Free functions with Symmetry

J. E. Pascoe

University of California, San Diego

jpascoe@math.ucsd.edu

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Matrix universe

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That is,

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Today we will be discussing function theory on $\mathcal{M}^d$.

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D is closed with respect to direct sums

$X = (X_1, \ldots, X_d) \in D$ and $Y = (Y_1, \ldots, Y_d) \in D$ if and only if

$$\begin{pmatrix} X & \\ & Y \end{pmatrix} = \left(\begin{pmatrix} X_1 & \\ & Y_1 \end{pmatrix}, \ldots, \begin{pmatrix} X_d & \\ & Y_d \end{pmatrix} \right) \in D.$$

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D is closed with respect to unitary similarity

If $X \in D \bigcap M_n^d$, $U \in U_n$, then

$$U^* X U = (U^* X_1 U, \ldots, U^* X_d U) \in D.$$

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f respects similarity

If $X, S^{-1}XS \in D \bigcap M_n^d, S \in GL_n$, then

$$f(S^{-1}XS) = S^{-1}f(X)S.$$

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- Gelfand et al and Diaconis et al studied the combinatorial and Hopf algebra structure of free symmetric polynomials during the 90s and 2000s.
- Recently, Agler and Young analyzed symmetric free functions in two variables on the domain,

$$\mathcal{B}^2 = \{(X_1, X_2) \in \mathcal{M}^2 | \|X_1\| < 1, \|X_2\| < 1\},$$

  and developed a theory of systems theoretic realizations.

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For example, we will establish a theory which allows us to understand free functions satisfying the following identities:

- Even: $p(x_1, x_2) = p(-x_1, -x_2)$,
- Cyclically symmetric: $p(x_1, x_2, x_3) = p(x_2, x_3, x_1)$,

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For example, we will establish a theory which allows us to understand free functions satisfying the following identities:

- Even: $p(x_1, x_2) = p(-x_1, -x_2)$,
- Cyclically symmetric: $p(x_1, x_2, x_3) = p(x_2, x_3, x_1)$,
- Symmetric: $p(x_1, x_2) = p(x_2, x_1)$.

Row ball

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$$\mathcal{C}^d = \{(X_1, \ldots, X_d) \in \mathcal{M}^d | \|X_1 X_1^* + \ldots + X_d X_d^*\| < 1\}.$$

The row ball is so called because

$$\|[X_1 \quad \ldots \quad X_d]\| < 1$$

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$$p(x_1, x_2) = 1 + 3x_1x_2 - 7x_1x_1 - x_2x_1x_2x_2$$

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However,

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is not even.

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An even free polynomial in two variables can be written *uniquely* as a free polynomial in four variables of the functions:

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$$p(x_1, x_2) = 1 + 3x_1x_2 - 7x_1x_1 - x_2x_1x_2x_2$$

can be written in terms of the u_i as

$$1 + 3u_2 - 7u_1 - u_3u_4.$$

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form a basis for the ring of even free polynomials.
So, the ring of even free polynomials in two variable is isomorphic to the ring of free polynomials in four variables.

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In fact, $\Phi : \mathcal{C}^2 \to \mathcal{C}^4$, and for any even free function $f : \mathcal{C}^2 \to \mathcal{C}^1$ there exists a unique free function $\hat{f} : \mathcal{C}^4 \to \mathcal{C}^1$ such that $f = \hat{f} \circ \Phi$.

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This is a bit peculiar, since $\Phi(\mathcal{C}^2)$ inside $\mathcal{C}^4$ is ostensibly a very thin set.

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More generally, this should be done with multivariate dilation theory, which on the row ball was essentially established by Gelu Popescu in the early 90s.

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Theorem (Cushing, P., Tully-Doyle)

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We say a free function is invariant if $p(x) = p(\pi(g)x)$.
For example, for even polynomials, the group is $\mathbb{Z}_2$ and

$$\pi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pi(1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Let G be a finite group. Let $\pi : G \to \mathcal{U}_d$ be a unitary representation. There exists a basis for the ring of invariant free functions $(u_\lambda)_\lambda$ such that the map Φ on $\mathcal{C}^d$ defined by the formula

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- *Moreover,*

$$\sup_{X \in \mathcal{C}^d} \|p(X)\| = \sup_{U \in \mathcal{C}^\Lambda} \|\hat{p}(U)\|.$$

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If we define an inner product of free polynomials as the inner product of their vectors of coefficients, we get (using some representation theory) that there is a basis of generators u_λ for the ring of invariant free polynomials which is not only orthogonal, but also satisfies $u_\lambda p \perp u_\mu q$ for arbitrary p and q.

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The geometry of the basis then essentially makes the results appear when coupled with dilation theory. (Fock space methods)

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We can reinterpret the main theorem as an isomorphism of function algebras analogous to Wolf's theorem. Let $H_\pi^\infty(\mathcal{C}^d)$ be the Banach algebra of bounded invariant free functions for π on $\mathcal{C}^d$.

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Corollary (Cushing, P., Tully-Doyle)

There is a Λ (perhaps infinite) such that

$$H^\infty_\pi(\mathcal{C}^d) \cong H^\infty(\mathcal{C}^\Lambda)$$

as Banach algebras.

Symmetric functions in two variables

Theorem (Cushing, P., Tully-Doyle)

Let

$$\Phi(X_1, X_2) = (A, B^2, BAB, \ldots, BA^nB, \ldots)$$

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If B^{-1} exists, we can write each of the basis elements in terms of $A, B^{-1}AB$ and B^2.
That is, if we consider functions on a domain where B^{-1} exists, the function theory will drastically change.

Cyclically symmetric functions in three variables

We now consider functions that satisfy

$$p(x_1, x_2, x_3) = \sigma \cdot p(x_1, x_2, x_3) = p(x_2, x_3, x_1).$$

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Let ω be a nontrivial third root of unity. Consider the following three linear polynomials:

$$u_0(X_1, X_2, X_3) = \frac{X_1 + X_2 + X_3}{\sqrt{3}}, u_1(X_1, X_2, X_3) = \frac{X_1 + \omega X_2 + \overline{\omega} X_3}{\sqrt{3}}$$

$$u_{-1}(X_1, X_2, X_3) = \frac{X_1 + \overline{\omega} X_2 + \omega X_3}{\sqrt{3}}$$

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$$p(x_1, x_2, x_3) = \sigma \cdot p(x_1, x_2, x_3) = p(x_2, x_3, x_1).$$

Let ω be a nontrivial third root of unity. Consider the following three linear polynomials:

$$u_0(X_1, X_2, X_3) = \frac{X_1 + X_2 + X_3}{\sqrt{3}}, u_1(X_1, X_2, X_3) = \frac{X_1 + \omega X_2 + \overline{\omega} X_3}{\sqrt{3}}$$

$$u_{-1}(X_1, X_2, X_3) = \frac{X_1 + \overline{\omega} X_2 + \omega X_3}{\sqrt{3}}$$

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It turns out a basis for the cyclically symmetric functions in three variables is the set of products $\prod_{j=1}^{n} u_{i_j}$ such that $\sum_{j=1}^{n} i_j \equiv_3 0$ such that for no $m < n$, $\sum_{j=1}^{m} i_j \equiv_3 0$.

Some domains have small bases

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In general, at least for abelian and solvable groups, it appears that on certain domains you can write all functions in the ring of invariant free functions in terms of some finite collection.

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$$g(z) = \sum_{n=0}^{\infty} g_n z^n = 1 - |G| \frac{\prod_{\sigma \in \mathcal{C}_G}(1 - \chi(\sigma)z)}{\sum_{\tau \in \mathcal{C}_G} \# C_\tau \prod_{\sigma \neq \tau}(1 - \chi(\sigma)z)}$$

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where $\# C_\tau$ denotes the number of elements in the conjugacy class of τ. Namely, the number of generators of a given degree is independent of the choice of basis.

Other topics

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Free functions with Symmetry

J. E. Pascoe

University of California, San Diego

jpascoe@math.ucsd.edu

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