed, assume $|z| \leq \frac{1}{p^{n+1}}$. Then

$$|\log_\gamma(1+\gamma z)| = \frac{|\log_p(1+\gamma z)|}{|\log_p(\gamma)|} = \frac{|\gamma z|}{|p|} = \frac{|z|}{|p|} \leq \frac{1}{p^n}$$

so $\log_\gamma(1+\gamma z)$ is in the radius of convergence for $G_{\mathfrak{q}}(z)$. Because we made the change of W so that $W \subset W(w, 1/p^{n+1})$ we then have that $F_{\mathfrak{q}}(z) \in R_{\mathbb{C}_p}$ since $F_{\mathfrak{q}}(z) \in \mathbb{C}_p[[z]]$ and $F_{\mathfrak{q}}(z)$ converges for all $z \in \mathbb{C}_p$ with $|z| \leq \frac{1}{p^{n+1}}$. The function $F_{\mathfrak{q}}(z)$ is the unique element of $R_{\mathbb{C}_p}$ with the property that for all $k \in \mathbb{Z}$ such that $\nu_k \in W$,

$$\begin{aligned} ev_{\nu_k}(F_{\mathfrak{q}}(z)) &= F_{\mathfrak{q}}(\gamma^{k-2} - \gamma^{-1}) \\ &= G_{\mathfrak{q}}(\log_\gamma(\gamma^{k-1})) \\ &= G_{\mathfrak{q}}(k-1) \\ &= \exp_p((k-1)\log_p(\lambda(\mathfrak{q}))) \\ &= (\lambda(\mathfrak{q}))^{k-1}. \end{aligned}$$

Furthermore, since $k \in \mathbb{Z}$ is such that $\nu_k \in W$ if and only if $k \equiv 1 \mod p^{r-1}(p-1)$ and $r > n$, $F_{\mathfrak{q}}(z)$ does not depend on the choice of extension λ of λ_0 since p^n divides $k-1$ so the exponent $k-1$ will kill any character of $I(\mathfrak{p})/P_1(\mathfrak{p})[p^\infty]$ (respectively $I(\overline{\mathfrak{p}})/P_1(\overline{\mathfrak{p}})[p^\infty]$).

Now let $\mathfrak{a} \subset \mathscr{O}_F$ be a nontrivial ideal of $\mathscr{O}_F$ such that $(\mathfrak{a}, \mathfrak{p}) = 1$ if $\alpha = \chi(\overline{\mathfrak{p}})$ (respectively $(\mathfrak{a}, \overline{\mathfrak{p}}) = 1$ if $\alpha = \chi(\mathfrak{p})$), and define

$$F_{\mathfrak{a}}(z) = \begin{cases} \prod_{\mathfrak{q}} F_{\mathfrak{q}}(z)^{val_{\mathfrak{q}}(\mathfrak{a})} & \text{if } (\mathfrak{a}, \mathfrak{p}) = 1 \text{ (respectively } (\mathfrak{a}, \overline{\mathfrak{p}}) = 1) \\ 0 & \text{else.} \end{cases}$$

Further, define $A_1(z) = 1$ and for $n \geq 2$ define

$$A_n(z) = \sum_{\substack{\mathfrak{a} \subset \mathscr{O}_F \\ N_{F/\mathbb{Q}}\mathfrak{a} = n}} \chi(\mathfrak{a}) F_{\mathfrak{a}}(z).$$

Define the formal q-expansion

$$\mathcal{F} = \sum_{n=1}^{\infty} A_n(z) q^n \in R_{\mathbb{C}_p}[[q]].$$

This formal q-expansion is the CM Hida family specializing to f_α in weight one.

Proposition 4.4.1. *For all $k \in \mathbb{Z}_{\geq 1}$, $\nu_k \in W$*

$$\mathcal{F}_k := \sum_{n=1}^{\infty} ev_{\nu_k}(A_n(z))q^n = \sum_{\mathfrak{a} \subset \mathscr{O}_F} \chi\lambda^{k-1}(\mathfrak{a})q^{N\mathfrak{a}}$$

is the q-expansion of a weight-k cusp form with level Γ_0 and character ε that is new at level N.

Proof. By definition of $A_n(z)$ we have that

$$\sum_{n=1}^{\infty} ev_{\nu_k}(A_n(z))q^n = \sum_{\mathfrak{a} \subset \mathscr{O}_F} \chi\lambda^{k-1}(\mathfrak{a})q^{N\mathfrak{a}}.$$

Shimura ([25]) showed that

$$\sum_{\mathfrak{a} \subset \mathscr{O}_F} \chi\lambda^{k-1}(\mathfrak{a})q^{N\mathfrak{a}}$$

is the q-expansion of a weight-k cusp form of level Γ_0 which is new at level N and has character defined by

$$\ell \longmapsto \frac{\chi((\ell))\lambda^{k-1}((\ell))}{\ell^{k-1}} = \chi((\ell)) \left(\frac{\lambda((\ell))}{\ell}\right)^{k-1}$$

for $\ell \in (\mathbb{Z}/N\mathbb{Z})^{\times}$ a prime not equal to p. We just need to show that this character is the character ε. To do this let $I_{\mathbb{Q}}(p)$ be the group of fractional ideals of $\mathbb{Q}$ prime to p, and we note that the function

$$\lambda : I_{\mathbb{Q}}(p) \longrightarrow \overline{\mathbb{Q}}^{\times}$$

$$\lambda((a)) = \lambda(a\mathscr{O}_F)$$

is a Hecke character of $\mathbb{Q}$ of infinity type 1 (where 1 has the meaning of the power of the embedding of $\mathbb{Q}$ into $\overline{\mathbb{Q}}$). The norm character

$$N : I_{\mathbb{Q}} \longrightarrow \mathbb{Q}^{\times}$$

$$N((a)) = |a|$$

also has infinity type 1. Therefore the quotient λ/N factors through $I_{\mathbb{Q}}(p)/P_{\mathbb{Q},1}(p) \cong (\mathbb{Z}/p\mathbb{Z})^{\times}$. Hence for primes $\ell \neq p$,

$$\frac{\lambda}{N}((\ell)) = \frac{\lambda((\ell))}{\ell}$$

is a $p-1$st root of unity. Now for $k \in \mathbb{Z}$ such that $\nu_k \in W$ we have that $k \equiv 1 \mod p-1$ so

$$\left(\frac{\lambda((\ell))}{\ell}\right)^{k-1} = 1.$$

Hence the character of

$$\sum_{\mathfrak{a} \subset \mathscr{O}_F} \chi\lambda^{k-1}(\mathfrak{a})q^{N\mathfrak{a}}$$

is the character defined for $\ell \in (\mathbb{Z}/N\mathbb{Z})^{\times}$ prime $\ell \neq p$

$$\ell \mapsto \chi((\ell)).$$

This is the character ε. $\square$

By the proposition, the functions $A_\ell(z) \in R_{\mathbb{C}_p}$ for $\ell \nmid Np$ and $A_p(z) \in R_{\mathbb{C}_p}$ satisfy the two properties that uniquely determine $T_\ell, U_p \in R_{\mathbb{C}_p}$. Hence $T_\ell = A_\ell$ for $\ell \nmid Np$ and $U_p = A_p$.

4.5 Two-variable p-adic L-function of the CM family

In this section we define and state the interpolation property of the two-variable p-adic L-function associated to $\mathcal{F}$.

Keeping the notation of the previous section, let $\Phi^{\pm}$ be generators for the rank one $T_{\mathbb{C}_p}$-module

$$\mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o \otimes_{\mathbb{T}^{\pm}_W} T_{\mathbb{C}_p} \subset \mathrm{Symb}^{\pm}_{\Gamma_0}(\mathbb{D}(R))^o.$$

The two-variable p-adic L-functions

$$\Lambda(\Phi^{\pm}, \cdot, \cdot) : W \times \mathcal{W} \longrightarrow \mathbb{C}_p$$

are defined using the second construction of section 2.7. We assume we have chosen periods $\Omega^{\pm}_{p,\kappa}$ and $\Omega^{\pm}_{\infty,\kappa}$ for $\kappa \in W$ for $\Lambda(\Phi^{\pm}, \cdot, \cdot)$. More will be said about these periods later. In order to prove conjecture 3.2.2 we restrict $\Lambda(\Phi^{\pm}, \cdot, \cdot)$ to a particular subset of $W \times \mathcal{W}$. Let $\psi = \eta\omega$ where η is a p-power order character of conductor $p^{m'}$. Let p^m be the conductor of ψ, so $m = m'$ if η is nontrivial and $m = 1$ if η is trivial. Let

$$U = \{t \in \mathbb{Z}_p : t \equiv 1 \bmod p^{r-1}\}.$$

Define the two-variable p-adic L-function

$$L_p(\chi\eta\omega, \alpha, \cdot, \cdot) : U \times \mathbb{Z}_p$$

$$L_p(\chi\eta\omega, \alpha, t, s) = \Lambda(\Phi^{-}, \omega^{-1}\langle\cdot\rangle^{t-2}, (\eta\omega)^{-1}\langle\cdot\rangle^{s-1})$$

so $L_p(\chi\eta\omega, \alpha, t, s)$ is a two-variable extension of the numerator of the function

$$L_p(\chi, \alpha, \psi\omega, \eta\omega, s)$$

that our conjecture is about.

The function $L_p(\chi\eta\omega, \alpha, t, s)$ is a two-variable p-adic analytic function of t and s. It is determined by the interpolation property (which comes from the interpolation property of $\Lambda(\Phi^-,\cdot,\cdot))$ that for all $k \in \mathbb{Z}_{\geq 2}$, $k \equiv 1 \bmod p^{r-1}$, and $j \in \mathbb{Z}$, $1 \leq j \leq k-1$, $j \equiv 1 \bmod 2(p-1)$

$$\frac{L_p(\chi\eta\omega,\alpha,k,j)}{\Omega^-_{p,\nu_k}} := \frac{\Lambda(\Phi^-,\nu_k,(\eta\omega)^{-1}\langle\cdot\rangle^{j-1})}{\Omega^-_{p,\nu_k}}$$

$$= E_p(\alpha,\eta\omega,k,j)\frac{L(\mathcal{F}_k,\eta\omega,j)}{\Omega^-_{\infty,\nu_k}}$$

$$= E_p(\alpha,\eta\omega,k,j)\frac{L(\chi\lambda^{k-1}\eta\omega,j)}{\Omega^-_{\infty,\nu_k}}$$

where

$$E_p(\alpha,\eta\omega,k,j) = \begin{cases} \dfrac{1}{\chi\lambda^{k-1}(\overline{\mathfrak{p}})^m}\left(1-\dfrac{(\eta\omega)^{-1}(p)p^{j-1}}{\chi\lambda^{k-1}(\overline{\mathfrak{p}})}\right)\times \\ \quad \times\dfrac{p^{m(j-1)}(j-1)!\tau((\eta\omega)^{-1})}{(2\pi)^{j-1}} & \text{if } \alpha = \chi(\overline{\mathfrak{p}}) \\ \dfrac{1}{\chi\lambda^{k-1}(\mathfrak{p})^m}\left(1-\dfrac{(\eta\omega)^{-1}(p)p^{j-1}}{\chi\lambda^{k-1}(\mathfrak{p})}\right)\times \\ \quad \times\dfrac{p^{m(j-1)}(j-1)!\tau((\eta\omega)^{-1})}{(2\pi)^{j-1}} & \text{if } \alpha = \chi(\mathfrak{p}). \end{cases}$$

These k and j are dense in $U\times\mathbb{Z}_p$ so this interpolation formula determines $L_p(\chi\eta\omega,\alpha,t,s)$ by continuity. To simplify notation, since the sign of the periods $\Omega^{\pm}_{p,\nu_k},\Omega^{\pm}_{\infty,\nu_k}$ is fixed as $-$ and since the periods are indexed by integers, for $k \in \mathbb{Z}_{\geq 2}$, $k \equiv 1 \bmod p^{r-1}$ we let $\Omega_{p,k} = \Omega^-_{p,\nu_k}$ and $\Omega_{\infty,k} = \Omega^-_{\infty,\nu_k}$.

4.6 Two-variable specialization of $L_{p,Katz}$

In this section we define a two-variable specialization of Katz's p-adic L-function that we compare to the two-variable p-adic L-function defined in the previous section.

Observe that the complex L-value appearing the interpolation formula in the previous section is

$$L(\chi\lambda^{k-1}\eta\omega, j) = L(\chi\lambda^{k-1}\eta\omega N^{-j}, 0).$$

By our choice of λ, the algebraic Hecke character $\chi\lambda^{k-1}\eta\omega N^{-j}$ has infinity type

$$(k-1-j, -j) \text{ if } \alpha = \chi(\overline{\mathfrak{p}})$$

$$(-j, k-1-j) \text{ if } \alpha = \chi(\mathfrak{p}).$$

If $\alpha = \chi(\mathfrak{p})$, then $\chi\lambda^{k-1}\eta\omega N^{-j}$ has infinity type in the range of interpolation for Katz's p-adic L-function. If $\alpha = \chi(\overline{\mathfrak{p}})$ we need to make a slight modification.

From here on, let c denote complex conjugation, so c is an automorphism of $\mathbb{C}$. Via our embedding ι_∞, c acts on ideals of F. We also have that c acts on $\mathbb{A}_F^\times$ and c acts on G_F via conjugation. These three actions are compatible with our conventions for the Artin map and our definitions of Hecke characters. There is a tautological relation of complex L-functions

$$L(\chi\lambda^{k-1}\eta\omega N^{-j}, s) = L(\chi\lambda^{k-1}\eta\omega N^{-j} \circ c, s)$$

that changes the infinity type. When $\alpha = \chi(\overline{\mathfrak{p}})$, $\chi\lambda^{k-1}\eta\omega N^{-j} \circ c$ has infinity type $(-j, k-1-j)$ which is in the range of interpolation of Katz's p-adic L-function. With these observations in hand we now specify a restriction of $L_{p,Katz}$.

Let κ_1 be the algebraic Hecke character

$$\kappa_1 = \begin{cases} \lambda \circ c & \text{if } \alpha = \chi(\overline{\mathfrak{p}}) \\ \lambda & \text{if } \alpha = \chi(\mathfrak{p}) \end{cases}$$

so by our choice of λ, κ_1 has infinity type $(0,1)$ and modulus $\overline{\mathfrak{p}}$. Further when we view κ_1 as a p-adic Hecke character, since λ takes values in $U = \{u \in \mathbb{C}_p^\times : |1-u| < 1\} \subset \mathbb{C}_p^\times$ we may consider the p-adic Hecke character $\kappa_1^{s_1}$ for any p-adic number $s_1 \in \mathbb{Z}_p$.

Let κ_2 be the algebraic Hecke character $\kappa_2 = \omega^{-1}N$. We then view κ_2 as a p-adic Hecke character and a Galois character. As a Galois character, κ_2 cuts our the cyclotomic $\mathbb{Z}_p$-extension of F. We say a few words about κ_2. If we consider the norm character N as a p-adic Hecke character and then as a Galois character the fixed field of the kernel of N is

$$F(\zeta_{p^\infty}) = \bigcup_{n=1}^{\infty} F(\zeta_{p^n})$$

and the map

$$N : \mathrm{Gal}(F(\zeta_{p^\infty})/F) \longrightarrow \mathbb{Z}_p^\times \subset \mathbb{C}_p^\times$$

is an isomorphism. That is, N when viewed as a p-adic Hecke character is the cyclotomic character. To get the cyclotomic $\mathbb{Z}_p$-extension of F we need to make $\mu_{p-1} \subset \mathbb{Z}_p^\times$ in the kernel of N, and multiplying by ω^{-1} does this. Since the image of κ_2 in $\mathbb{C}_p^\times$ is $1 + p\mathbb{Z}_p$, it makes sense to consider $\kappa_2^{s_2}$ as a p-adic Hecke character for any $s_2 \in \mathbb{Z}_p$.

Now define

$$\widetilde{\chi} = \begin{cases} \chi \circ c & \text{if } \alpha = \chi(\overline{\mathfrak{p}}) \\ \chi & \text{if } \alpha = \chi(\mathfrak{p}) \end{cases}$$

so $\widetilde{\chi}$ has conductor $\bar{\mathfrak{f}}$ if $\alpha = \chi(\bar{\mathfrak{p}})$ and conductor $\mathfrak{f}$ if $\alpha = \chi(\mathfrak{p})$.

Let $L_{p,Katz}$ be Katz's p-adic L-function with respect to the ideal $\mathfrak{m}$ where as in the notation of section 3.1, $\mathfrak{m}$ is the conductor of M/F. The ideal $\mathfrak{m}$ is divisible by all the primes that divide $\mathfrak{f}$ and $\bar{\mathfrak{f}}$. Let Ω_∞, Ω_p be the periods used to define $L_{p,Katz}$.

Define

$$L_{p,Katz}(\chi\eta, \alpha, \cdot, \cdot) : U \times \mathbb{Z}_p \longrightarrow \mathbb{C}_p$$

$$L_{p,Katz}(\chi\eta, \alpha, s_1, s_2) := L_{p,Katz}(\widetilde{\chi}\eta\kappa_1^{s_1-1}\kappa_2^{-s_2}).$$

Proposition 4.6.1. *$L_{p,Katz}(\chi\eta, \alpha, s_1, s_2)$ is determined by the following interpolation property: for all $k \in \mathbb{Z}_{\geq 2}, k \equiv 1 \bmod p^{r-1}$, $j \in \mathbb{Z}$, $1 \leq j \leq k-1$, $j \equiv 1 \bmod p-1$,*

$$\frac{L_{p,Katz}(\chi\eta, \alpha, k, j)}{\Omega_p^{k-1}} = E_p(\alpha, \eta\omega, k, j) \frac{-(2\pi)^{k-2}}{\sqrt{d_F}^{k-1-j}} \frac{L(\chi\lambda^{k-1}\psi\omega^{j-1}, j)}{\Omega_\infty^{k-1}}$$

where $E_p(\alpha, \eta\omega, k, j)$ is defined as in the previous section.

Proof. That $L_{p,Katz}(\chi\eta, \alpha, s_1, s_2)$ is determined by the interpolation property follows from the continuity of $L_{p,Katz}(\chi\eta, \alpha, s_1, s_2)$ and that the set of k's and j's is dense in $U \times \mathbb{Z}_p$. Let $k \in \mathbb{Z}_{\geq 2}$, $k \equiv 1 \bmod p^{r-1}$ and $j \in \mathbb{Z}$, $1 \leq j \leq k-1$, $j \equiv 1 \bmod p-1$. The first thing we need to observe is that the character plugged into $L_{p,Katz}(\cdot)$ is

$$\widetilde{\chi}\eta\kappa_1^{k-1}\kappa_2^{-j} = \begin{cases} \chi\eta\omega\lambda^{k-1}N^{-j} \circ c & \text{if } \alpha = \chi(\bar{\mathfrak{p}}) \\ \chi\eta\omega\lambda^{k-1}N^{-j} & \text{if } \alpha = \chi(\mathfrak{p}) \end{cases} \tag{4.4}$$

which has infinity type $(-j, k-1-j)$ so is in the range of interpolation for $L_{p,Katz}$. We do the two cases of $\alpha = \chi(\bar{\mathfrak{p}})$ and $\alpha = \chi(\mathfrak{p})$ separately.

Assume $\alpha = \chi(\overline{\mathfrak{p}})$. By the interpolation formula for $L_{p,Katz}$, we have (using that $c(\mathfrak{p}) = \overline{\mathfrak{p}}, c(\overline{\mathfrak{p}}) = \mathfrak{p}$, and $L(\chi' \circ c, 0) = L(\chi', 0)$ for any χ')

$$\frac{L_{p,Katz}(\chi\eta, \alpha, k, j)}{\Omega_p^{k-1}} = \frac{L_{p,Katz}(\chi\eta\omega\lambda^{k-1}N^{-j} \circ c)}{\Omega_p^{k-1}}$$

$$= \frac{(j-1)!(2\pi)^{k-1-j}}{\sqrt{d_F}^{k-1-j}} W_p(\chi\eta\omega\lambda^{k-1}N^{-j} \circ c)\times$$

$$\times \left(1 - \frac{(\chi\eta\omega)^{-1}\lambda^{1-k}N^j(\overline{\mathfrak{p}})}{p}\right)(1 - \chi\eta\omega\lambda^{k-1}N^{-j}(\mathfrak{p}))\times$$

$$\times \frac{L(\chi\eta\omega\lambda^{k-1}N^{-j}, 0)}{\Omega_\infty^{k-1}}$$

To simplify this we first observe that since λ has modulus $\mathfrak{p}$, $1 - \chi\eta\omega\lambda^{k-1}N^{-j}(\mathfrak{p}) = 1$. We also have that $(\eta\omega)^{-1}(\overline{\mathfrak{p}}) = (\eta\omega)^{-1}(p)$, $N^j(\overline{\mathfrak{p}}) = p^j$, and by the following lemma,

$$W_p(\chi\eta\omega\lambda^{k-1}N^{-j} \circ c) = \frac{-p^{m(j-1)}\tau((\eta\omega)^{-1})}{\chi\lambda^{k-1}(\overline{\mathfrak{p}})^m}.$$

Therefore the formula becomes

$$\frac{L_{p,Katz}(\chi\eta, \alpha, k, j)}{\Omega_p^{k-1}} = \frac{(j-1)!(2\pi)^{k-1-j}}{\sqrt{d_F}^{k-1-j}} \frac{-p^{m(j-1)}\tau((\eta\omega)^{-1})}{\chi\lambda^{k-1}(\mathfrak{p})^m}\times$$

$$\times \left(1 - \frac{(\eta\omega)^{-1}(p)p^{j-1}}{\chi\lambda^{k-1}(\overline{\mathfrak{p}})^m}\right) \frac{L(\chi\lambda^{k-1}\eta\omega, j)}{\Omega_\infty^{k-1}}$$

$$= E_p(\alpha, \eta\omega, k, j) \frac{-(2\pi)^{k-2}}{\sqrt{d_F}^{k-1-j}} \frac{L(\chi\lambda^{k-1}\psi\omega^{j-1}, j)}{\Omega_\infty^{k-1}}$$

When $\alpha = \chi(\mathfrak{p})$ we have

$$\frac{L_{p,Katz}(\chi\eta, \alpha, k, j)}{\Omega_p^{k-1}} = \frac{L_{p,Katz}(\chi\eta\omega\lambda^{k-1}N^{-j})}{\Omega_p^{k-1}}$$

$$= \frac{(j-1)!(2\pi)^{k-1-j}}{\sqrt{d_F}^{k-1-j}} W_p(\chi\eta\omega\lambda^{k-1}N^{-j})\times$$

$$\times\left(1 - \frac{(\chi\eta\omega)^{-1}\lambda^{1-k}N^j(\mathfrak{p})}{p}\right)(1 - \chi\eta\omega\lambda^{k-1}N^{-j}(\overline{\mathfrak{p}}))\times$$

$$\times\frac{L(\chi\eta\omega\lambda^{k-1}N^{-j}, 0)}{\Omega_\infty^{k-1}}$$

Similar to the other case, since λ modulus $\overline{\mathfrak{p}}$, $1 - \chi\eta\omega\lambda^{k-1}N^{-j}(\overline{\mathfrak{p}}) = 1$. We also have that $(\eta\omega)^{-1}(\mathfrak{p}) = (\eta\omega)^{-1}(p)$, $N^j(\mathfrak{p}) = p^j$, and by the following lemma

$$W_p(\chi\eta\omega\lambda^{k-1}N^{-j}) = \frac{-p^{m(j-1)}\tau((\eta\omega)^{-1})}{\chi\lambda^{k-1}(\mathfrak{p})^m}.$$

Therefore the formula becomes

$$\frac{L_{p,Katz}(\chi\eta, \alpha, k, j)}{\Omega_p^{k-1}} = \frac{(j-1)!(2\pi)^{k-1-j}}{\sqrt{d_F}^{k-1-j}} \frac{-p^{m(j-1)}\tau((\eta\omega)^{-1})}{\chi\lambda^{k-1}(\overline{\mathfrak{p}})^m}\times$$

$$\times\left(1 - \frac{(\eta\omega)^{-1}(p)p^{j-1}}{\chi\lambda^{k-1}(\mathfrak{p})^m}\right)\frac{L(\chi\lambda^{k-1}\eta\omega, j)}{\Omega_\infty^{k-1}}$$

$$= E_p(\alpha, \eta\omega, k, j)\frac{-(2\pi)^{k-2}}{\sqrt{d_F}^{k-1-j}}\frac{L(\chi\lambda^{k-1}\psi\omega^{j-1}, j)}{\Omega_\infty^{k-1}}$$

□

Lemma 4.6.2. *When* $\alpha = \chi(\overline{\mathfrak{p}})$ *we have*

$$W_p(\chi\eta\omega\lambda^{k-1}N^{-j} \circ c) = \frac{-p^{m(j-1)}\tau((\eta\omega)^{-1})}{\chi\lambda^{k-1}(\overline{\mathfrak{p}})^m}.$$

When $\alpha = \chi(\mathfrak{p})$ *we have*

$$W_p(\chi\eta\omega\lambda^{k-1}N^{-j}) = \frac{-p^{m(j-1)}\tau((\eta\omega)^{-1})}{\chi\lambda^{k-1}(\mathfrak{p})^m}.$$

Proof. By definition for an algebraic Hecke character χ' we view χ' as a complex Hecke character and define

$$W_p(\chi') = \frac{\chi'_{\mathfrak{p}}(\pi_{\mathfrak{p}}^{a_{\mathfrak{p}}})}{p^{a_{\mathfrak{p}}}} \sum_{u \in (\mathscr{O}_{F_{\mathfrak{p}}}/\mathfrak{p}^{a_{\mathfrak{p}}})^\times} \chi'_{\mathfrak{p}}(u) \exp(-2\pi i Tr_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u/\pi_{\mathfrak{p}}^{a_{\mathfrak{p}}}))$$

where $a_{\mathfrak{p}}$ is the power of $\mathfrak{p}$ in the conductor of χ' and $\pi_{\mathfrak{p}}$ is a uniformizer for $F_{\mathfrak{p}}$. Assume $\alpha = \chi(\overline{\mathfrak{p}})$. Then of the characters we need to consider, $\chi \circ c, \eta\omega \circ c = \eta\omega, \lambda \circ c, \mathbb{N} \circ c = N$, only $\eta\omega$ is ramified at $\mathfrak{p}$ so $\eta\omega$ is only character that will contribute to the sum in the formula for $W_p(\chi')$. Since $\eta\omega$ has conductor p^m with $m \geq 1$, $a_{\mathfrak{p}} = m$. We calculated before Theorem 4.1.5 that the sum becomes $\eta\omega(-1)\tau((\eta\omega)^{-1}) = -\tau((\eta\omega)^{-1})$. For the constant in front of the sum we calculate for each of the characters separately. For $\eta\omega$, we have $\eta\omega_{\mathfrak{p}}(\pi_{\mathfrak{p}}) = 1$. Then for the rest of the characters we have

$$\chi\lambda^{k-1}N^{-j} \circ c(\pi_{\mathfrak{p}}) = \chi\lambda^{k-1}(\overline{\mathfrak{p}})p^{-j}.$$

Therefore putting it all together we get

$$W_p(\chi\eta\omega\lambda^{k-1}N^{-j} \circ c) = \frac{-p^{m(j-1)}\tau((\eta\omega)^{-1})}{\chi\lambda^{k-1}(\overline{\mathfrak{p}})^m}.$$

Now assume $\alpha = \chi(\mathfrak{p})$. The calculation is similar. The character $\eta\omega$ contributes $-\tau((\eta\omega)^{-1})$ and

$$\chi\lambda^{k-1}N^{-j} \circ (\pi_{\mathfrak{p}}) = \chi\lambda^{k-1}(\mathfrak{p})p^{-j}$$

so we get

$$W_p(\chi\eta\omega\lambda^{k-1}N^{-j}) = \frac{-p^{m(j-1)}\tau((\eta\omega)^{-1})}{\chi\lambda^{k-1}(\mathfrak{p})^m}$$

□

4.7 Choice of periods and comparison

Let $S_{\mathbb{C}_p}$ be the fraction field of $T_{\mathbb{C}_p} = R_{\mathbb{C}_p}$.

Proposition 4.7.1. *There exists $\Psi \in \mathrm{Symb}^-_{\Gamma_0}(\mathbb{D}(R)) \otimes_{\mathbb{T}^{\pm}_W} S_{\mathbb{C}_p}$ such that the p-adic L-function*

$$L_p(\chi\eta\omega, \alpha, t, s) := \Lambda(\Psi, \omega^{-1}\langle\cdot\rangle^{t-2}, (\eta\omega)^{-1}\langle\cdot\rangle^{s-1})$$

is calculated with the p-adic and complex periods

$$(\Omega_{p,k}, \Omega_{\infty,k}) = \left(\Omega_p^{k-1}, \Omega_\infty^{k-1}\left(\frac{\sqrt{d_F}}{2\pi}\right)^{k-2}\right)$$

where Ω_p, Ω_∞ are the periods used to define Katz's p-adic L-function. We note that the domain of $L_p(\chi\eta\omega, \alpha, t, s)$ is as in the previous section.

Proof. Let $L_p(\chi\eta\omega, \alpha, t, s) = \Lambda(\Phi^-, \omega^{-1}\langle\cdot\rangle^{t-2}, (\eta\omega)^{-1}\langle\cdot\rangle^{s-1})$ be as in section 4.5. We determine a meromorphic function $P(t)$ on U such that $P(t)L_p(\chi\eta\omega, \alpha, t, s)$ has interpolation formula with the periods

$$\left(\Omega_p^{k-1}, \Omega_\infty^{k-1}\left(\frac{\sqrt{d_F}}{2\pi}\right)^{k-2}\right).$$

Let

$$P : U \times \mathbb{Z}_p \longrightarrow \mathbb{C}_p \cup \{\infty\}$$

be the p-adic meromorphic function defined by the ratio

$$P(t,s) = \frac{L_{p,Katz}(\chi\eta,\alpha,t,s)}{L_p(\chi\eta\omega,\alpha,t,s)}.$$

Then $P(t,s)$ has the interpolation property:

$$\frac{P(k,j)\Omega_{p,k}}{\Omega_p^{k-1}} = \frac{\Omega_{\infty,k}}{\Omega_\infty^{k-1}}\frac{-(2\pi)^{k-2}}{\sqrt{d_F}^{k-1-j}}$$

for k's and j's as in the previous section.

When choosing the periods for $L_p(\chi\eta\omega,\alpha,t,s)$ one way to choose them is to choose the $\Omega_{\infty,k}$ and Φ, and then this determines the $\Omega_{p,k}$. The only condition on the choice of the $\Omega_{\infty,k}$ is that the values in the interpolation formula for the p-adic L-function of the modular form $\mathcal{F}_k$ are algebraic. These values are

$$C_{alg}(\alpha,k,j)\frac{L(\chi\lambda^{k-1}\psi\omega^{j-1},j)}{(2\pi)^{j-1}\Omega_{\infty,k}}$$

for all odd finite order characters $\psi \in \mathcal{W}(\mathbb{C}_p)$, $k \in \mathbb{Z}_{\geq 2}$, $1 \leq j \leq k-1$ where

$$C_{alg}(\alpha,k,j) = \begin{cases} \dfrac{p^{m(j-1)}(j-1)!\tau(\psi^{-1}\omega^{1-j})}{\chi\lambda^{k-1}(\bar{\mathfrak{p}})^m}\left(1-\dfrac{\psi^{-1}\omega^{1-j}(p)}{\chi\lambda^{k-1}(\bar{\mathfrak{p}})p^{1-j}}\right)\dfrac{1}{i^{j-1}} & \text{if } \alpha = \chi(\bar{\mathfrak{p}}) \\ \dfrac{p^{m(j-1)}(j-1)!\tau(\psi^{-1}\omega^{1-j})}{\chi\lambda^{k-1}(\mathfrak{p})^m}\left(1-\dfrac{\psi^{-1}\omega^{1-j}(p)}{\chi\lambda^{k-1}(\mathfrak{p})p^{1-j}}\right)\dfrac{1}{i^{j-1}} & \text{if } \alpha = \chi(\mathfrak{p}) \end{cases}$$

and m is the power of p in the conductor of ψ.

We claim that we may take

$$\Omega_{\infty,k} = \Omega_\infty^{k-1}\left(\frac{\sqrt{d_F}}{2\pi}\right)^{k-2}.$$

Indeed, then the values in question are

$$C_{alg}(\alpha,k,j)\frac{(2\pi)^{k-1-j}L(\chi\lambda^{k-1}\psi\omega^{j-1},j)}{\sqrt{d_F}^{k-2}\Omega_\infty^{k-1}}$$

which by the interpolation property for Katz's p-adic L-function are algebraic. Therefore we can and do make this choice for the $\Omega_{\infty,k}$.

If we consider $P(t,s)$ with this choice of complex periods, then $P(t,s)$ satisfies the interpolation formula for $k \in \mathbb{Z}_{\geq 2}$ $k \equiv 1 \bmod p^{r-1}$, $j \in \mathbb{Z}$, $1 \leq j \leq k-1$, $j \equiv 1 \bmod 2(p-1)$

$$\frac{P(k,j)\Omega_{p,k}}{\Omega_p^{k-1}} = -\sqrt{d_F}^{j-1}.$$

We separate variables for the function $P(t,s)$. Since p splits in F, $\sqrt{d_F} \in \mathbb{Q}_p = F_{\mathfrak{p}}$. Then define the analytic function $Q(s)$ as

$$Q(s) = -\langle\sqrt{d_F}\rangle^{s-1},$$

and let $P(t) = P(t,s)/Q(s)$. The function $P(t)$ is a p-adic meromorphic function on U satisfying the relation that for all $k \in \mathbb{Z}_{\geq 2}$, $k \equiv 1 \bmod p^{r-1}$,

$$P(k)\Omega_{p,k} = \Omega_p^{k-1}.$$

Since $P(t)$ is a p-adic meromorphic function on U, there exists an element $\widetilde{P} \in S_{\mathbb{C}_p}$ such that for all $t \in U$

$$\widetilde{P}(\gamma^{t-2} - \gamma^{-1}) = P(t).$$

If we define $\Psi = \widetilde{P}\Phi^-$ and redefine the function

$$L_p(\chi\eta\omega, \alpha, t, s) = \Lambda(\Psi, \omega^{-1}\langle\cdot\rangle^{t-2}, (\eta\omega)^{-1}\langle\cdot\rangle^{s-1})$$

then $L_p(\chi\eta\omega, \alpha, t, s)$ satisfies the interpolation property that for all k, j as above,

$$\frac{L_p(\chi\eta\omega, \alpha, t, s)}{\Omega_p^{k-1}} = E_p(\alpha, \eta\omega, k, j)\frac{(2\pi)^{k-2}L(\chi\lambda^{k-1}\psi\omega^{j-1}, j)}{\sqrt{d_F}^{k-2}\Omega_\infty^{k-1}}.$$

That is, $L_p(\chi\eta\omega, \alpha, t, s)$ is calculated with the periods $(\Omega_p, \Omega_\infty)$. $\square$

Remarks 4.7.2. If $P(t)$ in the proof of the previous proposition does not have any zeros or poles, then Ψ is a generator for the free rank one $T_{\mathbb{C}_p}$-module $\mathrm{Symb}_{\Gamma_0}(\mathbb{D}(R))^o \otimes_{\mathbb{T}_W^\pm} T_{\mathbb{C}_p}$ and so Ψ would be a valid choice to define the p-adic L-function as in section 2.7.

We record the precise comparison of the p-adic L-function defined in the previous two sections that appeared in the proof of the previous proposition.

Corollary 4.7.3. *Let $L_{p,Katz}(\chi\eta, \alpha, t, s)$ and $L_p(\chi\eta\omega, \alpha, t, s)$ be defined as in the previous two sections, so*

$$L_p(\chi\eta\omega, \alpha, t, s) = \Lambda(\Phi^-, \omega^{-1}\langle\cdot\rangle^{t-2}, (\eta\omega)^{-1}\langle\cdot\rangle^{s-1})$$

where Φ^- is a generator of $\mathrm{Symb}^-_{\Gamma_0}(\mathbb{D}(R))^o \otimes_{\mathbb{T}_W^\pm} T_{\mathbb{C}_p}$ *as a $T_{\mathbb{C}_p}$-module. Then*

$$L_{p,Katz}(\chi\eta, \alpha, t, s) = P(\eta, t, s) L_p(\chi\eta\omega, \alpha, t, s)$$

where $P(\eta, t, s)$ is a p-adic meromorphic function determined by the interpolation property that for all $k \in \mathbb{Z}_{\geq 2}$, $k \equiv 1 \bmod p^{r-1}$, $j \in \mathbb{Z}$, $1 \leq j \leq k-1$, $j \equiv 1 \bmod 2(p-1)$,

$$\frac{P(\eta, k, j)\Omega_{p,k}}{\Omega_p^{k-1}} = \frac{\Omega_{\infty,k}}{\Omega_\infty^{k-1}} \frac{-(2\pi)^{k-2}}{\sqrt{d_F}^{k-1-j}}.$$

Remarks 4.7.4. We remark that $P(\eta, t, s)$ a priori depends on η and α, but as is clear from the interpolation formula does not actually depend on η or α. The reason for putting η in the notation will become clear in the next section.

4.8 Proof of the conjecture in this case

In this section we prove Conjecture 3.2.2 for χ. To this end we begin by recalling some notation. Let $n \in \mathbb{Z}_{\geq 0}$ and let $\mathbb{Q}_n/\mathbb{Q}$ denote the nth layer of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Let $\Gamma_n = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. Let K be the fixed field of the kernel of χ and let M be the Galois closure of K over $\mathbb{Q}$. Let $\Delta = \mathrm{Gal}(M/\mathbb{Q})$, $H = \mathrm{Gal}(M/F)$, and $G = \mathrm{Gal}(K/F)$. We note that $c \in \Delta - H$, and we choose c as our choice of τ. Let $K_n = \mathbb{Q}_n K$, $G_n = \mathrm{Gal}(K_n/F) = G \times \Gamma_n$, $M_n = \mathbb{Q}_n M$, $H_n = \mathrm{Gal}(M_n/F) = H \times \Gamma_n$, $\overline{K}_n = c(K_n)$, and $\overline{G}_n = \mathrm{Gal}(\overline{K}_n/F)$. Let $u_n = u_{M_n} \in M_n^\times$ be the Stark unit for M_n/F.

Let $\eta, \psi \in \mathcal{W}$ be finite order characters of conductors p^{m+1} and p^{n+1} and orders p^m and p^n respectively. Let $L_p(\chi\eta\omega, \alpha, t, s)$ and $L_p(\chi\psi\omega, \alpha, t, s)$ be as defined two sections ago.

The function Conjecture 3.2.2 is about is $L_p(\chi, \alpha, \eta\omega, \psi\omega, s)$, and by construction

$$L_p(\chi, \alpha, \eta\omega, \psi\omega, s) = \frac{L_p(\chi\eta\omega, \alpha, 1, s)}{L_p(\chi\psi\omega, \alpha, 1, s)}.$$

Theorem 4.8.1. *Conjecture 3.2.2 holds for χ, η, and ψ. That is,*

$$L_p(\chi, \alpha, \eta\omega, \psi\omega, 0) = \frac{\frac{\tau(\eta^{-1})}{p^{m+1}}\left(1 - \frac{\eta^{-1}(p)}{\alpha p}\right)(1 - \beta\eta(p))}{\frac{\tau(\psi^{-1})}{p^{n+1}}\left(1 - \frac{\psi^{-1}(p)}{\alpha p}\right)(1 - \beta\psi(p))} \frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}}.$$

Proof. We have that

$$L_p(\chi, \alpha, \eta\omega, \psi\omega, s) = \frac{L_p(\chi\eta\omega, \alpha, 1, s)}{L_p(\chi\psi\omega, \alpha, 1, s)}$$

$$= \frac{P(\eta, 1, s) L_{p,Katz}(\chi\eta\omega, \alpha, 1, s)}{P(\psi, 1, s) L_{p,Katz}(\chi\psi\omega, \alpha, 1, s)}.$$

Then by Corollary 4.7.3 $P(\eta, 1, s) = P(\psi, 1, s)$, so

$$L_p(\chi, \alpha, \eta\omega, \psi\omega, s) = \frac{L_{p,Katz}(\chi\eta\omega, \alpha, 1, s)}{L_{p,Katz}(\chi\psi\omega, \alpha, 1, s)}.$$

By definition

$$L_p(\chi\alpha\eta\omega, \psi\omega, 0) = \frac{L_{p,Katz}(\chi\eta\omega, \alpha, 1, 0)}{L_{p,Katz}(\chi\psi\omega, \alpha, 1, 0)}$$

$$= \begin{cases} \dfrac{L_{p,Katz}(\chi\eta \circ c)}{L_{p,Katz}(\chi\psi \circ c)} & \text{if } \alpha = \chi(\overline{\mathfrak{p}}) \\ \dfrac{L_{p,Katz}(\chi\eta)}{L_{p,Katz}(\chi\psi)} & \text{if } \alpha = \chi(\mathfrak{p}). \end{cases}$$

We now use Katz's Kronecker's second limit formula. We consider the two cases of α separately.

Assume $\alpha = \chi(\overline{\mathfrak{p}})$. Then by Katz's Kronecker's second limit formula (Theorem 4.1.5),

$$\frac{L_{p,Katz}(\chi\eta \circ c)}{L_{p,Katz}(\chi\psi \circ c)} = \frac{\dfrac{\tau(\eta^{-1})}{p^{m+1}} \left(1 - \dfrac{(\chi\eta)^{-1}(\overline{\mathfrak{p}})}{p}\right)(1 - \chi\eta(\mathfrak{p})) \displaystyle\sum_{\sigma \in H_m} (\chi\eta)_c(\sigma) \log_p(\sigma(u_{M_m}))}{\dfrac{\tau(\psi^{-1})}{p^{n+1}} \left(1 - \dfrac{(\chi\psi)^{-1}(\overline{\mathfrak{p}})}{p}\right)(1 - \chi\psi(\mathfrak{p})) \displaystyle\sum_{\sigma \in H_n} (\chi\psi)_c(\sigma) \log_p(\sigma(u_{M_n}))}.$$

Using the formula from corollary 3.2.6 for $\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}$ and $\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}$ we have

$$\frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}} = \frac{\displaystyle\sum_{\sigma\in H_m} \chi_\tau\eta(\sigma)\log_p(\sigma(u_{M_m}))}{\displaystyle\sum_{\sigma\in H_n} \chi_\tau\psi(\sigma)\log_p(\sigma(u_{M_n}))}$$

In our case, c is the choice for τ, and since ψ and η are even Dirichlet character, we have $\psi_c = \psi$ and $\eta_c = \eta$. Therefore

$$\frac{\log_p |\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p |\pi^*_{\rho\psi}(u_n)|_{1/\alpha}} = \frac{\displaystyle\sum_{\sigma\in H_m} (\chi\eta)_c(\sigma)\log_p(\sigma(u_{M_m}))}{\displaystyle\sum_{\sigma\in H_n} (\chi\psi)_c(\sigma)\log_p(\sigma(u_{M_n}))}.$$

Hence

$$\frac{L_{p,Katz}(\chi\eta\circ c)}{L_{p,Katz}(\chi\psi\circ c)} = \frac{\dfrac{\tau(\eta^{-1})}{p^{m+1}}\left(1-\dfrac{(\chi\eta)^{-1}(\overline{\mathfrak{p}})}{p}\right)(1-\chi\eta(\mathfrak{p}))\log_p|\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\dfrac{\tau(\psi^{-1})}{p^{n+1}}\left(1-\dfrac{(\chi\psi)^{-1}(\overline{\mathfrak{p}})}{p}\right)(1-\chi\psi(\mathfrak{p}))\log_p|\pi^*_{\rho\psi}(u_n)|_{1/\alpha}}.$$

To finish, we just note that since $\alpha = \chi(\overline{\mathfrak{p}})$, $\beta = \chi(\mathfrak{p})$ so

$$(\chi\eta)^{-1}(\overline{\mathfrak{p}}) = \eta^{-1}(p)/\alpha \text{ and } (\chi\psi)^{-1}(\mathfrak{p}) = \psi^{-1}(p)/\alpha$$

as well as

$$\chi\eta(\mathfrak{p}) = \beta\eta(p) \text{ and } \chi\psi(\mathfrak{p}) = \beta\psi(p).$$

Therefore

$$L_p(\chi,\alpha,\eta\omega,\psi\omega,0) = \frac{L_{p,Katz}(\chi\eta\circ c)}{L_{p,Katz}(\chi\psi\circ c)}$$

$$= \frac{\frac{\tau(\eta^{-1})}{p^{m+1}}\left(1-\frac{\eta^{-1}(p)}{\alpha p}\right)(1-\beta\eta(p))}{\frac{\tau(\psi^{-1})}{p^{n+1}}\left(1-\frac{\psi^{-1}(p)}{\alpha p}\right)(1-\beta\psi(p))}\frac{\log_p|\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p|\pi^*_{\rho\psi}(u_n)|_{1/\alpha}}$$

Now we assume $\alpha = \chi(\mathfrak{p})$. Then by Katz's Kronecker's second limit formula (Theorem 4.1.5),

$$\frac{L_{p,Katz}(\chi\eta)}{L_{p,Katz}(\chi\psi)} = \frac{\frac{\tau(\eta^{-1})}{p^{m+1}}\left(1-\frac{(\chi\eta)^{-1}(\mathfrak{p})}{p}\right)(1-\chi\eta(\overline{\mathfrak{p}}))\sum_{\sigma\in H_m}\chi\eta(\sigma)\log_p(\sigma(u_{M_m}))}{\frac{\tau(\psi^{-1})}{p^{n+1}}\left(1-\frac{(\chi\psi)^{-1}(\mathfrak{p})}{p}\right)(1-\chi\psi(\overline{\mathfrak{p}}))\sum_{\sigma\in H_n}\chi\psi(\sigma)\log_p(\sigma(u_{M_n}))}.$$

Using the formula from corollary 3.2.6 for $\log_p|\pi^*_{\rho\eta}(u_m)|_{1/\alpha}$ and $\log_p|\pi^*_{\rho\psi}(u_n)|_{1/\alpha}$ we have

$$\frac{\log_p|\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p|\pi^*_{\rho\psi}(u_n)|_{1/\alpha}} = \frac{\sum_{\sigma\in H_m}\chi_\tau\eta(\sigma)\log_p(\sigma(u_{M_m}))}{\sum_{\sigma\in H_n}\chi_\tau\psi(\sigma)\log_p(\sigma(u_{M_n}))}.$$

Therefore

$$\frac{L_{p,Katz}(\chi\eta)}{L_{p,Katz}(\chi\psi)} = \frac{\frac{\tau(\eta^{-1})}{p^{m+1}}\left(1-\frac{(\chi\eta)^{-1}(\mathfrak{p})}{p}\right)(1-\chi\eta(\overline{\mathfrak{p}}))\log_p|\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\frac{\tau(\psi^{-1})}{p^{n+1}}\left(1-\frac{(\chi\psi)^{-1}(\mathfrak{p})}{p}\right)(1-\chi\psi(\overline{\mathfrak{p}}))\log_p|\pi^*_{\rho\psi}(u_n)|_{1/\alpha}}.$$

To finish, we just note that since $\alpha = \chi(\mathfrak{p})$, we have that $\beta = \chi(\overline{\mathfrak{p}})$, so

$$(\chi\eta)^{-1}(\mathfrak{p}) = \eta^{-1}(p)/\alpha \text{ and } (\chi\psi)^{-1}(\mathfrak{p}) = \psi^{-1}(p)/\alpha$$

as well as

$$\chi\eta(\overline{\mathfrak{p}}) = \beta\eta(p) \text{ and } \chi\psi(\overline{\mathfrak{p}}) = \beta\psi(p).$$

Hence

$$L_p(\chi,\alpha,\eta\omega,\psi\omega,0) = \frac{L_{p,Katz}(\chi\eta)}{L_{p,Katz}(\chi\psi)}$$

$$= \frac{\frac{\tau(\eta^{-1})}{p^{m+1}}\left(1-\frac{\eta^{-1}(p)}{\alpha p}\right)(1-\beta\eta(p))}{\frac{\tau(\psi^{-1})}{p^{n+1}}\left(1-\frac{\psi^{-1}(p)}{\alpha p}\right)(1-\beta\psi(p))}\frac{\log_p|\pi^*_{\rho\eta}(u_m)|_{1/\alpha}}{\log_p|\pi^*_{\rho\psi}(u_n)|_{1/\alpha}}$$

□

Chapter 5

Conjecture 3.2.7 when F is real quadratic, χ is a mixed signature character, and $\operatorname{Ind}\chi = \operatorname{Ind}\psi$ for ψ a character of an imaginary quadratic field in which p splits

Let F be a real quadratic field and let χ be a mixed signature character. In this section, we prove that if there exists an imaginary quadratic field K such that p splits in K and such that there exists a ray class character ψ of K such that $\operatorname{Ind}\chi \cong \operatorname{Ind}\chi'$, then conjectures 1.1.9 and 3.2.7 are true for χ.

We make some remarks about when an imaginary quadratic K and ψ exist for a given real quadratic F and χ. Let $\rho = \operatorname{Ind}\chi$, so $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}})$, and we consider

$$\overline{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}) \longrightarrow \mathrm{PGL}_2(\overline{\mathbb{Q}}).$$

Then there exists another quadratic field K and a ray class character ψ of K such that $\rho = \operatorname{Ind}\psi$ if and only if the image of $\overline{\rho}$ in $\mathrm{PGL}_2(\overline{\mathbb{Q}})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In fact, if the image of $\overline{\rho}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ then for each of the three subgroups of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of index 2 we get a quadratic field and a ray class character such that ρ is the induction of that ray class character. In this chapter we assume that starting with a mixed signature character χ of a real quadratic field F such that the projective image of $\operatorname{Ind}\chi$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ then one of the other quadratic fields is imaginary quadratic and has p split.

5.1 Notation

For the next two sections, fix χ a mixed signature character of a real quadratic F and ψ a ray class character of an imaginary quadratic field K such that p is split in K with $\operatorname{Ind}\chi = \operatorname{Ind}\psi$. We let $\rho = \operatorname{Ind}\chi = \operatorname{Ind}\psi$ and $\rho^* = \operatorname{Ind}\chi^{-1} = \operatorname{Ind}\psi^{-1}$. Further, let M_ρ, E_χ, and L_ψ be the fixed fields of the kernels of ρ, χ, and ψ respectively, and let $G_\rho = \mathrm{Gal}(M_\rho/\mathbb{Q})$, $H_\chi = \mathrm{Gal}(M_\rho/F)$, and $H_\psi = \mathrm{Gal}(M_\rho/K)$. Fix a $\tau \in G_\rho - (H_\chi \cup H_\psi)$.

Fix a character $\eta \in \mathcal{W}$ of order p^m, and let $\chi_1 = \chi\eta$, $\chi_2 = \psi\eta$. Let M, E, and L be the fixed fields of the kernels of $\rho\eta, \chi_1$, and χ_2 respectively, and let

$$G = \mathrm{Gal}(M/\mathbb{Q}) = G_\rho \times \Gamma_m$$

$$H_1 = \mathrm{Gal}(M/F) = H_\chi \times \Gamma_m$$

$$H_2 = \mathrm{Gal}(M/K) = H_\psi \times \Gamma_m.$$

We view χ_1 and χ_2 as characters of the groups H_1 and H_2 respectively. We have the relations

$$\mathrm{Ind}\,\chi_2 = \mathrm{Ind}\,\psi \otimes \eta = \mathrm{Ind}\,\chi \otimes \eta = \mathrm{Ind}\,\chi_1.$$

Note that since H_1 and H_2 are not equal and have index 2 in G,

$$|G| = |H_1 H_2| = \frac{|H_1||H_2|}{|H_1 \cap H_2|}$$

so the index of $H_1 \cap H_2$ in G is 4. Since $\tau \in G_\rho$ we may view $\tau \in G$. Since $\tau \notin H_\chi \cup H_\psi$, $\tau \notin H_1 \cup H_2$. Then

$$\mathrm{Ind}\,\chi_1|_{H_1} = \chi_1 + \chi_{1,\tau} \text{ and } \mathrm{Ind}\,\chi_2|_{H_2} = \chi_2 + \chi_{2,\tau}.$$

This implies that

$$\chi_1|_{H_1 \cap H_2} = \chi_{1,\tau}|_{H_1 \cap H_2} = \chi_2|_{H_1 \cap H_2} = \chi_{2,\tau}|_{H_1 \cap H_2}.$$

We will use this fact many times.

Let w be the infinite place of M induced by ι_∞, and note that we are assuming that $w(E) \subset \mathbb{R}$. Let $\{1, \delta_w\}$ be the decomposition group of w in G. Note that $\{1, \delta_w\} \subset G_\rho$, so δ_w does not depend on η. Let D_p be the decomposition group of p in G_ρ induced by ι_p and let δ_p be the arithmetic Frobenius. We view D_p as a subgroup of G.

Let k be a finite extension of $\mathbb{Q}$ containing the values of χ_1 and χ_2.

5.2 Archimedean Stark conjecture

We note that in this section we do not need to assume that p is split in K. We just need to assume that $\operatorname{Ind}\chi = \operatorname{Ind}\psi$.

Let $u_M \in M^\times$ be the Stark unit for M/K, and let $u_E = N_{M/E}u_M$. We consider the rank one abelian Stark conjecture for the extension E/F, and prove a weaker version of Conjecture 1.1.1 with u_E. We will prove Conjecture 1.1.9 for χ_1. We begin with a lemma.

Lemma 5.2.1. *The representations* $\operatorname{Ind}\chi_1^a$ *and* $\operatorname{Ind}\chi_2^a$ *are isomorphic if and only if a is odd.*

Proof. To check whether or not $\operatorname{Ind}\chi_1^a = \operatorname{Ind}\chi_2^a$ it suffices to check that for all $\sigma \in G$,

$$Tr(\operatorname{Ind}\chi_1^a(\sigma)) = Tr(\operatorname{Ind}\chi_2^a(\sigma)).$$

The left coset decomposition of $H_1 \cap H_2$ in G is

$$G = H_1 \cap H_2 \cup (H_1 - (H_1 \cap H_2)) \cup (H_2 - (H_1 \cap H_2)) \cup G - (H_1 \cup H_2)$$

and we check the value of $Tr(\operatorname{Ind}\chi_1^a)$ and $Tr(\operatorname{Ind}\chi_2^a)$ on each coset of $H_1 \cap H_2$ in G separately. For $\sigma \in H_1 \cap H_2$, $\chi_1(\sigma) = \chi_2(\sigma)$, so $Tr(\operatorname{Ind}\chi_1^a(\sigma)) = Tr(\operatorname{Ind}\chi_2^a(\sigma))$. For $\sigma \in G - (H_1 \cup H_2)$, $Tr(\operatorname{Ind}\chi_1^a(\sigma)) = 0$ and $Tr(\operatorname{Ind}\chi_2^a(\sigma)) = 0$ since $\sigma \notin H_1$ and $\sigma \notin H_2$.

For the other two cosets, we need to consider $\operatorname{Ind}\chi_1$ and $\operatorname{Ind}\chi_2$. Since $\operatorname{Ind}\chi_1 = \operatorname{Ind}\chi_2$, for all $\sigma \in G$,

$$Tr(\operatorname{Ind}\chi_1(\sigma)) = Tr(\operatorname{Ind}\chi_2(\sigma)).$$

This gives the following relations: for $\sigma \in H_1 - H_1 \cap H_2$,

$$0 = Tr(\text{Ind}\,\chi_2(\sigma)) = Tr(\text{Ind}\,\chi_1(\sigma)) = \chi_1(\sigma) + \chi_{1,\tau}(\sigma)$$

so $\chi_1(\sigma) = -\chi_{1,\tau}(\sigma)$, and for $\sigma \in H_2 - H_1 \cap H_2$,

$$0 = Tr(\text{Ind}\,\chi_1(\sigma)) = Tr(\text{Ind}\,\chi_2(\sigma)) = \chi_2(\sigma) + \chi_{2,\tau}(\sigma)$$

so $\chi_2(\sigma) = -\chi_{2,\tau}(\sigma)$. Therefore for $\sigma \in H_1 - H_1 \cap H_2$,

$$Tr(\text{Ind}\,\chi_2^a(\sigma)) = 0$$

and

$$Tr(\text{Ind}\,\chi_1^a(\sigma)) = \chi_1^a(\sigma) + (-1)^a \chi_1^a(\sigma).$$

These are equal if and only if a is odd. Similarly, if $\sigma \in H_2 - H_1 \cap H_2$,

$$Tr(\text{Ind}\,\chi_1^a(\sigma)) = 0$$

and

$$Tr(\text{Ind}\,\chi_2^a(\sigma)) = \chi_2^a(\sigma) + (-1)^a \chi_2^a(\sigma)$$

which are equal if and only if a is odd. □

Remarks 5.2.2. The lemma is true for any two characters χ_1 and χ_2 of index two subgroups H_1 and H_2 of G such that $H_1 \neq H_2$ and $\text{Ind}\,\chi_1 = \text{Ind}\,\chi_2$. In our case, we can see that if a is even $\text{Ind}\,\chi_1^a \neq \text{Ind}\,\chi_2^a$ since χ_1^a is a totally even character of G_F, so $\text{Ind}\,\chi_1^a$ is an even representation of $G_{\mathbb{Q}}$ while $\text{Ind}\,\chi_2^a$ is an odd representation of $G_{\mathbb{Q}}$.

Proposition 5.2.3. *Let $\widetilde{\chi}$ be a mixed signature character of Gal(E/F). Then*

$$-\frac{1}{2e_M}\sum_{\sigma\in Gal(E/F)}\widetilde{\chi}(\sigma)\log|\sigma(u_E)|_w = L'(\widetilde{\chi},0)$$

where e_M is the number of roots of unity in M.

Proof. Since $\widetilde{\chi}$ is mixed signature and E is the fixed field of the kernel of χ_1, $\widetilde{\chi}=\chi_1^a$ for some odd integer a. Then by the Lemma 5.2.1, $\operatorname{Ind}\chi_1^a=\operatorname{Ind}\chi_2^a$, so

$$\sum_{\sigma\in G}Tr(\operatorname{Ind}\chi_1^a(\sigma))\log|\sigma(u_M)|_w=\sum_{\sigma\in G}Tr(\operatorname{Ind}\chi_2^a(\sigma))\log|\sigma(u_M)|_w.$$

Then since u_M is the Stark unit for M/K, we have

$$\sum_{\sigma\in G}Tr(\operatorname{Ind}\chi_2^a(\sigma))=\sum_{\sigma\in H_2}\chi_2^a(\sigma)\log|\sigma(u_M)|_w+\chi_{2,\tau}^a(\sigma)\log|\sigma(u_M)|_w$$

$$=-e_ML'(\chi_2^a,0)-e_ML'(\chi_{2,\tau}^a,0)$$

$$=-2e_ML'(\chi_2^a,0)$$

$$=-2e_ML'(\widetilde{\chi},0).$$

Let $\{1,\delta_w\}$ be the decomposition group of w in G. Since χ_1^a is a mixed signature character of Gal(E/F) we have that $\chi_1^a(\delta_w)=1$ and $\chi_{1,\tau}^a(\delta_w)=-1$. Then

$$\sum_{\sigma\in H_1}\chi_{1,\tau}^a(\sigma)\log|\sigma(u_M)|_w=\sum_{\sigma\in\{1,\delta_w\}\backslash H_1}(\chi_{1,\tau}^a(\sigma)+\chi_{1,\tau}^a(\delta_w\sigma))\log|\sigma(u_M)|_w=0,$$

so

$$\sum_{\sigma\in G} Tr(\mathrm{Ind}\,\chi_1^a(\sigma))\log|\sigma(u_M)|_w = \sum_{\sigma\in H_1}\chi_1^a(\sigma)\log|\sigma(u_M)|_w + \chi_{1,\tau}^a(\sigma)\log|\sigma(u_M)|_w$$

$$= \sum_{\sigma\in\mathrm{Gal}(E/F)}\chi_1^a(\sigma)\log|\sigma(N_{M/E}(u_M))|_w + 0$$

$$= \sum_{\sigma\in\mathrm{Gal}(E/F)}\widetilde{\chi}(\sigma)\log|\sigma(u_E)|_w$$

Hence

$$-\frac{1}{2e_M}\sum_{\sigma\in\mathrm{Gal}(E/F)}\widetilde{\chi}(\sigma)\log|\sigma(u_E)|_w = L'(\widetilde{\chi},0).$$

□

Remarks 5.2.4. By the Remarks 1.1.4, in order to prove Conjecture 1.1.1 we would need to show that the absolute value of u_E at the complex places of E is 1 and that u_E is a e_M/e_Eth power in E where e_E is the number of roots of unity in E.

Define

$$u_{\chi_1}^* = -\frac{1}{e_M}\sum_{\sigma\in\mathrm{Gal}(E/F)}\chi(\sigma)\otimes\sigma(u_E)\in(k\otimes\mathscr{O}_E^\times)^{\chi_1^{-1}}$$

and

$$u_{\chi_{1,\tau}}^* = \tau(u_{\chi_1}^*).$$

Corollary 5.2.5. *(Archimedean Stark Conjecture for χ_1 and $\chi_{1,\tau}$) We have that*

$$\log|u_{\chi_1}^*|_w = L'(\chi_1,0)\ \textit{and}\ \log|u_{\chi_{1,\tau}}^*|_{w^\tau} = L'(\chi_1,0)$$

so $u^*_{\chi_1}$ and $u^*_{\chi_1,\tau}$ are the units in Conjecture 1.1.9.

Proof. We first remark that $|\cdot|_w$ on E is the square root of $|\cdot|_w$ on M since w is a real place of E. Similarly for $|\cdot|_{w^\tau}$ on $\tau(E)$. Then this corollary follows immediately from Proposition 5.2.3 and the fact that

$$L(\chi_1, s) = L(\chi_{1,\tau}, s)$$

and

$$\log |u^*_{\chi_{1,\tau}}|_{w^\tau} = \log |u^*_{\chi}|_w.$$

□

Before we prove the p-adic conjecture for χ we have a proposition regarding the Stark unit $u^*_{\chi_{2,\tau}}$.

Proposition 5.2.6. *We have that $u^*_{\chi_{2,\tau}} = \delta_w(u^*_{\chi_2})$.*

Proof. Since u_M is the Stark unit for M/K, by definition, we have that

$$u^*_{\chi_2} = \sum_{\sigma \in H_2} \chi_2(\sigma) \otimes \sigma(u_M) \in (k \otimes \mathscr{O}_M^\times)^{\chi_2^{-1}}$$

and

$$u^*_{\chi_{2,\tau}} = \sum_{\sigma \in H_2} \chi_{2,\tau}(\sigma) \otimes \sigma(u_M) \in (k \otimes \mathscr{O}_M^\times)^{\chi_{2,\tau}^{-1}}.$$

Since $\delta_w \in G - H_2$, $\delta_w(u^*_{\chi_2}) \in (k \otimes \mathscr{O}_M^\times)^{\chi_{2,\tau}^{-1}}$ so $\delta_w(u^*_{\chi_2})$ is a $k^\times$-multiple of $u_{\chi_{2,\tau}}$ since $(k \otimes \mathscr{O}_M^\times)^{\chi_{2,\tau}^{-1}}$ is one dimensional as a k-vector space. We show $u^*_{\chi_{2,\tau}} = \delta_w(u^*_{\chi_2})$. To do this consider the map

$$\lambda : \mathbb{C} \otimes \mathscr{O}_M^\times \longrightarrow \mathbb{C}X$$

$$\lambda(\alpha \otimes u) = \alpha \sum_v \log |u|_v v$$

where X is Y modulo the vector $(1, 1, \ldots, 1)$, and Y is the free abelian group generated by the infinite places of M. By Dirichlet's equivariant unit theorem, λ is a G-module isomorphism. Let

$$e_{\chi_2} = \sum_{\sigma \in H_2} \chi_2(\sigma) w^\sigma \in (\mathbb{C}X)^{\chi_2^{-1}}$$

and

$$e_{\chi_{2,\tau}} = \sum_{\sigma \in H_2} \chi_{2,\tau}(\sigma) w^\sigma \in (\mathbb{C}X)^{\chi_{2,\tau}^{-1}}.$$

Since the character $\chi_{2,\tau}$ does not depend on τ, we may calculate e_{χ_τ} by taking $\tau = \delta_w$. Then since $w^{\delta_w} = w$ we have

$$\begin{aligned} e_{\chi_{2,\tau}} &= \sum_{\sigma \in H_2} \chi_{2,\tau}(\sigma) w^\sigma \\ &= \sum_{\sigma \in H_2} \chi_2(\delta_w \sigma \delta_w) w^\sigma \\ &= \sum_{\sigma \in H_2} \chi_2(\sigma) w^{\delta_w \sigma \delta_w} \\ &= \sum_{\sigma \in H_2} \chi_2(\sigma) w^{\delta_w \sigma} \\ &= \delta_w(e_{\chi_2}). \end{aligned}$$

At the same time, we have

$$\lambda(u^*_{\chi_2}) = \log|u^*_{\chi_2}|_w e_{\chi_2} = L'(\chi_2, 0)e_{\chi_2}$$

and

$$\lambda(u^*_{\chi_{2,\tau}}) = \log|u^*_{\chi_{2,\tau}}|_w e_{\chi_{2,\tau}} = L'(\chi_{2,\tau}, 0)e_{\chi_{2,\tau}} = L'(\chi_2, 0)e_{\chi_{2,\tau}}.$$

Then since λ is a G-equivariant isomorphism,

$$\lambda(\delta_w(u_{\chi_2})) = L'(\chi_2, 0)\delta_w(e_{\chi_2}) = L'(\chi_2, 0)e_{\chi_{2,\tau}}$$

implies $u^*_{\chi_{2,\tau}} = \delta_w(u^*_{\chi_2})$. $\square$

5.3 p-adic Stark conjecture

In this section we are again assuming that $p = \mathfrak{p}\overline{\mathfrak{p}}$ in K with ι_p picking out the prime $\mathfrak{p}$. We also let

$$f = \sum_{\mathfrak{a}\subset\mathscr{O}_F} \chi(\mathfrak{a})q^{N\mathfrak{a}} = \sum_{\mathfrak{a}\subset\mathscr{O}_K} \psi(\mathfrak{a})q^{N\mathfrak{a}}$$

be our usually weight one modular form along with everything that is associated to f. We fix a p-stabilization f_α of f. Since p splits in K, α is either $\psi(\mathfrak{p})$ or $\psi(\overline{\mathfrak{p}})$. Let ε be the character

$$\varepsilon : D_p \longrightarrow \overline{\mathbb{Q}}^\times$$

$$\varepsilon(\delta_p) = \alpha.$$

Note that if $\alpha = \psi(\mathfrak{p})$, then $\varepsilon = \chi_2|_{D_p}$ and if $\alpha = \psi(\overline{\mathfrak{p}})$, then $\varepsilon = \chi_{2,\tau}|_{D_p}$.

We begin this section with a couple lemmas.

Lemma 5.3.1. *The prime p is inert in F.*

Proof. We first make the observation that we are assuming the roots of the Hecke polynomial of f at p are distinct. Since the two roots of the polynomial are

$$\psi(\mathfrak{p}) = \psi(\delta_p) \text{ and } \psi(\overline{\mathfrak{p}}) = \psi_\tau(\delta_p)$$

we have $\psi(\delta_p) \neq \psi_\tau(\delta_p)$.

Now, p is inert in F if and only if $\delta_p \notin H_\chi$. Since p splits in K, $D_p \subset H_\psi$. Suppose $\delta_p \in H_\chi$. Then $\delta_p \in H_\chi \cap H_\psi$. All four of the characters ψ_τ, ψ, χ, and χ_τ are equal on $H_\chi \cap H_\psi$, so $\psi_\tau(\delta_p) = \psi(\delta_p)$, which is a contradiction. □

Lemma 5.3.2. *We have that*

$$\log_p |u^*_{\chi_1}|_{1/\alpha} = \log_p |u^*_{\chi_2} + u^*_{\chi_{2,\tau}}|_{1/\alpha} = \begin{cases} \log_p(u^*_{\chi_2}) & \text{if } \alpha = \psi(\mathfrak{p}) \\ \log_p(u^*_{\chi_{2,\tau}}) & \text{if } \alpha = \psi(\overline{\mathfrak{p}}) \end{cases}$$

Proof. The second equality follows from Proposition 3.2.9.

We use the facts that $\chi_1(\delta_w) = 1$ and that $\chi_1|_{H_1 \cap H_2} = \chi_2|_{H_1 \cap H_2} = \chi_{2,\tau}|_{H_1 \cap H_2}$

to see that

$$u_{\chi_1}^* = -\frac{1}{e_M} \sum_{\sigma \in \mathrm{Gal}(E/F)} \chi_1(\sigma) \otimes \sigma(u_E)$$

$$= -\frac{1}{e_M} \sum_{\sigma \in H_2} \chi_1(\sigma) \otimes \sigma(u_{M_n})$$

$$= -\frac{1}{e_M} \sum_{\sigma \in H_1 \cap H_2} \chi_1(\sigma) \otimes \sigma(u_{M_n}) + \chi_1(\sigma\delta_w) \otimes \sigma\delta_w(u_{M_n})$$

$$= -\frac{1}{e_M} \sum_{\sigma \in H_1 \cap H_2} \chi_2(\sigma) \otimes \sigma(u_{M_n}) + \chi_2(\sigma) \otimes \sigma\delta_w(u_{M_n}).$$

Now we note that since p is inert in F, $\delta_p \in H_2 - (H_1 \cap H_2)$. We also have that $D_p \cap H_2 \cap H_1 \subset D_p$ is index 2, and $\varepsilon | D_p \cap H_1 \cap H_2 = \chi_1 = \chi_{1,\tau} = \chi_2 = \chi_{2,\tau}$. Then $|u_{\chi_1}^*|_{1/\alpha}$ is

$$|u_\chi^*|_{1/\alpha} = \frac{1}{|D_p|} \sum_{\delta \in D_p} \varepsilon(\delta)\delta(u_\chi^*)$$

$$= \frac{1}{2}(u_{\chi_1}^* + \varepsilon(\delta_p)\delta_p(u_{\chi_1}^*)).$$

We now break the proof into the two choices of α. Assume first that $\alpha = \psi(\mathfrak{p}) = \chi_2(\delta_p)$. Then using

$$u_{\chi_1}^* = -\frac{1}{e_M} \sum_{\sigma \in H_1 \cap H_2} \chi_2(\sigma) \otimes \sigma(u_M) + \chi_2(\sigma) \otimes \sigma\delta_w(u_M)$$

and $\varepsilon(\delta_p) = \chi_2(\delta_p)$, we get that

$$|u^*_{\chi_1}|_{1/\alpha} = \frac{1}{2}(u^*_{\chi_1} + \varepsilon(\delta_p)\delta_p(u^*_{\chi_1}))$$

$$= -\frac{1}{2e_M} \sum_{\sigma \in H_1 \cap H_2} \chi_2(\sigma)\sigma(u_M) + \chi_2(\sigma)\sigma\delta_w(u_M) +$$

$$-\frac{1}{2e_M} \sum_{\sigma \in H_1 \cap H_2} \chi_2(\delta_p\sigma)\delta_p\sigma(u_M) + \chi_2(\delta_p\sigma)\delta_p\sigma\delta_w(u_M)$$

$$= -\frac{1}{2e_M} \left(\sum_{\sigma \in H_2} \chi_2(\sigma)\sigma(u_M) + \chi_2(\sigma)\sigma\delta_w(u_M) \right).$$

We have

$$-\frac{1}{e_M} \sum_{\sigma \in H_2} \chi_2(\sigma)\sigma(u_M) = u^*_{\chi_2}.$$

By rearranging the sum and using Proposition 5.2.6 we have

$$-\frac{1}{e_M} \sum_{\sigma \in H_2} (\chi_2(\sigma)\sigma\delta_w(u_M) = -\frac{1}{e_M} \sum_{\sigma \in H_2} \chi_{2,\tau}(\sigma)\delta_w\sigma(u_M) = \delta_w(u^*_{\chi_{2,\tau}}) = u^*_{\chi_2}.$$

Hence

$$|u^*_{\chi_1}|_{1/\alpha} = \frac{1}{2}(u^*_{\chi_2} + u^*_{\chi_2}) = u^*_{\chi_2}$$

so $\log_p |u^*_{\chi_1}|_{1/\alpha} = \log_p(u^*_{\chi_2})$.

Now assume that $\alpha = \psi(\overline{\mathfrak{p}}) = \chi_{2,\tau}(\delta_p)$. Since $\chi_2|_{H_1 \cap H_2} = \chi_{2,\tau}|_{H_1 \cap H_2}$, we have

$$u^*_{\chi_1} = -\frac{1}{e_M} \sum_{\sigma \in H_1 \cap H_2} \chi_{2,\tau}(\sigma) \otimes \sigma(u_M) + \chi_{2,\tau}(\sigma) \otimes \sigma\delta_w(u_M).$$

Using $\varepsilon(\delta_p) = \chi_{2,\tau}(\delta_p)$ we get

$$|u^*_{\chi_1}|_{1/\alpha} = \frac{1}{2}(u^*_{\chi_1} + \varepsilon(\delta_p)\delta_p(u^*_{\chi_1}))$$

$$= -\frac{1}{2e_M} \sum_{\sigma \in H_1 \cap H_2} \chi_{2,\tau}(\sigma)\sigma(u_M) + \chi_{2,\tau}(\sigma)\sigma\delta_w(u_M) +$$

$$-\frac{1}{2e_M} \sum_{\sigma \in H_1 \cap H_2} \chi_{2,\tau}(\delta_p\sigma)\delta_p\sigma(u_M) + \chi_{2,\tau}(\delta_p\sigma)\delta_p\sigma\delta_w(u_M)$$

$$= -\frac{1}{2e_M} \left(\sum_{\sigma \in H_2} \chi_{2,\tau}(\sigma)\sigma(u_M) + \chi_{2,\tau}(\sigma)\sigma\delta_w(u_M) \right).$$

We have

$$-\frac{1}{e_M} \sum_{\sigma \in H_2} \chi_{2,\tau}(\sigma)\sigma(u_M) = u^*_{\chi_{2,\tau}}.$$

By rearranging the sum and using Proposition 5.2.6 we have

$$-\frac{1}{e_M} \sum_{\sigma \in H_2} (\chi_{2,\tau}(\sigma)\sigma\delta_w(u_M) = -\frac{1}{e_M} \sum_{\sigma \in H_2} \chi_2(\sigma)\delta_w\sigma(u_M) = \delta_w(u^*_{\chi_2}) = u^*_{\chi_{2,\tau}}.$$

Hence

$$|u^*_{\chi_1}|_{1/\alpha} = \frac{1}{2}(u^*_{\chi_{2,\tau}} + u^*_{\chi_{2,\tau}}) = u^*_{\chi_{2,\tau}}$$

so $\log_p |u^*_{\chi_1}|_{1/\alpha} = \log_p(u^*_{\chi_{2,\tau}})$. □

Theorem 5.3.3. *Conjecture 3.2.7 holds for χ.*

Proof. Let $\eta, \eta' \in \mathcal{W}(\mathbb{C}_p)$ be of order p^n and p^m respectively. We need to show that

$$L_p(\chi, \alpha, \eta\omega, \eta'\omega, 0) = \frac{(1 - \beta\eta(p))\left(1 - \frac{\eta^{-1}(p)}{\alpha p}\right) \frac{\tau(\eta^{-1})}{p^{n+1}} \log_p |u^*_{\chi\eta} + u^*_{\chi_\tau\eta}|_{1/\alpha}}{(1 - \beta\eta'(p))\left(1 - \frac{\eta'^{-1}(p)}{\alpha p}\right) \frac{\tau(\eta'^{-1})}{p^{n+1}} \log_p |u^*_{\chi\eta'} + u^*_{\chi_\tau\eta'}|_{1/\alpha}}.$$

By the assumption that $\operatorname{Ind}\chi = \operatorname{Ind}\psi$, we have

$$L_p(\psi, \alpha, \eta\omega, \eta'\omega, s) = L_p(\chi, \alpha, \eta\omega, \eta'\omega, s)$$

so $L_p(\psi, \alpha, \eta\omega, \eta'\omega, 0) = L_p(\chi, \alpha, \eta\omega, \eta'\omega, 0)$. By theorem 12

$$L_p(\psi, \alpha, \eta\omega, \eta'\omega, 0) = \frac{(1-\beta\eta(p))\left(1-\frac{\eta^{-1}(p)}{\alpha p}\right)\frac{\tau(\eta^{-1})}{p^{n+1}}\log_p |u^*_{\psi\eta} + u^*_{\psi_\tau\eta}|_{1/\alpha}}{(1-\beta\eta'(p))\left(1-\frac{\eta'^{-1}(p)}{\alpha p}\right)\frac{\tau(\eta'^{-1})}{p^{n+1}}\log_p |u^*_{\psi\eta'} + u^*_{\psi_\tau\eta'}|_{1/\alpha}}.$$

Therefore we need to show that

$$\frac{\log_p |u^*_{\chi\eta} + u^*_{\chi_\tau\eta}|_{1/\alpha}}{\log_p |u^*_{\chi\eta'} + u^*_{\chi_\tau\eta'}|_{1/\alpha}} = \frac{\log_p |u^*_{\psi\eta} + u^*_{\psi_\tau\eta}|_{1/\alpha}}{\log_p |u^*_{\psi\eta'} + u^*_{\psi_\tau\eta'}|_{1/\alpha}}.$$

Since p is inert in F, by Proposition 3.2.9,

$$\frac{\log_p |u^*_{\chi\eta} + u^*_{\chi_\tau\eta}|_{1/\alpha}}{\log_p |u^*_{\chi\eta'} + u^*_{\chi_\tau\eta'}|_{1/\alpha}} = \frac{\log_p |u^*_{\chi\eta}|_{1/\alpha}}{\log_p |u^*_{\chi\eta'}|_{1/\alpha}}.$$

Then by Lemma 5.3.2

$$\frac{\log_p |u^*_{\chi\eta}|_{1/\alpha}}{\log_p |u^*_{\chi\eta'}|_{1/\alpha}} = \frac{\log_p |u^*_{\psi\eta} + u^*_{\psi_\tau\eta}|_{1/\alpha}}{\log_p |u^*_{\psi\eta'} + u^*_{\psi_\tau\eta'}|_{1/\alpha}}.$$

□

Chapter 6

Numerical evidence

6.1 $F = \mathbb{Q}(\sqrt{17})$, $K = \mathbb{Q}(\sqrt{4+\sqrt{17}})$, $p = 5$

In this section, we give numerical evidence of an example of Conjecture 3.2.3. Unfortunately this example falls into the setting for which we proved our conjecture in chapter 5. We hope to compute an example in a case where we have not proved our conjecture in the future. We do note though that our numerical evidence is for the conjecture at $s = 1$, while we proved the conjecture in chapter 5 at $s = 0$. We also note that we numerically verified the integral conjecture while in chapter 5 we prove the rational conjecture.

To begin we introduce the field extensions, modular forms, and Stark unit. Let $p = 5$, fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $F = \mathbb{Q}(a)$ where a is a root of $x^2 - 17$ and $K = \mathbb{Q}(b)$ where b is a root of $x^2 - (a-4)$. Then K is a degree 2 extension of F which is cut out by a quadratic character of G_F

of mixed signature. Let $\chi : G_F \to \{\pm 1\} \subset \overline{\mathbb{Q}}^\times$ be this quadratic character, so K is the fixed field of the kernel of χ. Let $\rho = \operatorname{Ind}_{G_F}^{G_\mathbb{Q}} \chi$ be the induction of χ. Let M be the fixed field of the kernel of ρ. The field M is the Galois closure of K over $\mathbb{Q}$. Explicitly

$$M = \mathbb{Q}(b, \overline{b}) \text{ where } \overline{b}^2 = -a + 4.$$

The Galois group of M over $\mathbb{Q}$ is D_8, the dihedral group with 8 elements. Let

$$f = \sum_{\mathfrak{a} \subset \mathscr{O}_F} \chi(\mathfrak{a}) q^{N\mathfrak{a}}$$

be the weight one modular form associated to ρ. The level of f is 68 and the character of f is the quadratic character of $(\mathbb{Z}/68\mathbb{Z})^\times$ of conductor 68. Because $p = 5$ is inert in F and then splits in K, the Hecke polynomial of f at p is

$$x^2 - 1 = (x-1)(x+1).$$

Let $\alpha = \pm 1$ be one of the roots of $x^2 - 1$ and let $\beta = -\alpha$ be the other root. Let

$$f_\alpha(z) = f(z) - \beta f(pz)$$

be the p-stabilization of f with U_p-eigenvalue α.

Let $\zeta_{p^2} = \zeta_{25} \in \overline{\mathbb{Q}}$ be the 25th root of unity which is mapped to $\exp(2\pi i/25)$ under ι_∞. Let $\zeta_5 = \zeta_{25}^5$ and let ψ be the character

$$\psi : (\mathbb{Z}/25\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

$$\psi(2) = \zeta_5.$$

We simultaneously view ψ as an element of weight space (so a character of $\mathbb{Z}_p^\times$ valued in $\mathbb{C}_p^\times$) and a Galois character. Let K_1 be the fixed field of the kernel of $\chi\psi$. Then K_1 is a degree $p = 5$ extension of K (so degree 10 over F). It is the first layer of the cyclotomic $\mathbb{Z}_5$ extension of K. We also use the characters ψ^2, ψ^3, and ψ^4 in the same role as ψ. Let $u_1 \in \mathscr{O}_{K_1}^{\times}$ be the Stark unit for the extension K_1/F which has positive sign under the embedding ι_∞. Stark's conjecture for K_1/F asserts that for $1 \leq i \leq 4$,

$$L'(\chi\psi^i, 0) = -\frac{1}{2} \sum_{\sigma \in \mathrm{Gal}(K_1/F)} \chi\psi^i(\sigma) \log |\sigma(u_1)|.$$

Let $M_1 = K_1 M$ be the compositum of K_1 and M and let $\mathbb{Q}_1$ be the fixed field of ψ as a character of $G_{\mathbb{Q}}$, so $\mathbb{Q}_1/\mathbb{Q}$ is the first layer of the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$ and M_1 is the first layer of the cyclotomic $\mathbb{Z}_p$ extension of M. Restriction gives an isomorphism

$$\mathrm{Gal}(M_1/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}_1/\mathbb{Q}) \times \mathrm{Gal}(M/\mathbb{Q}).$$

Let $\delta_p \in \mathrm{Gal}(M_1/\mathbb{Q})$ be the lift of the arithmetic Frobenius at p (with respect to ι_p) in $\mathrm{Gal}(M/\mathbb{Q})$ which acts trivially on $\mathbb{Q}_1$. Then δ_p has order 2. Let

$$|\cdot|_\beta : \mathscr{O}_{M_1}^{\times} \longrightarrow \mathscr{O}_{M_1}^{\times}$$

be the projection onto the subspace of $\mathscr{O}_{M_1}^{\times}$ where δ_p acts with eigenvalue β (so if $\beta = 1$, $|x|_\beta = x\delta_p(x)$, if $\beta = -1$, $|x|_\beta = x/\delta_p(x)$).

Now we describe the overconvergent modular forms we need. Let $\mathbb{D}$ be the space of locally analytic $\mathbb{Z}_p$-valued distributions on $\mathbb{Z}_p$. Let $\mathbb{D}_{-1}$ denote $\mathbb{D}$ with the weight -1 action of $\Sigma_0(p)$. Let $N = 68$, $\Gamma_0 = \Gamma_1(N) \cap \Gamma_0(p)$, and $\mathbb{W} = \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_{-1})$, so $\mathbb{W}$ is the space of overconvergent modular symbols of weight -1. Let $\mathbb{W}^{\pm} \subset \mathbb{W}$ be the $\pm$-subspace of $\mathbb{W}$. The Hecke eigensubspace of $\mathbb{W}^{\pm}$ with the same eigenvalues as f_α is a rank one $\mathbb{Z}_p$-module. Let $\varphi_\alpha^{\pm} \in \mathbb{W}^{\pm}$ be a generator for this rank one subspace, and let $\mu_\alpha = \varphi_\alpha(\{0\} - \{\infty\})$. We consider the analytic functions

$$L_p(\chi, \alpha, \psi^i, \psi^j, s) = \frac{\Lambda_{-1}(\varphi_\alpha^+, \psi^{-i}\langle\cdot\rangle^{s-1})}{\Lambda_{-1}(\varphi_\alpha^+, \psi^{-j}\langle\cdot\rangle^{s-1})}$$

for $1 \leq i, j \leq 4$. Note that for $1 \leq i \leq 4$, $\psi^i(68) = 1$ and the Euler like factors in Conjecture 3.2.8 are 1. Therefore the formula we verify is that for $1 \leq i, j \leq 4$,

$$L_p(\chi, \alpha, \psi^i, \psi^j, 1) = \frac{\tau(\psi^{-i})}{\tau(\psi^{-j})} \frac{\displaystyle\sum_{\sigma \in \mathrm{Gal}(K_1/F)} \chi\psi^i(\sigma^{-1}) \log_p |\sigma(u_1)|_\beta}{\displaystyle\sum_{\sigma \in \mathrm{Gal}(K_1/F)} \chi\psi^j(\sigma^{-1}) \log_p |\sigma(u_1)|_\beta}.$$

Now we give the numerical data. We begin with the Stark units. The Stark unit u_1 has minimal polynomial over F given by

$$h = z^{10} + (\frac{-2268731445425}{2} b^2 - \frac{279293603945}{2}) z^9 +$$

$$(509077626342089566600 b^2 + 62670319678796566645) z^8 +$$

$$(\frac{-908489137763713280149684575}{2} b^2 - \frac{111840123671250926011540715}{2}) z^7 +$$

$$(\frac{1212779745101402982169172133826675}{2} b^2 + \frac{149300009257135106668401653656195}{2}) z^6 +$$

$$(\frac{-518141421601118964495801146359795708750}{2} b^2 -$$

$$+ \frac{63786123864629766175002479114705820790}{2}) z^5 +$$

$$(\frac{1212779745101402982169172133826675}{2} b^2 + \frac{149300009257135106668401653656195}{2}) z^4 +$$

$$(\frac{-908489137763713280149684575}{2} b^2 - \frac{111840123671250926011540715}{2}) z^3 +$$

$$(509077626342089566600 b^2 + 62670319678796566645) z^2 +$$

$$(\frac{-2268731445425}{2} b^2 - \frac{279293603945}{2}) z + 1.$$

Over K, h factors as $h = h_1 h_2$, where

$$h_1 = z^5 +$$

$$(\frac{-796009073655}{4}b^3 - \frac{2268731445425}{4}b^2 - \frac{97993194995}{4}b - \frac{279293603945}{4})z^4 +$$

$$(310943152062735130b^3 + \frac{1772522746965065725}{2}b^2 +$$

$$+382788512662103700b + \frac{2182075216866668645}{2})z^3 +$$

$$(310943152062735130b^3 - \frac{1772522746965065725}{2}b^2 +$$

$$+382788512662103700b - \frac{2182075216866668645}{2})z^2 +$$

$$(\frac{-796009073655}{4}b^3 + \frac{2268731445425}{4}b^2 - \frac{97993194995}{4}b + \frac{279293603945}{4})z +$$

$$-1$$

and

$$h_2 = z^5 +$$

$$(\frac{796009073655}{4}b^3 - \frac{2268731445425}{4}b^2 + \frac{97993194995}{4}b - \frac{279293603945}{4})z^4 +$$

$$(-310943152062735130b^3 + \frac{1772522746965065725}{2}b^2 -$$

$$+382788512662103 70b + \frac{218207521686668645}{2})z^3 +$$

$$(-310943152062735130b^3 - \frac{1772522746965065725}{2}b^2 -$$

$$+38278851266210370b - \frac{218207521686668645}{2})z^2 +$$

$$(\frac{796009073655}{4}b^3 + \frac{2268731445425}{4}b^2 + \frac{97993194995}{4}b + \frac{279293603945}{4})z +$$

$$-1$$

We computed the spaces $\mathbb{W}^{\pm}$ and elements $\varphi_{\alpha}^{\pm}$ with the precision of 60 5-adic digits. For $1 \leq i, j \leq 4$, the p-adic numbers

$$L_p(\chi, \alpha, \psi^i, \psi^j, 1)$$

and

$$\frac{\displaystyle\sum_{\sigma\in\mathrm{Gal}(K_1/F)} \chi\psi^i(\sigma^{-1})\log_p|\sigma(u_1)|_\beta}{\displaystyle\sum_{\sigma\in\mathrm{Gal}(K_1/F)} \chi\psi^j(\sigma^{-1})\log_p|\sigma(u_1)|_\beta}$$

were computed with the precision of 60 5-adic digits. These ratios lie in the field $\mathbb{Q}_5(\zeta_{25})$. This field has ramification index 20 and was represented on the computer with respect to the uniformizer $\pi = \zeta_{25} - 1$. We note that 60 5-adic digits in $\mathbb{Q}_5(\zeta_{25})$ is $60 * 20 = 1200$ π-adic digits. We can compute the Gauss sum ratio to as high of a precision as we like. We then computed the π-adic valuation of the difference

$$L_p(\chi, \alpha\psi^i, \psi^j, 1) - \frac{\tau(\psi^{-i})}{\tau(\psi^{-j})} \frac{\displaystyle\sum_{\sigma\in\mathrm{Gal}(K_1/F)} \chi\psi^i(\sigma^{-1})\log_p|\sigma(u_1)|_\beta}{\displaystyle\sum_{\sigma\in\mathrm{Gal}(K_1/F)} \chi\psi^j(\sigma^{-1})\log_p|\sigma(u_1)|_\beta} \tag{6.1}$$

and recorded the valuations in the following table:

α	(i,j)	π-adic valuation of (6.1)
1	(1,2)	1135
1	(1,3)	1136
1	(1,4)	1135
1	(2,3)	1135
1	(2,4)	1135
1	(3,4)	1135
-1	(1,2)	1142
-1	(1,3)	1140
-1	(1,4)	1141
-1	(2,3)	1140
-1	(2,4)	1141
-1	(3,4)	1140

Considering the precision we are working at, a number in our computer representation of $\mathbb{Q}_5(\zeta_{25})$ is 0 if it has π-adic valuation 1200. Therefore the table says that the value of (6.1) is extremely close to 0. A certain amount of rounding error is expected because we have done somewhat complicated calculations in $\mathbb{Q}_5(\zeta_{25})$. The discrepancy between the third column in the table and 1200 is most likely from rounding error. With our original precision of 60 5-adic digits, the third column of the table is less than 4 5-adic digits away from 0.

Chapter 7

Appendix

7.1 Rigid analytic geometry

Rigid analytic geometry is used in this thesis to give a precise account of the results of Bellaïche and Dmitrov's work on the eigencurve at weight one points ([2]), and how to use their results to construct p-adic L-functions on the eigencurve following Bellaïche ([1]). Every rigid analytic space we consider will be an affinoid space except for the open unit disk. In this appendix, we introduce the relevant notions that are used in the thesis. The main reference for this appendix is the book [4].

Let k be a field that is complete with respect to a nonarchimidean absolute value. The two main cases we consider are when k is $\mathbb{Q}_p$ or $\mathbb{C}_p$ with normalized absolute value $|\cdot|$ so that $|p| = 1/p$.

Definition 7.1.1. *The* ***Tate algebra*** *in m variables over k is the subalgebra $k\langle x_1, \ldots, x_m\rangle$*

of $k[[x_1, \ldots, x_m]]$ *consisting of formal power series whose coefficients tend to* 0*:*

$$k\langle x_1, \ldots, x_m\rangle = \left\{ f = \sum_{\alpha \in \mathbb{N}^m} a_\alpha x^\alpha : |a_\alpha| \to 0 \text{ as } |\alpha| \to \infty \right\}$$

where if $\alpha = (\alpha_1, \ldots, \alpha_n)$, x^α *means* $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *and* $|\alpha| = \alpha_1 + \cdots \alpha_n$.

Proposition 7.1.2. $k\langle x_1, \ldots, x_n\rangle$ *is Noetherian, Jacobson, a unique factorization domain, and has Krull dimension* n*.*

Proof. For Noetherian and unique factorization domain, theorem 1 on page 207 of [4]. For Jacobson, theorem 3 on page 208 of [4]. For Krull dimension see the following proposition. □

Corollary 7.1.3. $k\langle x\rangle$ *is a principal ideal domain.*

Definition 7.1.4. *An* ***affinoid algebra*** *is a* k*-algebra isomorphic to* $k\langle x_1, \ldots, x_m\rangle/I$ *for some* m *and some ideal* I*. By the proposition an affinoid algebra is Noetherian and Jacobson. For an affinoid algebra* A*, let* $M(A)$ *denote the set of maximal ideals of* A*.*

Proposition 7.1.5. *(Noether normalization) Let* A *be an affinoid algebra. Then there exists for some* d*, an injective map* $k\langle x_1, \ldots, x_d\rangle \hookrightarrow A$ *that makes* A *a finite* $k\langle x_1, \ldots, x_d\rangle$*-module. The number* d *is the Krull dimension of* A*.*

Proof. Theorem 1, corollary 2, and remark after corollary 2 page 227 of [4]. □

By the proposition, for an affinoid algebra, A, the nullstellensatz holds. That is, for each maximal ideal $\mathfrak{m} \in M(A)$, $A/\mathfrak{m}$ is a finite extension of k. Let $X = M(A)$

and for a finite extension L of k, let $X(L) = \mathrm{Hom}_{k-alg}(A, L)$. We define

$$X(\overline{k}) = \bigcup_{\substack{L \subset \overline{k} \\ \text{finite}}} X(L)$$

and we have an action of the absolute Galois group G_k on $X(\overline{k})$. The map

$$\begin{aligned} X(\overline{k}) &\longrightarrow X \\ \varphi &\longmapsto \ker(\varphi) \end{aligned}$$

is surjective by the nulltellensatz, and G_k acts transitively on each fiber. There is therefor a bijection

$$X(\overline{k})/G_k = X.$$

We note that when $k = \mathbb{C}_p$, then $X = X(\mathbb{C}_p)$.

Definition 7.1.6. *If A is a k-algebra, a **k-algebra norm** on A is a non-archimedean norm $|\cdot| : A \to \mathbb{R}_{\geq 0}$ such that $|\lambda x| = |\lambda||x|$ for all $\lambda \in k$, $x \in A$, and such that $|xy| \leq |x||y|$ for all $x, y \in A$. If A is complete for $|\cdot|$ we say that $|\cdot|$ is a **Banach-algebra norm** and that A is a **Banach space**.*

Definition 7.1.7. *For $f = \sum_\alpha a_\alpha x^\alpha \in k\langle x_1, \ldots, x_n \rangle$ define its **Gauss norm** to be*

$$|f| = \sup_\alpha |a_\alpha|.$$

The Gauss norm makes $k\langle x_1, \ldots, x_n \rangle$ a Banach space (proposition 1, page 192 [4]).

Definition 7.1.8. *Let A be an affinoid algebra and let $X = M(A)$. For $a \in A$ and $x \in X$ it makes sense to define $|a(x)|$ as the absolute value of the image of a in A/x*

since A/x is a finite extension of k so there is a unique extension of the absolute value on k. For $a \in A$, define

$$|a|_{sup} = \sup_{x \in X} |a(x)|.$$

The next two propositions will show that $|\cdot|$ is finite and defines a Banach-algebra norm on A.

Proposition 7.1.9. *Any ideal of the Tate algebra is closed with respect to the topology induced by the Gauss norm. Therefore any affinoid algebra is a Banach space as well once we choose a presentation as a quotient of a Tate-algebra.*

Proof. Corollary 2, page 208 of [4]. □

Proposition 7.1.10. *Any two Banach-algebra norms on an affinoid algebra are equivalent. Furthermore, any k-algebra morphism between two affinoid algebras is continuous. For a reduced affinoid algebra, $|\cdot|_{sup}$ makes the affinoid algebra a Banach space.*

Proof. Theorem 1 and proposition 2 page 229 of [4] for first two assertions. Theorem 1, page 242 of [4] for third assertion. □

Theorem 7.1.11. *Let A be an affinoid algebra and B an A-algebra of finite type as an A-module. Then B is an affinoid algebra.*

Proof. Proposition 5, page 223 of [4]. □

Now given an affinoid algebra, A we give the construction of the rigid analytic space $Sp(A)$.

Definition 7.1.12. *Let A be an affinoid algebra and let $X = M(A)$. A subset $U \subset X$ is called an **affinoid subdomain**, if there exists an affinoid algebra A' and a k-algebra morphism $g : A \to A'$ such that the following holds: For any affinoid algebra B and k-algebra homomorphism $f : A \to B$, we have that $M(f)(M(B)) \subset U$ if and only if there exists k-algebra homomorphism $h : A' \to B$ such that $f = h \circ g$.*

If U is an affinoid subdomain, then the A' is unique and we define the ring of rigid analytic functions on U to be $\mathscr{O}_X(U) = A'$.

*A subset $U \subset X$ is called an **admissible open** if it has a set-theoretic covering by affinoid subdomains $\{U_\alpha\}$ of X such that the following holds: For any $f : A \to B$ with $M(f)(M(B)) \subset U$, the covering $\{M(f)(U_\alpha)\}$ of $M(B)$ has a finite subcover.*

*A collection $\{U_\alpha\}$ of admissible open subsets of X is an **admissible covering** of its union $U = \bigcup_\alpha U_\alpha$, if the following holds: For any $f : A \to B$ with $M(f)(M(B)) \subset U$, the covering $\{M(f)(U_\alpha)\}$ of $M(B)$ admits a refinement which is a finite covering of $M(B)$ by affinoid subdomians.*

*The definitions of admissible opens and admissible coverings define a Grothendieck topology on X, which we will call the **Tate topology** on X.*

Theorem 7.1.13. *(Tate's acyclicity theorem) Let A be an affinoid algebra and $X = M(A)$. The function $U \mapsto \mathscr{O}_X(U)$ from affinoid subdomains to k-algebras extends uniquely to a sheaf with respect to the Tate topology. In particular, if U is an affinoid subdomain and $U = \cup U_\alpha$ is a covering of U by affinoid subdomains, then the* *sequence*

$$0 \longrightarrow \mathscr{O}_X(U) \longrightarrow \prod_{\alpha} \mathscr{O}_X(U_\alpha) \longrightarrow \prod_{\alpha,\beta} \mathscr{O}_X(U_\alpha \cap U_\beta)$$

is exact.

Proof. Section 8.2 of [4]. □

Definition 7.1.14. *Let A be an affinoid algebra. Then the **affinoid** (or **affinoid variety** or k-**affinoid variety** if we want to make k explicit)* $\mathrm{Sp}(A)$ *is defined to be the pair $(X, \mathscr{O}_X)$ where $X = M(A)$ with its Tate topology and $\mathscr{O}_X$ is the sheaf of k-algebras on X just defined.*

Definition 7.1.15. *Let $X = \mathrm{Sp}(A)$ be an affinoid. Then for $x \in X$, we define the **stalk** of X at x to be*

$$\mathscr{O}_{X,x} = \varinjlim_{x \in U} \mathscr{O}_X(U)$$

*where the direct limit is over admissible opens. For an affinoid algebra A and a maximal ideal $\mathfrak{m}$ of A we define the **rigid analytic localization** of A at $\mathfrak{m}$ to be the stalk of* $\mathrm{Sp}(A)$ *at the point corresponding to $\mathfrak{m}$.*

The k-algebra $\mathscr{O}_{X,x}$ is a local ring.

Proposition 7.1.16. *Let $X = \mathrm{Sp}\, A$ be an affinoid, let $x \in X$ and let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x. Then the canonical map $A \to \mathscr{O}_{X,x}$ factors through an injective map $A_{\mathfrak{m}} \to \mathscr{O}_{X,x}$ and this injective map induces an isomorphism*

$$\widehat{A}_{\mathfrak{m}} \cong \widehat{\mathscr{O}}_{X,x}$$

between the $\mathfrak{m}$-adic completion of A and the maximal-adic completion of $\mathscr{O}_{X,x}$.

Proof. [4] page 298. □

To make the definition of a rigid analytic space we work in the category of locally ringed G-spaces. A locally ringed G-space is a pair $(X, \mathscr{O}_X)$ where X is a set with a mild Grothendieck topology (so all the sets in the Grothendieck topology are subsets of X) and $\mathscr{O}_X$ is a sheaf on the Grothendieck topology whose stalks are all local rings. A morphism from a locally ringed G-space $(Y, \mathscr{O}_Y)$ to another locally ringed G-space $(X, \mathscr{O}_X)$ is a pair $(f, f^{\#})$ where $f : Y \to X$ is a continuous map with respect to the Grothendieck topologies and $f^{\#} : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is a sheaf morphism which induces homomorphisms of local rings on the stalks. For more details on the category of locally ringed G-spaces we refer the reader to ([4]).

Definition 7.1.17. *A* ***rigid analytic variety*** *(or* ***rigid analytic space****) is a locally ringed G-space $(X, \mathscr{O}_X)$ admitting a covering $\{U_\alpha\}$ such that for each α, $(U_\alpha, \mathscr{O}|_{U_\alpha})$ is isomorphic to an affinoid variety. A morphism $X \to Y$ between two rigid analytic spaces is a morphism between the associated locally ring G-spaces.*

We remark that the category of affinoid varieties is a subcategory of the category of rigid analytic varieties, and that any morphism $f : Y = \mathrm{Sp}(B) \to X = \mathrm{Sp}(A)$ of affinoid varieties is induced by a morphism of k-algebras from A to B. More precisely, the category of affinoid varieties is the opposite category to the category of affinoid algebras where morphisms of affinoid algebras are k-algebra homomorphisms.

Definition 7.1.18. *Given affinoid varieties $X = \mathrm{Sp}(A)$, $Y = \mathrm{Sp}(B)$ and $Z = \mathrm{Sp}(C)$ with morphisms $X \to Z$ and $Y \to Z$, then the* ***fiber product*** *of X and Y over Z is* *given by the completed tensor product of A and B over C:*

$$X \times_Z Y = \mathrm{Sp}(A \widehat{\otimes}_C B).$$

Furthermore, this definition glues to define fiber products in the category of rigid analytic spaces.

Definition 7.1.19. *A rigid-analytic space X is **disconnected** if there exists an admissible open covering $\{U, V\}$ of X with $U, V \neq \emptyset$ and $U \cap V = \emptyset$. We say that X is **connected** if X is not disconnected.*

We record some facts about connectedness.

Proposition 7.1.20. *1. Let $X = \mathrm{Sp}(A)$ be an affinoid. Then X is connected if and only if $M(A)$ is connected with respect to the Zariski topology.*

2. Let $X_1, \ldots, X_n$ be the connected components of $M(A)$ with respect to the Zariski topology. Then the X_i are affinoid subdomains of X and there exist affinoid subalgebras A_i of A such that $X_i = M(A)$ and

$$A = A_1 \times A_2 \times \cdots \times A_n.$$

*The X_i are called the **connected components** of X.*

Proof. [4] page 345. □

Definition 7.1.21. *Let X be a rigid analytic space. We say that X is **reduced, normal**, or smooth at a point $x \in X$ if the local ring $\mathscr{O}_{X,x}$ is reduced, normal, or regular respectively.*

Proposition 7.1.22. *Let $X = \operatorname{Sp} A$ be an affinoid and let $x \in X$ be a point corresponding to the maximal ideal $\mathfrak{m} \subset A$. If one of the rings $A_{\mathfrak{m}}, \mathscr{O}_{X,x}$, or $\widehat{A}_{\mathfrak{m}} = \mathscr{O}_{X,x}$ is reduced, normal, or regular, then all three rings satisfy this property.*

Proof. [4] page 301. □

Definition 7.1.23. *Let X and Y be rigid analytic spaces. A morphism $f : Y \to X$ is called* ***étale at*** *$y \in Y$, if the induced morphism on the stalks:*

$$\mathscr{O}_{X,f(y)} \longrightarrow \mathscr{O}_{Y,y}$$

is flat and unramified. We recall that unramified means $\mathscr{O}_{Y,y}/\mathfrak{m}_{f(y)}\mathscr{O}_{Y,y}$ is a finite separable extension of $\mathscr{O}_{X,f(y)}/\mathfrak{m}_{f(y)}$ where $\mathfrak{m}_{f(y)}$ is the maximal ideal of $\mathscr{O}_{X,f(y)}$. The morphism $f : Y \to X$ is called ***étale*** *if f is étale at y for all $y \in Y$.*

Assume X, Y are rigid analytic spaces over an algebraically closed field. For $f : Y \to X$ a morphism and $y \in Y$, the ***ramification index*** *of f at $y \in Y$ is defined to be the length of the $\mathscr{O}_{X,f(y)}/\mathfrak{m}_{f(y)}$-module $\mathscr{O}_{Y,y}/\mathfrak{m}_{f(y)}\mathscr{O}_{Y,y}$.*

Definition 7.1.24. *Let A be an affinoid algebra. We define the* ***relative Tate algebra*** *to be*

$$A\langle x_1, \ldots, x_n\rangle = \left\{ f = \sum a_\alpha x^\alpha \in A[[x_1, \ldots, x_n]] : a_\alpha \to 0 as |\alpha| \to \infty \right\}.$$

If $A \cong k\langle y_1, \ldots y_m\rangle/I$, then the obvious map

$$k\langle y_1, \ldots, y_m, x_1, \ldots, x_n\rangle \longrightarrow A\langle x_1, \ldots, x_n\rangle$$

is surjective giving $A\langle x_1, \ldots, x_n\rangle$ the structure of an affinoid algebra.

Proposition 7.1.25. *Let $Y = \mathrm{Sp}(B)$ and $X = \mathrm{Sp}(A)$ be affinoids and let $f : Y \to X$ be a morphism from Y to X. Then f is étale if and only if B has a presentation of the form*

$$B = A\langle x_1, \ldots, x_n\rangle/(f_1, f_2, \ldots, f_n)$$

such that the determinant of the $n \times n$ matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ is invertible in B.

Proof. Proposition 8.1.1 of [12]. □

Proposition 7.1.26. *Let A be a k-affinoid algebra. The canonical k-algebra homomorphism*

$$A\langle x_1, \ldots, x_n\rangle \longrightarrow A\widehat{\otimes}_k k\langle x_1, \ldots, x_n\rangle$$

is an isomorphism.

Proof. Proposition 7, page 224 of [4] □

For the next two corollaries, let k' be a field extension of k that is complete with respect to an absolute value that extends the absolute value on k (for example $k = \mathbb{Q}_p$ and $k' = \mathbb{C}_p$).

Corollary 7.1.27. *There are canonical isomorphisms*

$$k\langle x_1, \ldots, x_n\rangle \widehat{\otimes}_k k\langle y_1, \ldots, y_m\rangle \cong k\langle z_1, \ldots, z_{n+m}\rangle$$

and

$$k'\widehat{\otimes}_k k\langle x_1, \ldots, x_n\rangle \cong k'\langle x_1, \ldots, x_n\rangle.$$

Proof. Corollary 8, page 224 of [4]. □

Corollary 7.1.28. *Let A and B be k-affinoid algebras. Then $A\widehat{\otimes}_k B$ is a k-affinoid algebra and $k'\widehat{\otimes}_k B$ is a k'-affinoid algebra. The canonical homomorphism $B \to k'\widehat{\otimes}_k B$ is injective.*

Proof. Corollary 9, page 224 of [4]. □

In this thesis, we consider $\mathbb{Q}_p$ and $\mathbb{C}_p$ rigid analytic varieties, and we repeatedly do the following construction. Given a $\mathbb{Q}_p$-affinoid $X = \mathrm{Sp}(A)$, we consider the base change to $\mathbb{C}_p$, $X_{\mathbb{C}_p} = \mathrm{Sp}(A_{\mathbb{C}_p})$ where $A_{\mathbb{C}_p} := \mathbb{C}_p\widehat{\otimes}_{\mathbb{Q}_p} A$. Define $X(\mathbb{C}_p) = \mathrm{Hom}_{\mathbb{Q}_p-alg}(A, \mathbb{C}_p)$. Then we have the relation

$$X(\mathbb{C}_p) = \mathrm{Hom}_{\mathbb{Q}_p-alg}(A, \mathbb{C}_p) = \mathrm{Hom}_{\mathbb{C}_p-alg}(A_{\mathbb{C}_p}, \mathbb{C}_p) = X_{\mathbb{C}_p}(\mathbb{C}_p) = M(A_{\mathbb{C}_p}).$$

We will be concerned with rigid analytic functions on $X_{\mathbb{C}_p}$, which are the elements of $A_{\mathbb{C}_p}$. We view A as a subring of $A_{\mathbb{C}_p}$ via the natural map $A \to A_{\mathbb{C}_p}$, and call elements of A in $A_{\mathbb{C}_p}$, $\mathbb{Q}_p$-rigid analytic functions on $X_{\mathbb{C}_p}$.

Example 7.1.29. *1. The closed disk (of dimension one) and radius r with $r \in p^{\mathbb{Q}}$:*

We start with the affinoid closed disk of radius p^r with $r \in \mathbb{Z}$. This is given by $\mathrm{Sp}(A)$*, where*

$$A = \mathbb{Q}_p\langle p^r x\rangle$$

or more explicitly we have that

$$A = \left\{ f = \sum a_n x^n \in \mathbb{Q}_p[[x]] : |a_n p^{rn}| = p^{-rn}|a_n| \to 0 \ as \ n \to \infty \right\}.$$

We define the affinoid closed disk of radius $p^{-a/b}$, where $\gcd(a, b) = 1$ *to be* $\mathrm{Sp}(A)$

where

$$A = \mathbb{Q}_p\langle x, y\rangle/(p^a x - y^b).$$

More explicitly we may view A *as a subring of* $\mathbb{Q}_p[[x]]$ *as*

$$A = \left\{ f = \sum a_n x^n : p^{an/b}|a_n| \to 0 \text{ as } n \to \infty \right\}.$$

We remark that letting $X = \mathrm{Sp}(A)$*, then we have,*

$$X(\mathbb{C}_p) = \mathrm{Hom}_{\mathbb{Q}_p-alg}(A, \mathbb{C}_p) = \{z \in \mathbb{C}_p : |z| \le p^{a/b}\}.$$

In the case when $a/b = -1/m$*, for* $m \in \mathbb{Z}_{\ge 0}$ *we write down the isomorphism of the two rings explicitly. Let* $R \subset \mathbb{Q}_p[[x]]$ *be the subring*

$$R = \left\{ \sum_{n=0}^{\infty} a_n x^n : |p^{n/m} a_n| \to 0 \text{ as } n \to \infty \right\}$$

where we view $p^{n/m}a_n \in \mathbb{C}_p$*. Also set*

$$A = \mathbb{Q}_p\langle x, y\rangle/(py - x^m).$$

We have an isomorphism $A \cong R$ *given by the following relation. Let*

$$\sum_{\alpha,\beta} a_{\alpha\beta} y^\alpha x^\beta \in \mathbb{Q}_p\langle y, x\rangle/(yp - x^m).$$

Then in A *we have*

$$\begin{aligned} \sum_{\alpha,\beta} a_{\alpha\beta} y^\alpha x^\beta &= \sum_{\alpha,\beta} a_{\alpha\beta} (x^m/p)^\alpha x^\beta \\ &= \sum_{k=0}^{\infty} \left(\sum_{\beta=0}^{k} \frac{a_{(k-\beta)/m,\beta}}{p^{(k-\beta)/m}} \right) x^k \end{aligned}$$

where we set $a_{(k-\beta)/m,\beta} = 0$ *if* $m \nmid (k - \beta)$*. This defines a map from* A *to* R *which is an isomorphism for affinoid algebras.*

2. *Union of affinoids. Let X_n, $n \geq 1$ be a collection of affinoids, and assumed we have morphisms $X_n \to X_{n+1}$ for all n such that X_n is an affinoid subdomain of X_{n+1}. Then there is a unique rigid analytic space X which is the admissible increasing union of the X_n. Furthermore, for such an X, the ring of rigid analytic functions on X is the inverse limit of the rings of analytic functions on the X_n,*

$$\mathscr{O}(X) = \varprojlim_n \mathscr{O}(X_n).$$

3. *The open unit disk (of dimension one). We make use of the previous construction where X_m is the closed unit disk of radius $1/p^{1/m}$ around 0. Let*

$$R_m = \left\{ f = \sum a_n x^n \in \mathbb{Q}_p[[x]] : |p^{n/m} a_n| \to 0 \text{ as } n \to \infty \right\}$$

be the ring of $\mathbb{Q}_p$-rigid analytic functions on X_m, so $X_m = \operatorname{Sp} R_m$. For $k < m$, we have maps $X_k \to X_m$ induced by the inclusions $R_m \to R_k$. Then we define the open unit disk to be

$$X = \bigcup_{m=1}^{\infty} X_m = \operatorname{Sp}(\varprojlim_m R_m).$$

Since the maps $R_m \to R_k$ are inclusions the ring of analytic functions on X is

$$R := \varprojlim_m R_m = \bigcap_{m=1}^{\infty} R_m = \left\{ f = \sum a_n x^n : |p^{n/m} a_n| \to 0 \text{ as } n \to \infty, \forall m \in \mathbb{Z}_{\geq 1} \right\}.$$

To finish, we note that $\mathbb{Z}_p[[x]] \subset R$ so we may view elements of $\mathbb{Z}_p[[x]]$ as rigid analytic functions on the open unit disk. We call these rigid analytic functions the $\mathbb{Q}_p$-bounded rigid analytic function on the open unit disk. We have that $\mathbb{Z}_p[[x]]$ is the closed unit disk in R:

$$\mathbb{Z}_p[[x]] = \left\{ f = \sum a_n x^n \in R : \sup_n |a_n| \leq 1 \right\}.$$

7.2 Topological rings, modules, and completed tensor product

In this section for completeness, we record the notion of completed tensor product.

Let R be a topological ring and let M, N be topological R-modules whose topologies are determined by basis of open submodules or 0, say $\{M_\alpha\}_{\alpha \in I}$, $\{N_\beta\}_{\beta \in J}$. Then we define the completed tensor product to be

$$M \widehat{\otimes}_R N = \varprojlim_{\alpha,\beta} M/M_\alpha \otimes_R N/N_\beta.$$

Then $M \widehat{\otimes}_R N$ is the completion of the usual tensor product $M \otimes_R N$ with respect to the topology determined by the basis of open submodules of $M \otimes_R N$ given by

$$M_\alpha \otimes_R N + M \otimes_R N_\beta \subset M \otimes_R N$$

as α and β run through I and J.

If A and B are continuous R algebras where the topology of A and B are given by a family of open submodules, then $A \widehat{\otimes}_R B$ as a topological R-algebra is defined by viewing A and B as topological R-modules and using the above construction.

7.3 Hecke characters

In this appendix, we record our definition and conventions for Hecke characters following [23]. Let F be a number field and let $\mathbb{A}_F$ denote the adeles of F. Given an integral ideal $\mathfrak{f}$ of F that factors as $\mathfrak{f} = \prod_{v|\mathfrak{f}} \mathfrak{f}_v$, we define the subgroups $U_\mathfrak{f}, U_{\mathfrak{f},p}, U_{\mathfrak{f},\infty} \subset \mathbb{A}_F^\times$ as

$$U_\mathfrak{f} = \left\{ (x_v)_v \in \mathbb{A}_F^\times : \begin{array}{c} x_v > 0 \text{ if } v \text{ is real} \\ x_v \equiv 1 \bmod \mathfrak{f}_v \text{ if } v \mid \mathfrak{f} \\ x_v \in \mathscr{O}_{F_v}^\times \text{ if } v \nmid \mathfrak{f} \text{ and is finite} \end{array} \right\}$$

$$U_{\mathfrak{f},p} = \left\{ (x_v)_v \in \mathbb{A}_F^\times : \begin{array}{c} x_v > 0 \text{ if } v \text{ is real} \\ x_v \equiv 1 \bmod \mathfrak{f}_v \text{ if } v \mid \mathfrak{f} \\ x_v \in \mathscr{O}_{F_v}^\times \text{ if } v \nmid \mathfrak{f}p \text{ and is finite} \\ x_v = 1 \text{ if } v \mid p \end{array} \right\}$$

$$U_{\mathfrak{f},\infty} = \left\{ (x_v)_v \in \mathbb{A}_F^\times : \begin{array}{c} x_v = 1 \text{ if } v \mid \infty \\ x_v \equiv 1 \bmod \mathfrak{f}_v \text{ if } v \mid \mathfrak{f} \\ x_v \in \mathscr{O}_{F_v}^\times \text{ if } v \nmid \mathfrak{f} \text{ and is finite} \end{array} \right\}$$

Definition 7.3.1. *An element*

$$T = \sum_{\sigma \in \mathrm{Hom}(F,\overline{\mathbb{Q}})} n_\sigma \sigma \in \mathbb{Z}[\mathrm{Hom}(F, \overline{\mathbb{Q}})]$$

is called an ***infinity type*** *of F. An infinity type T induces a group homomorphism*

$$T : F^\times \longrightarrow E^\times$$

$$T(\alpha) = \prod_\sigma \sigma(\alpha)^{n_\sigma}$$

where E is any extension of $\mathbb{Q}$ that contains the values $\prod_\sigma \sigma(\alpha)^{n_\sigma}$ for all $\alpha \in F$.

Definition 7.3.2. *1. Let E be a finite extension of $\mathbb{Q}$, $\mathfrak{f} \subset \mathscr{O}_F$ an ideal, and T an infinity type of F. An* ***E-valued algebraic Hecke character of infinity type T and modulus*** *$\mathfrak{f}$ is a group homomorphism*

$$\chi : \mathbb{A}_F^\times \longrightarrow E^\times$$

such that $U_\mathfrak{f} \subset \ker(\chi)$ and $\chi|_{F^\times} = T$. The smallest $\mathfrak{f}$ with respect to divisibility such that $U_\mathfrak{f} \subset \ker(\chi)$ is called the ***conductor*** *of χ.*

Let χ is an algebraic Hecke character of modulus $\mathfrak{f}$ and $\mathfrak{a}$ an ideal of F such that $(\mathfrak{a}, \mathfrak{f}) = 1$ and that factors as

$$\mathfrak{a} = \prod_{(\mathfrak{p},\mathfrak{a})=1} \mathfrak{p}^{a_\mathfrak{p}}$$

then we define $\chi(\mathfrak{a})$ to be

$$\prod_{(\mathfrak{p},\mathfrak{f})=1} \chi(\pi_\mathfrak{p})^{a_\mathfrak{p}}$$

where $\pi_\mathfrak{p}$ denotes a uniformizer of $F_\mathfrak{p}$.

2. *A p**-adic Hecke character** is a continuous group homomorphism*

$$\chi : \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}_p^\times.$$

*By continuity, there exists an integral ideal $\mathfrak{f}$ of F such that $(\mathfrak{f}, p) = 1$ and $U_{\mathfrak{f},p} \subset \ker(\chi)$. Any $\mathfrak{f}$ for which this is true is called a **modulus** of χ and we say that χ is a p-adic Hecke character of modulus $\mathfrak{f}$. The smallest $\mathfrak{f}$ with respect to divisibility such that $U_{\mathfrak{f},p} \subset \ker(\chi)$ is called the **tame conductor** of χ.*

3. *A **complex Hecke character** is a continuous group homomorphism*

$$\chi : \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times.$$

*By continuity, there exists an integral ideal $\mathfrak{f}$ of F such that $U_{\mathfrak{f},\infty} \subset \ker(\chi)$. Any $\mathfrak{f}$ for which this is true is called a **modulus** of χ and we say χ is a complex Hecke character of modulus $\mathfrak{f}$. The smallest $\mathfrak{f}$ with respect to divisibility such that $U_{\mathfrak{f},\infty} \subset \ker(\chi)$ is called the **conductor** of χ.*

If χ is an algebraic, p-adic, or complex Hecke character and v is a place of F, then we let χ_v denote χ restricted to $F_v^\times \subset \mathbb{A}_F^\times$.

Example 7.3.3. *An example of an algebraic Hecke character that appears multiple times in this thesis is the* ***norm character****, which we denote by* N*. The norm character is defined as*

$$N : \mathbb{A}_F^\times \longrightarrow \mathbb{Q}^\times$$

$$N((x)_v) = \prod_{v\text{-finite}} |x_v|_v^{-1} \prod_{v\text{-real}} sgn(|x_v|_v).$$

It is clear that N *is a group homomorphism and*

$$U_{(1)} = \prod_{v\text{-finite}} \mathscr{O}_{F_v}^\times \subset \ker(\chi)$$

so N *has conductor* (1)*. By construction,* N *has the property that for all nonzero ideals* $\mathfrak{a} \subset \mathscr{O}_F$*,*

$$N(\mathfrak{a}) = |\mathscr{O}_F/\mathfrak{a}|.$$

Furthermore, for $\alpha \in \mathscr{O}_F - \{0\}$*,*

$$N(\alpha) = |\mathscr{O}_F/(\alpha)| \prod_{v\text{-real}} sgn(|\alpha|_v)$$

$$= |N_{F/\mathbb{Q}}(\alpha)| \prod_{v\text{-real}} sgn(|\alpha|_v)$$

$$= \prod_{v\text{-real}} v(\alpha) \prod_{v\text{-complex}} v(\alpha)\overline{v(\alpha)}$$

$$= \prod_{\sigma : F \hookrightarrow \mathbb{C}} \sigma(\alpha)$$

so N has infinity type $T = \sum_{\sigma: F \hookrightarrow \overline{\mathbb{Q}}} \sigma$.

Remarks 7.3.4. The notion of an algebraic Hecke characters was introduced and studied by Weil, who called them characters of type A_0. Complex Hecke characters are also known as grossencharacters or idele class characters, while p-adic Hecke characters are known as p-adic idele class characters. Since we need all three notions of algebraic, p-adic, and complex Hecke characters we introduce and use the definitions given to avoid confusion.

We give a second definition of an algebraic Hecke character that will be used in section 4.4.

Let $\mathfrak{f}$ be an ideal of $\mathscr{O}_F$ and let $\alpha \in F^\times$ be an element such that $((\alpha), \mathfrak{m}) = 1$ and say that $\mathfrak{f}$ factors as

$$\mathfrak{f} = \prod_{i=1}^{m} \mathfrak{p}_i^{f_i}.$$

We define $\alpha \equiv 1 \bmod \mathfrak{f}$ to mean that $\alpha \equiv 1 \bmod \mathfrak{p}_i^{f_i}$ in $\mathscr{O}_{F_{\mathfrak{p}_i}}$ for all i. When $\alpha \in \mathscr{O}_F$ this agrees with the usual definition of congruence.

Let $I(\mathfrak{f})$ denote the group of fractional ideals of F that are coprime with $\mathfrak{f}$. Let

$$P_1(\mathfrak{f}) = \{(\alpha) \in I(\mathfrak{f}) : \alpha \equiv 1 \bmod \mathfrak{f} \text{ and } \alpha \text{ is totally positive}\}$$

The second definition of an algebraic Hecke character is the following. An **E-valued algebraic Hecke character** of modulus $\mathfrak{f}$ and infinity type $T = \sum n_\sigma \sigma$ is a group homomorphism

$$\chi : I(\mathfrak{f}) \longrightarrow E^\times$$

such that for all $\mathfrak{a} \in P_1(\mathfrak{f})$ such that $\mathfrak{a} = (\alpha)$ with $\alpha \equiv 1 \bmod \mathfrak{f}$ and α totally positive

$$\chi((\alpha)) = T(\alpha) = \prod_{\sigma} \sigma(\alpha)^{n_\sigma}.$$

Given an E-valued algebraic Hecke character of modulus $\mathfrak{f}$ and infinity type T, χ using the second definition, we get an algebraic Hecke character of the same modulus and infinity type, $\chi_{\mathbb{A}}$ using the first definition by defining $\chi_{\mathbb{A}}$ to be the unique group homomorphism

$$\chi_{\mathbb{A}} : A_F^\times \longrightarrow E^\times$$

such that

(i) For all primes $\mathfrak{p} \in I(\mathfrak{f})$, $\chi_{\mathbb{A}}|_{\mathscr{O}_{F_\mathfrak{p}}^\times} = 1$ and $\chi_{\mathbb{A}}(\pi_\mathfrak{p}) = \chi(\mathfrak{p})$ for any uniformizer in $F_\mathfrak{p}$.

(ii) $\chi_{\mathbb{A}}|_{F^\times} = T$.

(iii) $U_\mathfrak{f} \subset \ker(\chi_{\mathbb{A}})$. This gives a one-to-one correspondence between E-valued algebraic Hecke characters of modulus $\mathfrak{f}$ and infinity type T using the first and second definitions.

Remarks 7.3.5. We make a remark about defining algebraic Hecke characters using the second definition. Let $\mathfrak{f}$ be an integral ideal of F and let χ_0 be a group homomorphism

$$\chi_0 : P_1(\mathfrak{f}) \longrightarrow \overline{\mathbb{Q}}^\times$$

such that for all $\mathfrak{a} = (\alpha) \in P_1(\mathfrak{f})$ where $\alpha \equiv 1 \bmod \mathfrak{f}$ and α totally positive

$$\chi_0((\alpha)) = T(\alpha)$$

for some infinity type T. We claim that we can always extend χ_0 to an E-valued Hecke character of modulus $\mathfrak{f}$ and infinity type T where E is some finite extension of $\mathbb{Q}$. Indeed, we consider the short exact sequence

$$1 \longrightarrow P_1(\mathfrak{f}) \longrightarrow I(\mathfrak{f}) \longrightarrow I(\mathfrak{f})/P_1(\mathfrak{f}) \longrightarrow 1$$

Applying $\mathrm{Hom}(\cdot, \overline{\mathbb{Q}}^{\times})$ to this short exact sequence the long exact sequence becomes

$$1 \to \mathrm{Hom}(I(\mathfrak{f})/P_1(\mathfrak{f}), \overline{\mathbb{Q}}^{\times}) \to \mathrm{Hom}(I(\mathfrak{f}), \overline{\mathbb{Q}}^{\times}) \to \mathrm{Hom}(P_1(\mathfrak{f}), \overline{\mathbb{Q}}^{\times}) \to 1$$

since $\overline{\mathbb{Q}}^{\times}$ is divisible so $\mathrm{Ext}^1(\cdot, \overline{\mathbb{Q}}^{\times}) = 0$. Hence the there exists a group homomorphism $\chi \in \mathrm{Hom}(I(\mathfrak{f}, \overline{\mathbb{Q}}^{\times})$ such that $\chi|_{P_1(\mathfrak{f})} = \chi_0$. Furthermore we see that any two extensions of χ_0 from $P_1(\mathfrak{f})$ to $I(\mathfrak{f})$ differ by a character of the ray class group $I(\mathfrak{f})/P_1(\mathfrak{f})$.

Given an E-valued Hecke character, χ of infinity type T and modulus $\mathfrak{f}$ as well as embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ we obtain p-adic and complex Hecke characters χ_p and χ_∞ which are defined as follows. Let E_p be the completion of the image of E in $\mathbb{C}_p$ and E_∞ be the completion of E in $\mathbb{C}$ under the embeddings ι_p and ι_∞. Define

$$\chi_p : \mathbb{A}_F^{\times}/F^{\times} \longrightarrow E_p^{\times} \subset \mathbb{C}_p^{\times}$$

at places v of F not dividing p as χ, so $\chi_p|_{F_v^{\times}} = \chi|_{F_v^{\times}}$. At places above p we define χ_p to be the group homomorphism

$$\chi_p : (F \otimes \mathbb{Q}_p)^{\times} \longrightarrow E_p^{\times} \subset \mathbb{C}_p^{\times}$$

$$\chi_p(\alpha \otimes 1) = \chi(\alpha)/\iota_p(T(\alpha)).$$

Since the image of $F^\times$ in $(F \otimes \mathbb{Q}_p)^\times$ is dense this defines χ_p on $(F \otimes \mathbb{Q}_p)^\times$. We do something similar for χ_∞. Define

$$\chi_\infty : \mathbb{A}_F^\times / F^\times \longrightarrow E_\infty^\times \subset \mathbb{C}^\times$$

at the places v of F not dividing ∞ as χ, so $\chi_\infty|_{F_v^\times} = \chi|_{F_v^\times}$. At places above ∞ we define χ_∞ to be the group homomorphism

$$\chi_\infty : (F \otimes \mathbb{R})^\times \longrightarrow E_\infty^\times \subset \mathbb{C}^\times$$

$$\chi_\infty(\alpha \otimes 1) = \chi(\alpha)/\iota_\infty(T(\alpha)).$$

Since the image of $F^\times$ in $(F \otimes \mathbb{R})^\times$ is dense this defines χ_∞ on $(F \otimes \mathbb{R})^\times$.

Given an algebraic Hecke character χ when it is clear from context when we are considering χ_p or χ_∞, we may drop the subscripts p and ∞. Given a p-adic (or complex) Hecke character ψ we may say by abuse of language that ψ is an algebraic Hecke character of infinity type T if there exists an algebraic Hecke character χ of infinity type T such that $\psi = \chi_p$ (or $\psi = \chi_\infty$).

If ψ is a complex Hecke character such that $\psi = \chi_\infty$ for a algebraic Hecke character then the conductor of ψ and χ are the same. If ψ is a p-adic Hecke character of tame conductor $\mathfrak{m}$ such that $\psi = \chi_p$ for an algebraic Hecke character of conductor $\mathfrak{f}$, then $\mathfrak{f}$ and $\mathfrak{m}$ will agree at the primes of F not dividing p. By definition no primes of F above p will divide $\mathfrak{m}$, while $\mathfrak{f}$ may be divisible by primes above p.

7.4 Hecke L-functions

In this appendix, we record the functional equation of the complex L-function associated to a complex Hecke character. A reference for this section is [22].

Let

$$\chi : \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times$$

be a complex Hecke character of infinity type T and conductor $\mathfrak{f} = \prod_{v\text{-finite}} \mathfrak{p}_v^{a_v}$. For each finite place v of F fix a uniformizer π_v of $\mathscr{O}_{F_v}$. Let $\mathfrak{d} = \prod_{v\text{-finite}} \mathfrak{p}_v^{d_v}$ be the different of $F/\mathbb{Q}$ and let d_F be the discriminant of F.

Definition 7.4.1. *We define* ***(primitive) complex** L**-function*** *of χ for $Re(s) > 1$, by the Euler product*

$$L(\chi, s) = \prod_{v \nmid \infty} L_v(\chi, s)$$

where the product is over all finite places of F and

$$L_v(\chi, s) = \begin{cases} (1 - \chi(\pi_v) N v^{-s})^{-1} & \text{if } v \nmid \mathfrak{f} \\ 1 & \text{else.} \end{cases}$$

We note that for $v \nmid \mathfrak{f}$, $\chi_v|_{\mathscr{O}_{F_v}^\times} = 1$ so $L_v(\chi, s)$ does not depend on the choice of uniformizer π_v.

We will state the functional equation of $L(\chi, s)$. To do this, we define the Euler factors of χ at the infinite places of F and the local root numbers of χ at all the places of F.

Let v be a real place of F. Then $\chi_v : \mathbb{R}^\times \to \mathbb{C}^\times$ must be of the form

$$\chi_v(t) = |t|^{s_0}(t/|t|)^m$$

for some $s_0 \in \mathbb{C}$ and $m \in \{0, 1\}$. We define the Euler factor of χ at v to be

$$L_v(\chi, s) = \pi^{-(s+s_0+m)}\Gamma((s + s_0 + m)/2)$$

and the local root number of χ at v to be

$$W(\chi_v) = i^{-m}.$$

Let v be a complex place of F. We then identify F_v with $\mathbb{C}^\times$ (there are two ways to do this) and view χ_v as a character of $\mathbb{C}^\times$: $\chi_v : \mathbb{C}^\times \to \mathbb{C}^\times$. Then χ_v is of the form $\chi_v(z) = |z|^{2s_0}(z/|z|)^m$ for some $s_0 \in \mathbb{C}$ and $m \in \mathbb{Z}$. We define the Euler factor of χ at v to be

$$L_v(\chi, s) = 2(2\pi)^{-(s+s_0+|m|)/2}\Gamma(s + s_0 + |m|/2)$$

and the local root number of χ at v to be

$$W(\chi_v) = i^{-|m|}.$$

The root number and Euler facts, $W(\chi_v)$ and $L_v(\chi, s)$ do not depend on the choice of identification of F_v with $\mathbb{C}^\times$.

At finite places $v \nmid \mathfrak{f}$ of F, define the local root number to be

$$W(\chi_v) = \frac{\chi(\pi_v^{d_v})}{|\chi(\pi_v^{d_v})|}$$

so $W(\chi_v) = 1$ if $v \nmid \mathfrak{d}\mathfrak{f}\infty$.

Define the local root number of χ at a place $v \mid \mathfrak{f}$ to be

$$W(\chi_v) = \frac{\chi_v(\pi_v^{a_v+d_v})}{|\chi_v(\pi_v^{a_v+d_v})|} \frac{1}{Nv^{a_v/2}} \sum_{u \in (\mathscr{O}_{F_v}/v^{a_v})^\times} \chi_v^{-1}(u) \exp(2\pi i Tr_{F_v/\mathbb{Q}_p}(u/\pi_v^{a_v+d_v}))$$

where $Tr_{F_v/\mathbb{Q}_p}(u/\pi_v^{a_v+d_v})$ is viewed as an element of $\mathbb{Q}/\mathbb{Z}$ via the maps $\mathbb{Q}_p \mapsto \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Q}/\mathbb{Z}$. Define the local root number of χ at a place $v \nmid \mathfrak{f}$ to be 1 (we are following Rohrlich).

Let c be the real number such that we have the relation $\chi/|\chi| = N^c$ where N is the norm character (see example above) viewed as a complex Hecke character, and define $k = 2c + 1$.

Define the completed L-function of χ for $Re(s) > 1$ to be

$$\Lambda(\chi, s) = (|d_F| N_{K/\mathbb{Q}}\mathfrak{f})^{s/2} L(\chi, s) \prod_{v|\infty} L_v(\chi, s)$$

and the global root number to be

$$W(\chi) = \prod_v W(\chi_v).$$

Then $\Lambda(\chi, s)$ has a meromorphic (analytic if $\chi \neq 1$) continuation to all of $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(\chi, k - s) = W(\chi)\Lambda(\overline{\chi}, s).$$

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