# Research Statement

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My research interests lie primarily in extremal combinatorics. A typical problem in this area concerns the extremal values of a graph parameter in a specific set of graphs. For example, the Turán type problems concern the maximum number of edges of an $n$-vertex graph without a certain graph $F$. Much of my research concern the minimum independence number of an $n$-vertex hypergraph forbidding certain structures, which are closely related to the off-diagonal hypergraph Ramsey numbers.

A *hypergraph* $H$ is a pair $(V(H), E(H))$, where $V = V(H)$ is a set of *vertices*, and $E = E(H)$ is a set of subsets of $V$ called *edges*. An $r$*-uniform hypergraph* (or $r$-graph for short) is a hypergraph whose edges have size $r$. When $r = 2$, this gives the usual definition of graph. Let $v$ be a vertex of $H$. The *degree* of $v$ in $H$, denoted by $d(v)$, is the number of edges containing $v$. A hypergraph is $d$*-regular* if all of its vertices have degree $d$. An *independent set* of $H$ is a set of vertices which does not contain any edge of $H$. The size of the largest independent set in $H$ is called the *independence number* of $H$, usually denoted by $\alpha(H)$.

The classic Turán's Theorem [15] says if the number of edges in a graph is small enough, then the graph must contain an independent set of a certain size. More accurately, a graph $G$ on $n$ vertices with average degree $d$ has independence number $\alpha(G) \geq n/(d + 1)$, with equality only for disjoint union of cliques $K_{d+1}$.

In general, we try to solve the following type of problems:

**Problem.** *Let $\mathcal{H}$ be a family of hypergraphs, determine*

$$\min_{G\in\mathcal{H}} \alpha(G).$$

In the rest of this document, I will introduce some of my results and some conjectures.

## 1 Hypergraphs with large girth and fixed average degree

For triangle-free graphs $G$ on $n$-vertices with average degree $d$, Ajtai, Komlós and Szemerédi [2] showed that $\alpha(G) = \Omega(\log d \cdot n/d)$, which is later improved by Shearer [14], who showed that $\alpha(G) \geq (1 - o(1)) \log d \cdot n/d$.

The *girth* of a graph is the length of a shortest cycle in the graph. For graphs with high girth, Lauer and Wormald [8] and Gamarnik and Goldberg [6] showed that there exist a function $\delta = \delta(g, d)$ such that $lim_{g\to\infty}\delta(g, d) = 0$ and if $G$ is an $n$-vertex $d$-regular graph of girth $g$, then

$$\alpha(G) \geq (f(d) - \delta)n, \qquad f(d) = \frac{1}{2}(1 - (d-1)^{-\frac{2}{d-2}}). \tag{1}$$

Note that $f(d) \sim \log d/d$ as $d \to \infty$.

To define *girth* in hypergraph, we first need to define what is a *cycle* in hypergraph. There are many different ways to define cycle in hypergraph–see, e.g., a survey by Verstraëte [16]. Here we chose to work with the *Berge-cycle*. For $k \geq 3$, a *Berge $k$-cycle* is a hypergraph with $k$ edges $e_1, e_2, \ldots, e_k$ and there exist $k$ distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_k, v_1\} \in e_1, \{v_1, v_2\} \in e_2, \ldots, \{v_{k-1}, v_k\} \in e_k$. When $k = 2$, this corresponds to $v_1, v_2 \in e_1 \cap e_2$.

The *girth* of a hypergraph is the smallest $g$ such that the hypergraph contains a Berge $g$-cycle. In particular, the girth of a non-linear hypergraph is 2. For $(r + 1)$-uniform hypergraphs $G$ with girth $g \geq 5$, Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] established the following lower bounds.

$$\alpha(G) \geq 0.36 \cdot 10^{-\frac{5}{r}} \left(\frac{\log d}{rd}\right)^{\frac{1}{r}} n. \tag{2}$$

Based on this result, Duke, Lefmann and Rödl [4] showed that the same bound (with different constant) holds for linear hypergraphs.

The author and Verstraëte [13] extended the ideas of Gamarnik and Goldberg [6] to hypergraphs. First, it is convenient to define the following: Let $u(d,r)$ be the only positive real number that satisfies the following equation:

$$\sum_{n\geq 0}\binom{n+d-2}{d-2}\frac{u(d,r)^{rn+1}}{rn+1}=1. \tag{3}$$

**Theorem 1.** *For any integers $r\geq 1$, $d\geq 2$ and $g\geq 4$, let $G$ be an $(r+1)$-uniform $d$-regular hypergraph with $n$ vertices and girth $g$. Let*

$$f(d,r)=u(d,r)-\frac{u(d,r)^{r+1}}{r+1}. \tag{4}$$

*Then there exists a function $\epsilon=\epsilon(g,d,r)$ such that $lim_{g\to\infty}\epsilon(g,d,r)=0$ and*

$$\alpha(G)\geq (f(d,r)-\epsilon)n. \tag{5}$$

For $r=1$, this coincides with (1). Indeed, $f(d,r)\sim(\log d/rd)^{1/r}$ as $d\to\infty$, and so if $g$ is large enough relative to $d$, then this slightly improves the constant in (2) asymptotically as $d\to\infty$.

## 2 Hypergraph Ramsey numbers for cycles versus cliques

Let $\mathcal{F}$ be a family of $r$-graphs and $t\geq 1$. The Ramsey numbers $R(t,\mathcal{F})$ denote the minimum $n$ such that every $n$-vertex $r$-graph contains either a hypergraph in $\mathcal{F}$ or an independent set of size $t$. If we let $\mathcal{H}$ be the set of $n$-vertex $\mathcal{F}$-free $r$-graphs, then Question  is equivalent to evaluating $R(t,\mathcal{F})$.

For $k\geq 2$, an $r$-uniform *loose $k$-cycle* is an $r$-graph, denoted $C_k^r$, with edges $e_1,e_2,\dots,e_k$ (Indices are modulo $k$) and there are $k$ distinct vertices $v_1,v_2,\dots,v_k$ such that $e_i\cap e_{i+1}=v_i$, and for any $i$,$j$ such that $i-j\neq\pm 1$, $e_i\cap e_j=\emptyset$.

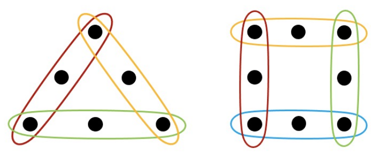

Figure 1: Loose cycles $C_3^3$ and $C_4^3$.

For $r\geq 3$, Kostochka, Mubayi and Verstraëte [7] showed that $R(t,C_3^r)=t^{3/2+o(1)}$. They proposed The following conjecture:

**Conjecture 1.** *For $r,k\geq 3$,*

$$R(t,C_k^r)=t^{\frac{k}{k-1}+o(1)}. \tag{6}$$

Towards this conjecture, Méroueh [10] showed $R(t,C_k^3)=O(t^{1+1/\lfloor(k+1)/2\rfloor})$ for $k\geq 3$. In particular, this gives $R(t,C_4^3)=O(t^{3/2})$. Myself and Verstraëte [11] showed the following:

**Theorem 2.**

$$R(t,C_4^3)\leq t^{\frac{4}{3}+o(1)}. \tag{7}$$

Note that this upper bound matches Conjecture 1. We believe that the proof of Theorem 2 above should be extended to $r\geq 4$ and even values of $k\geq 6$.

For $k\geq 2$, recall that a *Berge $k$-cycle* is a family of sets $e_1,e_2,\dots,e_k$ such that $e_1\cap e_2,e_2\cap e_3,\dots,e_k\cap e_1$ has a system of distinct representatives. A Berge cycle is *non-trivial* if $e_1\cap e_2\cap\cdots\cap e_k=\emptyset$. Let $\hat{\mathcal{B}}_k^r$ denote the family of non-trivial $r$-uniform Berge $k$-cycles. In support of Conjecture 1, myself and Verstraëte [12] proved the following theorem for non-trivial Berge cycles of even length:

**Theorem 3.** *For $k \geq 2$, and $t$ large enough,*

$$R(t, \hat{\mathcal{B}}^3_{2k}) \leq t^{\frac{2k}{2k-1}+o(1)}. \tag{8}$$

Erdős and Simonovits [5] conjectured that there exists an $n$-vertex graph of girth more than $2k$ with $\Theta(n^{1+1/k})$ edges. This notoriously difficult conjecture remains open, except when $k \in \{2, 3, 5\}$, largely due to the existence of generalized polygons [3, 9, 16]. Myself and Verstraëte [12], via random block construction, proved the following theorem relating this conjecture to lower bounds on Ramsey numbers for non-trivial Berge cycles:

**Theorem 4.** *Let $k \geq 2$, $r \geq 3$. If the Erdős-Simonovits conjecture is true, then*

$$R(t, \hat{\mathcal{B}}^r_k) \geq t^{\frac{k}{k-1}-o(1)}. \tag{9}$$

This shows that if the Erdős-Simonovits Conjecture is true, then Theorem 3 is tight up to a $t^{o(1)}$ factor. We believe that Theorem 3 should extend to odd values of $k$ and all $r \geq 3$:

**Conjecture 2.** *For all $r, k \geq 3$,*

$$R(t, \hat{\mathcal{B}}^r_k) \leq t^{\frac{k}{k-1}+o(1)}. \tag{10}$$

Note that this conjecture supports Conjecture 1. It can be viewed as a possible intermediate step.

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