This is **not** a practice midterm– some of these problems are inappropriate for the exam setting (too long/short/difficult/easy), and this collection does not necessarily represent the ratio of each topic you will see on the midterm. However, they are relevant problems to the material covered and should be manageable with what you've learned.

1.a Find (up to isomorphism) all abelian groups of order $1400 = 2^3 * 5^2 * 7$.

1.b Suppose G is an abelian group with $|G| = 90 = 2 * 3^2 * 5$ and G has exactly two elements of order 6. Is this enough information to uniquely identify G? Either find G, find two groups satisfying these properties, or show that no such group exists.

2. Which of the following are normal subgroups?

2.a $\langle 3 \rangle \times \langle 15 \rangle \times \langle \pi \rangle \times \langle 18 \rangle$ in $\mathbb{Z}/54 \times \mathbb{Z}/160 \times \mathbb{R} \times \mathbb{C}^*$

2.b Inn(G) in Aut(G) [Recall: $\text{Aut}(G) := \{\phi : G \to G : \phi \text{ is an isomorphism}\}$, $\text{Inn}(G) := \{\phi \in \text{Aut}(G) : \exists g \in G, \phi(x) = g^{-1}xg\}$]

2.c K in $H \times K$ (where both H and K are groups)

2.d D_4 in S_4

3.a If H is a normal subgroup of G and both H and G/H are abelian, must G be abelian?

3.b Let G be a group, and H and K be normal subgroups of G such that $H \cong K$. Must $G/H \cong G/K$?

4. Let $G = \mathbb{Z}[x] = \{\text{polynomials } p(x) = \sum_{i=0}^{\infty} a_i x^i : a_i \in \mathbb{Z} \text{ and only finitely many } a_i \text{ are nonzero}\}$. G forms a group under addition of polynomials [e.g. $(3x^2+4x+1)+(2x^3+2x^2+7) = 2x^3+5x^2+4x+8$].

4.a Let $\phi : \mathbb{Z}[x] \to \mathbb{R}$ where $\phi(p(x)) = p(0)$. Prove that ϕ is a homomorphism. [ϕ is called the "evaluation homomorphism"]

4.b Describe $\ker \phi$ and $(x^2 + 3) + \ker \phi$.

4.c What (familiar) group is isomorphic to $\mathbb{Z}[x]/\ker \phi$? (Defend your claim)

5. Prove that if H is a normal subgroup of G and $\phi : G \to G'$ is a homomorphism then $\phi(H)$ is a normal subgroup of $\phi(G')$. [Need $\phi(H)$ be a normal subgroup of G'?]

6. Let G be a group with subgroups H and N, and further let N be normal.

6.a Show $HN := \{hn : h \in H, n \in N\}$ is a subgroup of G. [Need HN be normal?]

6.b Prove the map $\phi : H \to HN/N, \phi(h) \mapsto hN$ is a surjective homomorphism.

6.c Prove $H/(H \cap N)$ is isomorphic to HN/N.

7.a Prove the kernel of a homomorphism is a normal subgroup (you can assume it's a subgroup).

7.b Given a normal subgroup H of G, construct a homomorphism ϕ (and prove it's a homomorphism) with domain G and $\ker \phi = H$.

7.c Prove A_n is normal in S_n using part (a) . [A HW question ("if G has exactly one subgroup H of a given order n then H is normal") also provides a quick way of proving this claim, but I want you to find the homomorphism]

8. Find a nontrivial homomorphism between the following groups, or prove none (other than the trivial homomorphism) exist:

8.a $\phi : \mathbb{Z}/12 \to \mathbb{Z}$

8.b $\psi : \mathbb{Z}/p \to \mathbb{Z}/p^2$ where p is a prime number

8.c $\eta : \mathbb{C} \to \mathbb{R}$

8.d $\phi : G \to H$ where G and H are (finite) groups having coprime order