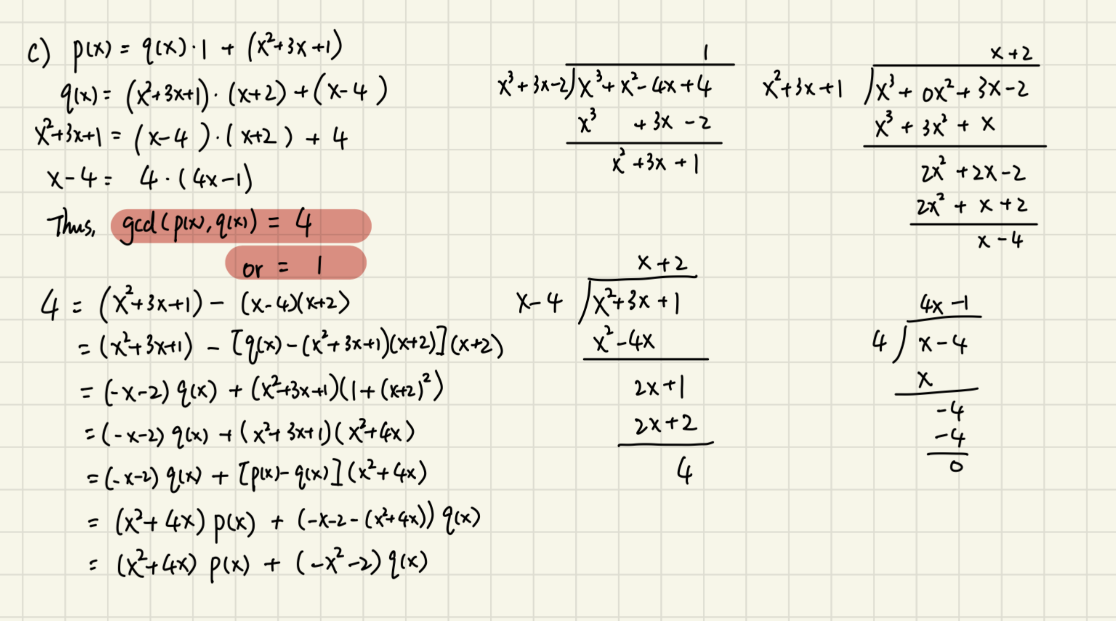
17.5.4(d):

(c) $\gcd(p(x), q(x)) = 1 = p(x)(4x^2 + x) + q(x)(x^2 + 2)$
Process:

$$\begin{aligned} p(x) &= q(x)(1) + (x^2 + 3x + 1) \\ q(x) &= (x^2 + 3x + 1)(x + 2) + (x + 1) \\ (x^2 + 3x + 1) &= (x + 1)(x + 2) + 4 \\ (x + 1) &= 4(4x + 4) + 0 \end{aligned}$$

Rewriting:

$$\begin{aligned} 4 &= (x^2 + 3x + 1) - (x + 1)(x + 2) \\ &= p(x) - q(x) - (q(x) - (x^2 + 3x + 1)(x + 2))(x + 2) \\ &= p(x) - q(x)(x + 3) + (x^2 + 3x + 1)(x + 2)^2 \\ &= p(x) - q(x)(x + 3) + (p(x) - q(x))(x + 2)^2 \\ &= p(x)(1 + (x + 2)^2) + q(x)(-x - 3 - (x + 2)^2) \\ &= p(x)(x^2 + 4x) + q(x)(-x^2 - 2) \\ 1 &= p(x)(4x^2 + x) + q(x)(x^2 + 2) \end{aligned}$$



c) $p(x) = q(x) \cdot 1 + (x^2+3x+1)$

$q(x) = (x^2+3x+1) \cdot (x+2) + (x-4)$

$x^2+3x+1 = (x-4) \cdot (x+2) + 4$

$x-4 = 4 \cdot (4x-1)$

Thus, $\gcd(p(x), q(x)) = 4$

or $= 1$

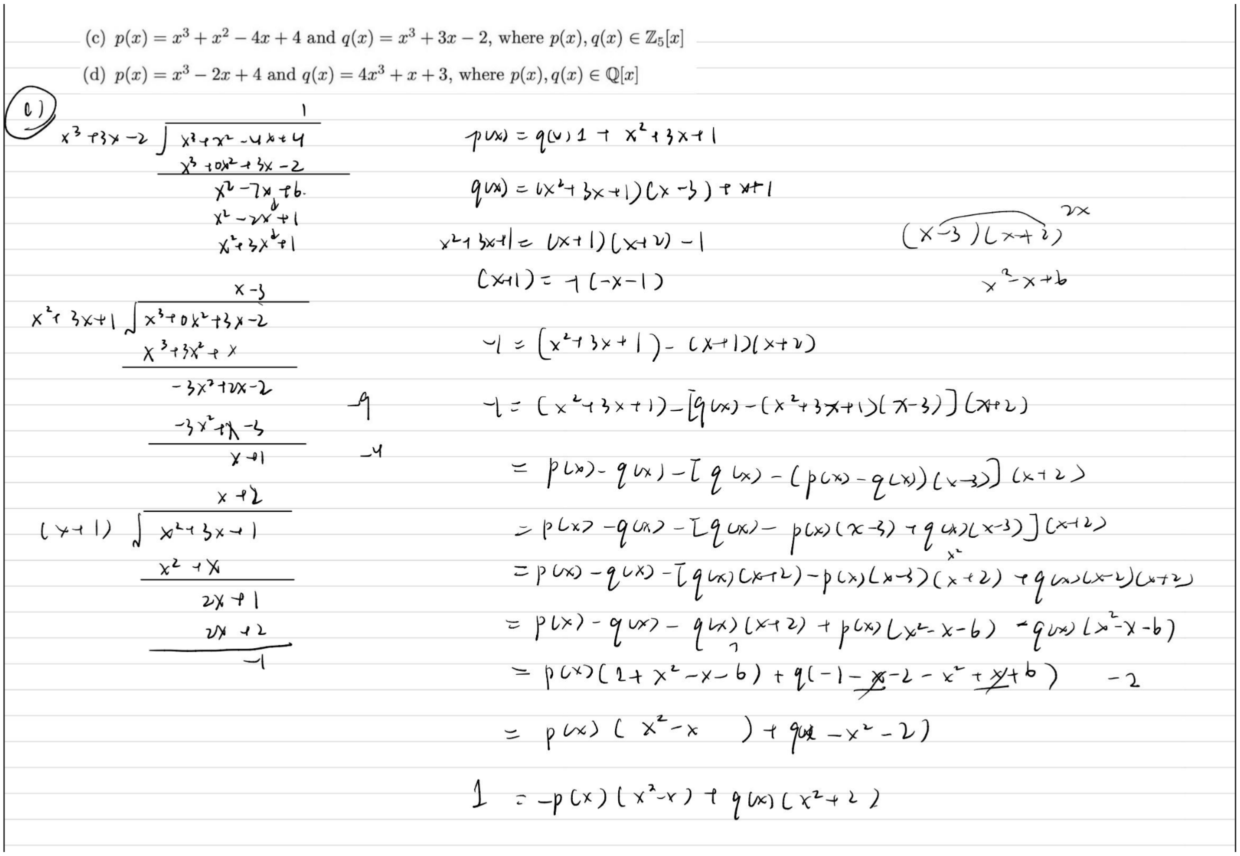
$$\begin{aligned} 4 &= (x^2+3x+1) - (x-4)(x+2) \\ &= (x^2+3x+1) - [q(x) - (x^2+3x+1)(x+2)](x+2) \\ &= (-x-2)\, q(x) + (x^2+3x+1)(1+(x+2)^2) \\ &= (-x-2)\, q(x) + (x^2+3x+1)(x^2+4x) \\ &= (-x-2)\, q(x) + [p(x) - q(x)](x^2+4x) \\ &= (x^2+4x)\, p(x) + (-x-2-(x^2+4x))\, q(x) \\ &= (x^2+4x)\, p(x) + (-x^2-2)\, q(x) \end{aligned}$$

$$\begin{array}{rl} & \qquad\qquad\quad 1 \\ x^3+3x-2 \,\big) & \overline{x^3+x^2-4x+4} \\ & \underline{x^3 \qquad +3x-2} \\ & \quad x^2+3x+1 \end{array}$$

$$\begin{array}{rl} & \qquad\qquad\quad x+2 \\ x^2+3x+1 \,\big) & \overline{x^3+0x^2+3x-2} \\ & \underline{x^3+3x^2+x} \\ & \quad 2x^2+2x-2 \\ & \quad \underline{2x^2+x+2} \\ & \qquad\quad x-4 \end{array}$$

$$\begin{array}{rl} & \qquad x+2 \\ x-4 \,\big) & \overline{x^2+3x+1} \\ & \underline{x^2-4x} \\ & \quad 2x+1 \\ & \quad \underline{2x+2} \\ & \qquad 4 \end{array}$$

$$\begin{array}{rl} & 4x-1 \\ 4 \,\big) & \overline{x-4} \\ & \underline{x} \\ & -4 \\ & \underline{-4} \\ & 0 \end{array}$$



(c) $p(x) = x^3 + x^2 - 4x + 4$ and $q(x) = x^3 + 3x - 2$, where $p(x), q(x) \in \mathbb{Z}_5[x]$

(d) $p(x) = x^3 - 2x + 4$ and $q(x) = 4x^3 + x + 3$, where $p(x), q(x) \in \mathbb{Q}[x]$

c)

$$x^3+3x-2 \,\overline{\big)\, x^3+x^2-4x+4}^{\;1}$$
$$x^3+0x^2+3x-2$$
$$x^2-7x+6.$$
$$x^2-2x+1$$
$$x^2+3x+1$$

$$x^2+3x+1 \,\overline{\big)\, x^3+0x^2+3x-2}^{\;x-3}$$
$$x^3+3x^2+x$$
$$-3x^2+2x-2 \qquad -9$$
$$-3x^2+x-3$$
$$x+1 \qquad -4$$

$$(x+1) \,\overline{\big)\, x^2+3x+1}^{\;x+2}$$
$$x^2+x$$
$$2x+1$$
$$2x+2$$
$$-1$$

$p(x) = q(x)1 + x^2+3x+1$

$q(x) = (x^2+3x+1)(x-3) + x+1$

$x^2+3x+1 = (x+1)(x+2) - 1$

$(x+1) = 1(-x-1)$

$(x-3)(x+2)$ $2x$

$x^2-x+6$

$-1 = (x^2+3x+1) - (x+1)(x+2)$

$-1 = (x^2+3x+1) - [q(x) - (x^2+3x+1)(x-3)](x+2)$

$= p(x) - q(x) - [q(x) - (p(x)-q(x))(x-3)](x+2)$

$= p(x) - q(x) - [q(x) - p(x)(x-3) + q(x)(x-3)](x+2)$

$= p(x) - q(x) - [q(x)(x+2) - p(x)(x-3)(x+2) + q(x)(x-2)(x+2)$

$= p(x) - q(x) - q(x)(x+2) + p(x)(x^2-x-6) - q(x)(x^2-x-6)$

$= p(x)(2+x^2-x-6) + q(-1-\cancel{x}-2-x^2+\cancel{x}+6)$ $-2$

$= p(x)(x^2-x) + q(x)(-x^2-2)$

$1 = -p(x)(x^2-x) + q(x)(x^2+2)$

In this question, some students calculated the wrong gcd.

17.5.12:

12.

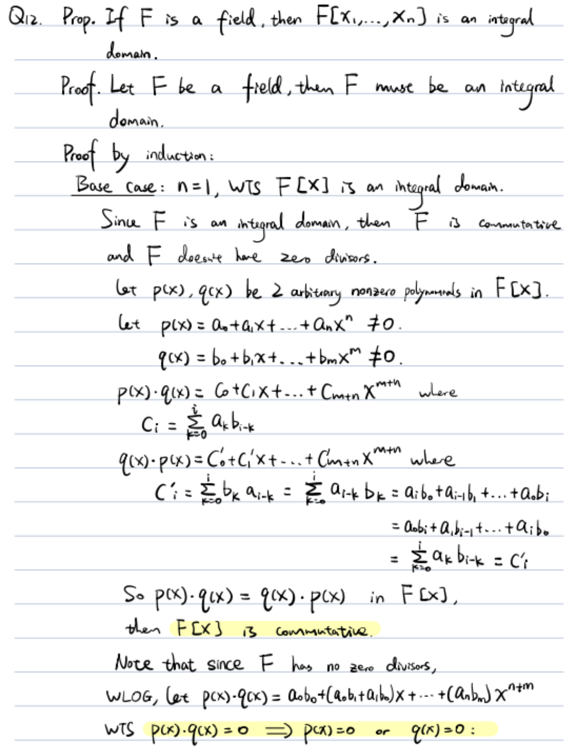
**Proposition 2.** *If $F$ is a field, then $F[x_1, ..., x_n]$ is an integral domain.*

*Proof.* Suppose $F$ is a field.
We will proceed with proof by induction on $n$.
**Base Case**:
We know that $F[x]$ is an integral domain as $F$ is an integral domain, and Proposition 17.4 gives us that a polynomial ring over an integral domain is an integral domain.
**Inductive Step**:
Suppose for some $n \geq 1$ that $F[x_1, ..., x_n]$ is an integral domain.
Let $R = F[x_1, ..., x_n]$.
Then, we know that $F[x_1, ..., x_{n+1}] \cong F[x_1, ..., x_n][x_{n+1}] = R[x_{n+1}]$.
Furthermore, Proposition 17.4 gives us that $R[x_{n+1}]$ is an integral domain because $R$ is an integral domain by the inductive hypothesis.
Hence, we have that $F[x_1, ..., x_{n+1}]$ is an integral domain.
Therefore, we have shown by induction on $n$ that $F[x_1, ..., x_n]$ is an integral domain for all $n \geq 1$. ☐

12.

**Proposition.** *If $F$ is a field, then $F[x_1, ..., x_n]$ is an integral domain.*

*Proof.* Let F be a field.
We will show that $F[x_1, ..., x_n]$ is an integral domain for all $n \in \mathbb{N}$.
By Theorem 17.4, we know if F is an integral domain, then $F[x]$ is an integral domain.
Thus, **our base case:**
when n=1, $F[x_1]$ is an integral domain since F is a field, i.e. F is an integral domain.
When n=2, $F[x_1, x_2]$ is also an integral domain as $(F[x_1])[x_2]$, the ring of polynomials in two indeterminates $x_1$ and $x_2$ with coefficient F as what's in the textbook.
Then, **Our Induction steps:**
Assume $F[x_1, x_2, ..., x_k]$ is an integral domain for some $k \in \mathbb{N}$.
We will show that $F[x_1, ..., x_k, x_{k+1}]$ is also an integral domain.
Since $F[x_1, .., x_k, x_{k_1}] = (F[x_1, ..., x_k])[x_{k+1}]$ as ring of polynomials in $k+1$ indeterminates with coefficients in F, we know it is an integral domain.
Therefore, we can conclude that $F[x_1, ..., x_n]$ is an integral domain for all $n \in \mathbb{N}$. ☐



Q12. Prop. If F is a field, then $F[x_1,\dots,x_n]$ is an integral domain.

Proof. Let F be a field, then F must be an integral domain.

Proof by induction:

Base case: $n=1$, WTS $F[X]$ is an integral domain.

Since F is an integral domain, then F is commutative and F doesn't have zero divisors.

Let $p(x), q(x)$ be 2 arbitrary nonzero polynomials in $F[x]$.

Let $p(x) = a_0+a_1x+\dots+a_nx^n \neq 0$.

$q(x) = b_0+b_1x+\dots+b_mx^m \neq 0$.

$p(x)\cdot q(x) = c_0+c_1x+\dots+c_{m+n}x^{m+n}$ where

$$c_i = \sum_{k=0}^{i} a_k b_{i-k}$$

$q(x)\cdot p(x) = c_0'+c_1'x+\dots+c_{m+n}'x^{m+n}$ where

$$c_i' = \sum_{k=0}^{i} b_k a_{i-k} = \sum_{k=0}^{i} a_{i-k} b_k = a_ib_0+a_{i-1}b_1+\dots+a_0b_i$$
$$= a_0b_i+a_1b_{i-1}+\dots+a_ib_0$$
$$= \sum_{k=0}^{i} a_k b_{i-k} = c_i'$$

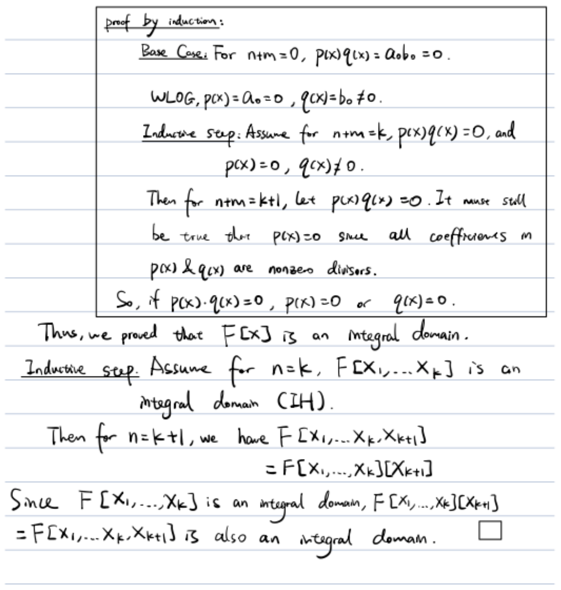
So $p(x)\cdot q(x) = q(x)\cdot p(x)$ in $F[x]$,

then $F[X]$ is commutative.

Note that since F has no zero divisors,

WLOG, let $p(x)\cdot q(x) = a_0b_0+(a_0b_1+a_1b_0)x+\dots+(a_nb_m)x^{n+m}$

WTS $p(x)\cdot q(x) = 0 \implies p(x)=0$ or $q(x)=0$ :



proof by induction:

Base Case: For $n+m=0$, $p(x)q(x) = a_0b_0 = 0$.

WLOG, $p(x)=a_0=0$, $q(x)=b_0\neq 0$.

Inductive step: Assume for $n+m=k$, $p(x)q(x)=0$, and $p(x)=0$, $q(x)\neq 0$.

Then for $n+m=k+1$, let $p(x)q(x)=0$. It must still be true that $p(x)=0$ since all coefficients in $p(x)$ & $q(x)$ are nonzero divisors.

So, if $p(x)\cdot q(x)=0$, $p(x)=0$ or $q(x)=0$.

Thus, we proved that $F[x]$ is an integral domain.

Inductive step. Assume for $n=k$, $F[X_1,\dots X_k]$ is an integral domain (IH).

Then for $n=k+1$, we have $F[X_1,\dots X_k,X_{k+1}]$
$= F[X_1,\dots,X_k][X_{k+1}]$

Since $F[X_1,\dots,X_k]$ is an integral domain, $F[X_1,\dots,X_k][X_{k+1}]$ $= F[X_1,\dots X_k,X_{k+1}]$ is also an integral domain. □

Only a few of the students use the proof by induction, and I give some partial credits to students who tried hard to solve the problem and let them see the solution.

17.5.25(d):

(d)

**Proposition 6.**

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

*Proof.* Suppose $f(x), g(x) \in F[x]$, where the coefficients of $f(x)$ are labeled $a_i$ and the coefficients of $g(x)$ are labeled $b_i$.
Then, we have that $(fg)(x) = \sum_{k=0}^{m+n} c_k x^k$ where $c_k = \sum_{i+j=k} a_i b_j$.
Thus by $(a)$, we have

$$(fg)'(x) = (\sum_{k=0}^{m+n} c_k x^k)' = \sum_{k=1}^{m+n} k c_k x^{k-1}$$

Now, we also have $f'(x)g(x) = \sum_{k=1}^{m+n} c_k' x^{k-1}$ where $c_i' = \sum_{i+j=k} i a_i b_j$.
Similarly, we have $f(x)g'(x) = \sum_{k=1}^{m+n} c_k'' x^{k-1}$ where $c_i'' = \sum_{i+j=k} j a_i b_j$.
Hence, adding these together, we get

$$f'(x)g(x) + f(x)g'(x) = \sum_{k=1}^{m+n} (c_k' + c_k'')x^k$$

To show this is equivalent to $(fg)'(x)$ from above, it suffices to show that $c_k' + c_k'' = k c_k$.
We have that

$$\begin{aligned} c_k' + c_k'' &= \sum_{i+j=k} i a_i b_j + j a_i b_j \\ &= \sum_{i+j=k} (i+j) a_i b_j \\ &= k \sum_{i+j=k} a_i b_j \\ &= k c_k \end{aligned}$$

We have thus shown that the sums are identical, and therefore

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

□

(d)

**Proposition 5.** *(fg)'(x) = f'(x)g(x) + f(x)g'(x)*

*Proof.* First let f(x) = $a_n x^n$ and g(x) = $b_m x^m$

$$\begin{aligned}(fg)'(x) &= ((a_n x^n)(b_m x^m))' \\ &= n a_n x^{n-1}(b_m x^m) + (a_n x^n) m b_m x^{m-1} \\ &= f'(x)g(x) + f(x)g'(x)\end{aligned}$$

For monomials this is true now let f(x) = $a_n x^n$ and g(x) = $b_0 + b_1 x + ... + b_m x^m$

$$\begin{aligned}(fg)'(x) &= ((a_n x^n)(b_0 + b_1 x + ... + b_m x^m))' \\ &= n a_n x^{n-1}(b_0 + b_1 x + ... + b_m x^m) + a_n x^n (b_1 + 2b_2 x + ... + m b_m x^{m-1}) \\ &= f'(x)g(x) + f(x)g'(x)\end{aligned}$$

We have shown that if we have one monomial and one polynomial this is true now let f(x) = $a_0 + a_1 x + ... + a_n x^n$ and g(x) = $b_0 + b_1 x + ... + b_m x^m$

$$\begin{aligned}(fg)'(x) &= ((a_0 + a_1 x + ... + a_n x^n)(b_0 + b_1 x + ... + b_m x^m))' \\ &= (a_1 + 2a_2 x + ... + n a_n x^{n-1})(b_0 + b_1 x + ... + b_m x^m) \\ &+ (a_0 + a_1 x + ... + a_n x^n)(b_1 + 2b_2 x + ... + m b_m x^{m-1}) \\ &= f'(x)g(x) + f(x)g'(x)\end{aligned}$$

Lastly we have shown that f'(x)g(x) + f(x)g'(x) also holds when we have two polynomials which means that we have shown that f'(x)g(x) + f(x)g'(x) is always true. ☐

Many students did the proof without using monomial for f(x), and I gave them a little partial credit due to their hard work.