Math 103A W23 HW 5

30a. **Suggested answer**:

Proof. Suppose we take some $1 \leq i < k$.
Then, we have that

$$\begin{aligned} \sigma\tau\sigma^{-1}(\sigma(a_i)) &= \sigma\tau(\sigma^{-1}\sigma)(a_i) \\ &= \sigma\tau(a_i) \\ &= \sigma(a_{i+1}) \end{aligned}$$

Likewise, the same argument provides that $\sigma\tau\sigma^{-1}(\sigma(a_k)) = \sigma(a_1)$.
Therefore, we have that $\sigma\tau\sigma^{-1}$ creates a cycle of the form $(\sigma(a_1), \sigma(a_2), ..., \sigma(a_{\!}$
Additionally, we may observe that σ is a permutation and thus by definition is injective, meaning that the $\sigma(a_i)$ are distinct, and thus that the cycle is of length k.
Now, it remains to be shown that there are no other cycles in $\sigma\tau\sigma^{-1}$.
That is, that if $j \notin \{\sigma(a_1), \sigma(a_2), ..., \sigma(a_k)\}$ then $\sigma\tau\sigma^{-1}(j) = j$.
Suppose that $j \notin \{\sigma(a_1), \sigma(a_2), ..., \sigma(a_k)\}$.
Then, we have that $\sigma^{-1}(j) \neq a_i$ for any $1 \leq i \leq k$ as if it were then we could multiply both sides to get that $j = \sigma(a_i)$ which would be a contradiction.
Thus, we have that $\tau\sigma^{-1}(j) = \sigma^{-1}(j)$ and finally that $\sigma\tau\sigma^{-1}(j) = \sigma\sigma^{-1}(j) = j$.
Therefore, there are no other cycles.
Thus, we have shown that $\sigma\tau\sigma^{-1} = (\sigma(a_1), \sigma(a_2), ..., \sigma(a_k))$ is a k-cycle for any permutation σ. $\square$

Comments:
Clear two-step proof. The first part shows the existence of a cycle of length k, and the second part starting with "it remains to be shown that there are no other cycles in $\sigma\tau\sigma^{-1}$" does what it claims to do.

Common Mistakes:
Some people neglected to show that the elements not in $(\sigma(a_1), \sigma(a_2), ..., \sigma(a_k))$ are fixed under $\sigma\tau\sigma^{-1}$.

32b. **Suggested answer**:

(b) (1) orbits of $\alpha = (1254)$, $\mathcal{O}_{x,\alpha} = \{y : x \sim y\} = \{\alpha^i(x) : i = 0,1,2,3\}$ since α is order 4

Then, $\mathcal{O}_{1,\alpha} = \{1,2,5,4\}$ $\qquad$ $\mathcal{O}_{4,\alpha} = \{4,1,2,5\}$

$\mathcal{O}_{2,\alpha} = \{2,5,4,1\}$ $\qquad$ $\mathcal{O}_{5,\alpha} = \{5,4,1,2\}$

$\mathcal{O}_{3,\alpha} = \{3\}$

so, orbits of α are $\{3\}$, $\{1,2,5,4\}$, $\{n\}$ for $n \geq 6$.

(2) orbits of $\beta = (123)(45)$, $\mathcal{O}_{x,\beta} = \{y : x \sim y\} = \beta^i(x) : i = 0,1,2,3,4,5\}$ since β is order 6

$\mathcal{O}_{1,\beta} = \{1,2,3\}$ $\qquad$ $\mathcal{O}_{4,\beta} = \{4,5\}$

$\mathcal{O}_{2,\beta} = \{1,2,3\}$ $\qquad$ $\mathcal{O}_{5,\beta} = \{4,5\}$

$\mathcal{O}_{3,\beta} = \{1,2,3\}$

so, orbits of β are $\{1,2,3\}$, $\{4,5\}$, $\{n\}$ for $n \geq 6$.

(3) orbits of $\gamma = (13)(25)$. $\mathcal{O}_{x,\gamma} = \{y : x \sim y\} = \{\gamma^i(x) : i = 0,1\}$ since γ is order 2

$\mathcal{O}_{1,\gamma} = \{1,3\}$ $\qquad$ $\mathcal{O}_{4,\gamma} = \{4\}$

$\mathcal{O}_{2,\gamma} = \{2,5\}$ $\qquad$ $\mathcal{O}_{5,\gamma} = \{2,5\}$

$\mathcal{O}_{3,\gamma} = \{1,3\}$

so, orbits of γ are $\{1,3\}$, $\{2,5\}$, $\{4\}$, $\{n\}$ for $n \geq 6$

Comments:
The final answers are clearly and completely stated.
Common Mistakes:
Many neglected to include the fixed elements (namely, 3 for α and 4 for γ) in their answers.

36. **Suggested answer**:

36. Show that $\alpha^{-1}\beta^{-1}\alpha\beta$ is even

Let $\beta = (a_1\ a_2)(a_3\ a_4)\cdots(a_{m-1}\ a_m)$

$\beta^{-1} = [(a_1\ a_2)(a_3\ a_4)\cdots(a_{m-1}\ a_m)]^{-1}$

$= (a_{m-1}\ a_m)(a_{m-3}\ a_{m-2})\cdots(a_1\ a_2)$

therefore β & β^{-1}, α & α^{-1} have the same number of transpositions.

let the number of transpositions for β be g, for α be m

$\alpha^{-1}\beta^{-1}\alpha\beta \Rightarrow m + g + m + g$

$= 2(m+g)$

even $\Downarrow$

Comments:

Concise but complete answer for this question.

Common Mistakes:

You need to include the reasoning shown in the suggested answer to use the fact that α^{-1}, β^{-1} can also be written as m and g transpositions. Also, many people divide the problem into different cases that consider different parity combinations of αandβ, which leads to a correct but long solution.