Solve the following problems, and turn in the solutions to at least four of them. Note that some of the Hartshorne exercises are now from Chapter III rather than Chapter II.

Throughout this assignment, for $A$ a ring and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^n_A$, let $H^i(\mathbb{P}^n_A, \mathcal{F})$ be the cohomology of the complex

$$0 \to \bigoplus_{0 \leq i \leq n} \Gamma(U_i, \mathcal{O}(d)) \to \bigoplus_{0 \leq i < j \leq n} \Gamma(U_i \cap U_j, \mathcal{O}(d)) \to \bigoplus_{0 \leq i < j < k \leq n} \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}(d)) \to \cdots \, .$$

1. Let $f : Y \to X$ be a qcqs morphism of schemes. Let $\mathcal{F}$ be a quasicoherent sheaf on $X$.
   
   (a) Show that there is a natural map $\mathcal{F} \to f_* f^* \mathcal{F}$ of sheaves of $\mathcal{O}_X$-modules. (Reminder: the qcqs condition is there to guarantee that $f_*$ preserves quasicoherence.)

   (b) Now assume that $X$ is reduced and $f$ is surjective. Prove that $\mathcal{F} \to f_* f^* \mathcal{F}$ is injective. (Hint: this is similar to a lemma in the proof that projective morphisms are proper.)

2. Let $f : X \to S$ be a morphism of schemes with $S = \text{Spec } A$ for some noetherian ring $A$. Let $\mathcal{F}$ be a coherent sheaf on $X$.

   (a) It was shown in class that if $X = \mathbb{P}^n_S$, then $\Gamma(X, \mathcal{F})$ is a finite $A$-module. Deduce the same when $X$ admits a closed immersion into $\mathbb{P}^n_S$.

   (b) Using Chow’s lemma and the previous exercise, prove the same when $f$ is proper.

3. Compute the cohomology of the sheaf $\mathcal{O}(d)$ on $\mathbb{P}^n_A$. There are many references for this (including Hartshorne), but try on your own first!

4. Hartshorne exercise III.5.2(a).

5. Let $\mathcal{F}$ be a locally free coherent sheaf on $\mathbb{P}^1_K$ of rank 2 (for $K$ a field).

   (a) Suppose that that there exists a short exact sequence

   $$0 \to \mathcal{O}(n_1) \to \mathcal{F} \to \mathcal{O}(n_2) \to 0$$

   in which $n_1 \geq n_2$. Prove that the exact sequence splits. Hint: reduce to the case $n_2 = 0$, then use the long exact sequence in cohomology.

   (b) Suppose that there exists a short exact sequence

   $$0 \to \mathcal{O}(n_1) \to \mathcal{F} \to \mathcal{O}(n_2) \to 0$$

   in which $n_1 \leq n_2 - 1$. Prove that there also exists a short exact sequence

   $$0 \to \mathcal{O}(n_1 + c) \to \mathcal{F} \to \mathcal{O}(n_2 - c) \to 0$$

   for some positive integer $c$. Hint: this time, reduce to the case $n_1 = -1$ and remember that every line bundle on $\mathbb{P}^1_K$ of degree $n$ is isomorphic to $\mathcal{O}(n)$. But be careful: the quotient of two vector bundles is not always a vector bundle!
6. Let $\mathcal{F}$ be a locally free coherent sheaf on $\mathbb{P}^1_K$ of rank $d$ (for $K$ a field).

(a) Prove that there exists a filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_d = \mathcal{F}$ of $\mathcal{F}$ by vector subbundles such that each quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to $\mathcal{O}(n_i)$ for some $n_i \in \mathbb{Z}$. Hint: use the fact that $\mathcal{F}(n)$ is generated by global sections for $n$ sufficiently large.

(b) Using (a) and the previous exercise, prove that there exists an isomorphism

$$\mathcal{F} \cong \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_d)$$

for some $n_1, \ldots, n_d \in \mathbb{Z}$ (but not necessarily the same ones you found in (a)). Hint: note that the sum of the degrees of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ is independent of the filtration. Then use the previous exercise to raise the degrees of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ for small values of $i$ at the expense of larger values.