Q1:

Ch 21.5 Q1

(a) We have $\alpha = \sqrt{1/3 + \sqrt{7}}$

$$\alpha^2 = 1/3 + \sqrt{7}$$

$$\alpha^2 - 1/3 = \sqrt{7}$$

$$(\alpha^2 - 1/3)^2 = 7$$

$\sqrt{1/3 + \sqrt{7}}$ is a root of $x^4 - \frac{2}{3}x^2 - \frac{62}{9} = 0.$

Minimal polynomial:

$$\mathbb{Q}(\sqrt{1/3+\sqrt{7}})$$
$$|\,2$$
$$\mathbb{Q}(\sqrt{7})$$
$$|\,2$$
$$\mathbb{Q}$$

(c) We have $\alpha = \sqrt{3} + \sqrt{2}\,i$

$$\alpha^2 = 3 + 2\sqrt{6}\,i - 2 = 1 + 2\sqrt{6}\,i$$

$$\alpha^2 - 1 = 2\sqrt{6}\,i$$

$$(\alpha^2 - 1)^2 = -24$$

$\sqrt{3} + \sqrt{2}\,i$ is a root of $\cdot\ x^4 - 2x^2 + 25 = 0.$

Minimal polynomial:

$$\mathbb{Q}(\sqrt{3} + \sqrt{2}\,i)$$
$$|\,2$$
$$\mathbb{Q}(\sqrt{6}\,i)$$
$$|\,2$$
$$\mathbb{Q}$$

Q14:

14.

Proposition. *Let K be an algebraic extension of E, and E an algebraic extension of F. Then K is algebraic over F.*

Proof. Suppose E is algebraic over F and K is algebraic over E. Let $\alpha \in K$. Since we know that α is algebraic over E, there is some polynomial $f(x) \in E[x]$ where $f(x) = \beta_0 + \beta_1 x + ... + \beta_n x^n$ and $f(\alpha) = 0$. We know that since $f(x) \in E[x]$, each coefficient $\beta_i \in E$. This means that each coefficient is algebraic over F so $L = F(\beta_0, \beta_1, ..., \beta_n)$ is a finite extension of F. We can see that $f(x) \in L[x]$ since all the coefficients are in L and we know that $f(\alpha) = 0$ so α is algebraic over L as well and $L(\alpha)$ is a finite extension of L. We can see that $[L(\alpha) : F] = [L(\alpha) : L][L : F]$ so we know that $L(\alpha)$ is a finite extension of F. Therefore, we know that α is algebraic over F. Since $\alpha \in K$, we know that K is then an algebraic extension of F. $\square$

14) Prop: Let K be an algebraic extension of E, and E an algebraic extension of F. Prove K is algebraic over F

Proof: Assume K is an algebraic extension of E, and E is an algebraic extension of F.

Let α be some arbitrary element in K.

Let $p(x) = \beta_0 + \beta_1 x + \cdots + \beta_n x^n$ and $p(x) \in E[x]$ where $p(\alpha) = 0$.

Let $L = F(\beta_0, \beta_1, \ldots, \beta_n)$ be a field extension over F.

Then α is algebraic over L since the coefficients of $p(x)$ form the basis of L.

It follows that $L(\alpha)$ is algebraic over L since α is algebraic over L and L is inherently algebraic over itself.

There is a finite number of coefficients $\beta_0, \ldots, \beta_n$, and adjoining finite amount of algebraic elements means the field extension is finite. Thus, L/F is finite.

Similarly, $L(\alpha)/L$ is finite

We know, for finite extensions, $[L(\alpha):F] = [L(\alpha):L][L:F]$.

Thus, for all $\alpha \in K$, α is algebraic over F.

Therefore K is algebraic over F.

$\square$

Q21:

Q21.

Proposition: Let E be an algebraic extension of field F, and let σ be an automorphism of E leaving F fixed. Let $\alpha \in E$. Then σ induces a permutation of the set of all zeros of the minimal polynomial of α that are in E.

Proof: Suppose $\alpha \in E$. Then there exists minimal polynomial $f(x) \in F[x]$ such that $f(\alpha)=0$.

Since σ is an automorphism, $\sigma(f(\alpha)) = \sigma(0) = 0$.

Assume $\sigma(\alpha) = \beta$ where $\beta \in E$. Then

$\sigma(f(\alpha)) = \sigma(a_0 + a_1\alpha + \cdots + a_n\alpha^n)$. Since F is fixed, $a_0, a_1, \ldots, a_n$ are fixed, $\sigma(f(\alpha)) = a_0 + a_1\beta + \cdots + a_n\beta^n = 0$.

Therefore, β must be a zero of the minimal polynomial of α.

We also know that if $S = \{\beta \in E \mid f(\beta) = 0\}$, then $\sigma: S \to S$ is bijective.

It is because σ is a automorphism, $\sigma: E \to E$ must be bijective.

And for S, any element $\beta \in S$ must has $\sigma(\beta) \in S$ because otherwise $\sigma(f(\beta)) \neq 0$. Therefore, $\sigma: S \to S$ is bijective.

Therefore, σ induces a permutation of the set of all zeros of the minimal polynomial of α that are in E.