Math 103A W23 HW 2

7. **Suggested answer**:

1.5 Question 7

Proposition The set $\mathbb{S} = \mathbb{R}\backslash\{-1\}$ combined with the operation $a \circ b = a + b + ab$ makes an abelian group.

Proof. In order to prove that this set makes an abelian group, we must prove 5 things: closedness, associativity, identity, inverse, and commutativity.

Closedness In order to prove that the operation described is a mapping onto itself, we must prove that no (a, b) where a and b are real map onto -1. This is logically equivalient to saying that, if $a \circ b = -1$, and a and b are real and a is not -1, b must be -1(and vice versa, however a and b are symmetric).

We can use algebra to manipulate the expression $-1 = a + b + ab$
$-1 = a + b(1 + a)$
$-1 - a = b(1 + a)$
$\frac{-1-a}{1+a} = b$
$(-1)\frac{1+a}{1+a} = b = -1$

This proves that no two numbers (a, b) inside the set S result in the output -1, and since the products and sums of real numbers must be real, we can conclude that $\circ$ is a closed operation under $\mathbb{S}$.

Associativity We can prove associativity by stating that, as $a+b+ab+c+c(a+b+ab) = a+(b+c+bc)+a(b+c+bc)$, $a\circ(b\circ c)$ is by definition equal to $(a\circ b)\circ c$.

Identity We can prove that 0 is the identity by proving that, by definition, $a \circ 0 = a + 0 + 0a = a$, and similarly $0 \circ a = 0 + a + 0a = a$.

Inverse We can prove that every element in $\mathbb{S}$ has an inverse by writing out the equation $a \circ a^{-1} = a + a^{-1} + aa^{-1} = 0$
$a^{-1}(1 + a) + a = 0$
$a^{-1} = \frac{-a}{1+a}$

This proves a^{-1} exists, and because $a = 1 + a$ has no solutions, we know that all a in $\mathbb{S}$ has an a^{-1} that is also within $\mathbb{S}$.

Lastly, we can prove commutativity by stating that, as $b+a+ba = a+b+ab$, $b \circ a$ is by definition equal to $a \circ b$.

This proves all the axioms required to conclude that the group given is both a group and an abelian group.

□

Comments:
The solution clearly states the elements needed to prove the proposition: closedness(closure), associativity, identity, inverse and commutativity.

Common Mistakes:
Most students correctly calculated the inverse but neglected to show that the inverse resides in the group. For this question, we need to show that $\frac{-a}{1+a}$ can't equal -1.

16. **Suggested answer**:

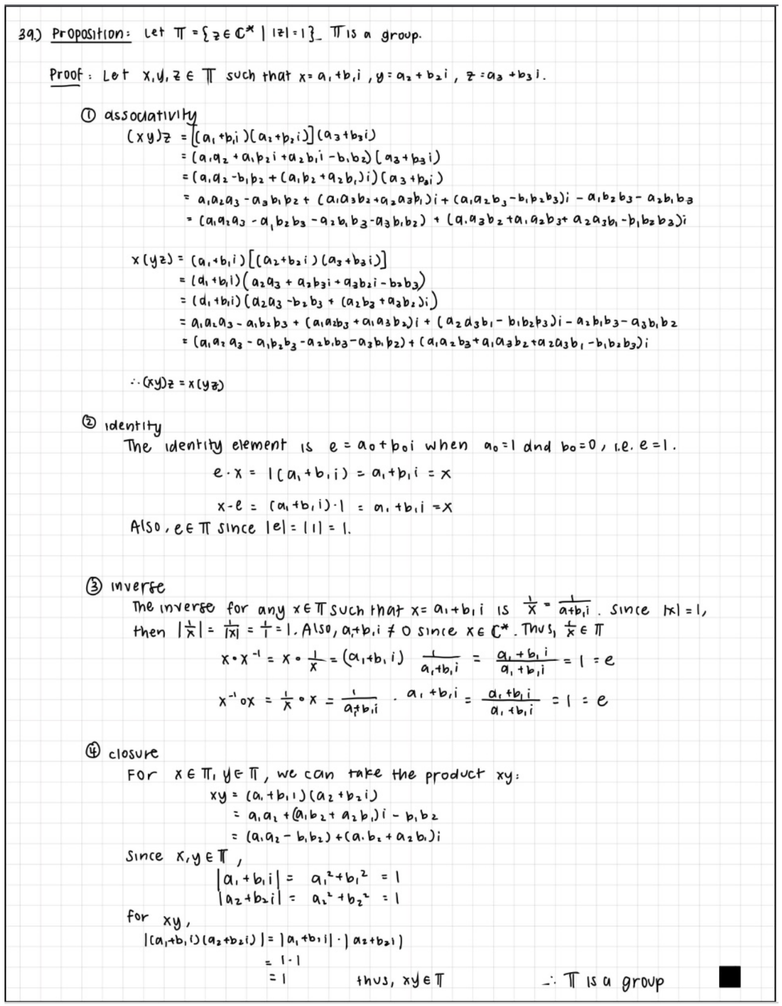
16.

Proposition. *There exists a group G and elements $g, h \in G$ such that $(gh)^n \neq g^n h^n$.*

Proof. Let G be the symmetry group of an equilateral triangle, D_3. Let $g = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$ and $h = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}$. Then $g \circ h = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$ so $(g \circ h)^2 = \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}$. We also have $g^2 = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$ and $h^2 = \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}$ so $g^2 \circ h^2 = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$. Thus, $(g \circ h)^2 \neq g^2 \circ h^2$. □

Comments:
A very straightforward answer/example.

39. **Suggested answer**:



39.) Proposition: Let $\mathbb{T} = \{z \in \mathbb{C}^* \mid |z| = 1\}$. $\mathbb{T}$ is a group.

Proof: Let $x, y, z \in \mathbb{T}$ such that $x = a_1 + b_1 i$, $y = a_2 + b_2 i$, $z = a_3 + b_3 i$.

① associativity

$$
\begin{aligned}
(xy)z &= [(a_1+b_1 i)(a_2+b_2 i)](a_3+b_3 i) \\
&= (a_1a_2 + a_1b_2 i + a_2b_1 i - b_1b_2)(a_3+b_3 i) \\
&= (a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1)i)(a_3+b_3 i) \\
&= a_1a_2a_3 - a_3b_1b_2 + (a_1a_3b_2 + a_2a_3b_1)i + (a_1a_2b_3 - b_1b_2b_3)i - a_1b_2b_3 - a_2b_1b_3 \\
&= (a_1a_2a_3 - a_1b_2b_3 - a_2b_1b_3 - a_3b_1b_2) + (a_1a_3b_2 + a_1a_2b_3 + a_2a_3b_1 - b_1b_2b_3)i
\end{aligned}
$$

$$
\begin{aligned}
x(yz) &= (a_1+b_1 i)[(a_2+b_2 i)(a_3+b_3 i)] \\
&= (a_1+b_1 i)(a_2a_3 + a_2b_3 i + a_3b_2 i - b_2b_3) \\
&= (a_1+b_1 i)(a_2a_3 - b_2b_3 + (a_2b_3 + a_3b_2)i) \\
&= a_1a_2a_3 - a_1b_2b_3 + (a_1a_2b_3 + a_1a_3b_2)i + (a_2a_3b_1 - b_1b_2b_3)i - a_2b_1b_3 - a_3b_1b_2 \\
&= (a_1a_2a_3 - a_1b_2b_3 - a_2b_1b_3 - a_3b_1b_2) + (a_1a_2b_3 + a_1a_3b_2 + a_2a_3b_1 - b_1b_2b_3)i
\end{aligned}
$$

$\therefore (xy)z = x(yz)$

② identity

The identity element is $e = a_0 + b_0 i$ when $a_0 = 1$ and $b_0 = 0$, i.e. $e = 1$.

$$e \cdot x = 1(a_1 + b_1 i) = a_1 + b_1 i = x$$

$$x \cdot e = (a_1 + b_1 i) \cdot 1 = a_1 + b_1 i = x$$

Also, $e \in \mathbb{T}$ since $|e| = |1| = 1$.

③ inverse

The inverse for any $x \in \mathbb{T}$ such that $x = a_1 + b_1 i$ is $\frac{1}{x} = \frac{1}{a_1 + b_1 i}$. Since $|x| = 1$, then $|\frac{1}{x}| = \frac{1}{|x|} = \frac{1}{1} = 1$. Also, $a_1 + b_1 i \neq 0$ since $x \in \mathbb{C}^*$. Thus, $\frac{1}{x} \in \mathbb{T}$

$$x \circ x^{-1} = x \circ \frac{1}{x} = (a_1 + b_1 i) \frac{1}{a_1 + b_1 i} = \frac{a_1 + b_1 i}{a_1 + b_1 i} = 1 = e$$

$$x^{-1} \circ x = \frac{1}{x} \circ x = \frac{1}{a_1 + b_1 i} \cdot a_1 + b_1 i = \frac{a_1 + b_1 i}{a_1 + b_1 i} = 1 = e$$

④ closure

For $x \in \mathbb{T}$, $y \in \mathbb{T}$, we can take the product xy:

$$
\begin{aligned}
xy &= (a_1 + b_1 i)(a_2 + b_2 i) \\
&= a_1a_2 + (a_1b_2 + a_2b_1)i - b_1b_2 \\
&= (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i
\end{aligned}
$$

Since $x, y \in \mathbb{T}$,

$$|a_1 + b_1 i| = a_1^2 + b_1^2 = 1$$
$$|a_2 + b_2 i| = a_2^2 + b_2^2 = 1$$

for xy,

$$
\begin{aligned}
|(a_1 + b_1 i)(a_2 + b_2 i)| &= |a_1 + b_1 i| \cdot |a_2 + b_2 i| \\
&= 1 \cdot 1 \\
&= 1
\end{aligned}
$$

thus, $xy \in \mathbb{T}$ $\therefore \mathbb{T}$ is a group ∎

Comments:
Again, the elements of the proof are very clear: associativity, identity, inverse and closure.

Common Mistakes:
1. Similar to the common mistake in Q7, we need to show that the identity/inverse we found resides within the group. For this question, we need to show that the identity 1 has $|1| = 1$, and the inverse $\frac{1}{x}$ has $|\frac{1}{x}| = 1$.
2. Notice that you can utilize the properties of modulus of complex number to simplify your proof for closure and inverse.