Math 170B Final

June 12, 2019

- Please put your name, ID number, and sign and date.
- There are 8 problems worth a total of 200 points.
- **You must show your work to receive credit**.

Print Name: _____

Student ID: _____

Seat Number: _____

Signature and Date: _____

Problem	Score
1	/25
2	/25
3	/25
4	/25
5	/25

Problem	Score
6	/25
7	/25
8	/25
Total	/200

Formulas and Definitions:

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- Taylor series: $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!}(x-y)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-y)^{n+1}$, if $f \in C^{n+1}[a,b]$, $x, y \in [a,b]$, and for some ξ between x, y.
- Newton's method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ and $x_{n+1} - r = \frac{1}{2}\frac{f''(\xi_n)}{f'(x_n)}(x_n - r)^2$, for some ξ_n between x_n, r.
- Method of false position is a variation of bisection method where, for each interval $[a_n, b_n]$, approximations and interval cut locations are both chosen at $b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$.
- Fixed point iterations: $x_{n+1} = F(x_n)$ and $x_{n+1} - s = F'(s)(x_n - s) + \frac{F''(s)}{2!}(x_n - s)^2 + \ldots$, for s a fixed point of $F(x)$.
- If $\{x_n\}_{n=0}^{\infty}$ converges to r, then it has order of convergence $q > 0$ if $\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^q}$ exists and is nonzero.
- Divided differences satisfy $f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$.
- If $x_0, \ldots, x_n \in [a,b]$, then $f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$, for some $\xi \in (a,b)$.
- Newton form:
$$p(x) = f[x_0] + \sum_{k=1}^{n} \left(f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j) \right).$$
- The Chebyshev polynomial of degree n, $T_n(x)$, satisfies: $T_n(x) = \cos(n \cos^{-1} x)$; and $T_0(x) = 1$, $T_1(x) = x$, $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$; and $|T_n(x)| \le 1$ in $[-1,1]$, with $T_n\left(\cos \frac{j\pi}{n}\right) = (-1)^j$, for $j = 0, \ldots, n$; and $T_n\left(\cos \frac{2j-1}{2n}\pi\right) = 0$, for $j = 1, \ldots, n$.
- If $p(x)$ is a monic polynomial of degree n, then $\max_{x\in[-1,1]} |p(x)| \ge \max_{x\in[-1,1]} \left|\frac{T_n(x)}{2^{n-1}}\right|$, for $n \ge 1$.
- Degree $\le d$ spline interpolants, with knots $t_0 < \ldots < t_n$, are interpolating functions in $C^k[t_0, t_n]$ that are composed of degree $\le d$ polynomials in $[t_i, t_{i+1}]$, with k as large as possible for general data and n.
- A natural spline satisfies $S''(t_0) = S''(t_n) = 0$.
- If E is a normed vector space, with norm $||\cdot||$, and G is a subspace of E, the best approximation $g \in G$ of an $f \in E$ in G is defined to satisfy $||f - g|| \le ||f - h||$, for all $h \in G$.
- If E is an inner product space, with inner product $< \cdot, \cdot >$, and G is a subspace of dimension n, the best approximation $g \in G$ of an $f \in E$ in G uses the norm $||h|| = \sqrt{< h, h >}$, and satisfies the normal equations $A\vec{c} = \vec{b}$, where $g = \sum_{i=1}^{n} c_i g_i$, $a_{ij} =< g_i, g_j >$, and $b_i =< f, g_i >$, for any basis $g_1, \ldots, g_n$ of G.
- $f \perp G$ means $< f, h >= 0$ for all $h \in G$.
- $||f + g||^2 = ||f||^2 + 2 < f, g > + ||g||^2$.
- A two-dimensional interpolation polynomial coming from the tensor product method is a linear combination of $u_i(x)v_j(y)$, where u_i, v_j are one-dimensional cardinal functions.

Do not remove this page. You can use below and back for scratch work.

1. (25 pts) Given the following header for a **Matlab** function:

   function [p] = NewtonForm(m,x,y,z)

   that inputs

   - number of data points m;
   - m-component vectors x and y, the x- and y-coordinates, respectively, of the data points;
   - location z;

   and uses divided difference tables and **Newton form** to output the value of the interpolation polynomial, for the given data points, at z.

2. (25 pts) Let $f \in C^2[r, b]$, where r is a **root** of f and $r < b$. Suppose, further, that $f''(x) < 0$ for all $x \in [r, b]$, and $f'(r) < 0$. Prove **Newton's method**'s sequence of approximations **converges** to r for all initial guesses $x_0 \in [r, b]$.

3. (25 pts)

(a) Find approximations of the form $\frac{p}{q}$, for p, q integers with $q \neq 0$, for $\sqrt{5}$ using c_0, c_1 of the method of **false position** with initial interval $[a_0, b_0]$, where $a_0 < b_0$ are the two integers closest to $\sqrt{5}$.

(b) Let $F(x) = \frac{1}{2x} - \frac{1}{8x^2}$, which has fixed point at $s = \frac{1}{2}$, and let $\{x_n\}_{n=0}^{\infty}$ denote a sequence of **fixed point iterations**, using fixed point function $F(x)$, satisfying $x_n \neq s$ for all $n \geq 0$, that converges to s. Determine the **order of convergence** of the sequence.

4. (25 pts) Given evenly spaced nodes

$$0 = x_0 < x_1 < \ldots < x_{10} = 1$$

and the values of $f(x_i) = (6.1)x_i - 1.8$, $f'(x_i) = 27x_i - 6$, and $f''(x_i) = 60x_i$, for all $i = 0, \ldots, 10$, let $p(x)$ be the **piecewise quintic** (degree ≤ 5) interpolant for the data (each piece of polynomial only uses two nodes). Find $p(0.401)$.

5. (25 pts)

(a) Find a, b, c, d (they exist) such that $S(x)$ is a **natural cubic spline** interpolant on $[0, 3]$, where

$$S(x) = \begin{cases} 2a + x - x^3, & x \in [0, 1], \\ 1 - b(x-1) - 3(x-1)^2 + 4(x-1)^3, & x \in [1, 2], \\ 4(x-2) - c(x-2)^2 - 2d(x-2)^3, & x \in [2, 3]. \end{cases}$$

(b) Find the degree ≤ 2 polynomial that **best approximates**

$$f(x) = -5x^3 + 3x^2 - x + 2$$

under the norm $||g|| = \max_{x \in [-1,1]} |g(x)|$.

6. (25 pts)

(a) Find the line $c_0 + c_1 x$ that **best approximates** $f(x) = 5x^3 - 3x$ under the norm $||g|| = \left(\int_0^1 [g(x)]^2 x \, dx\right)^{1/2}$.

(b) Consider nodes (x_i, y_j), for $i = 1, 2, 3$ and $j = 1, 2$, where $x_1 = 0, x_2 = 1, x_3 = 3$ and $y_1 = 0, y_2 = 2$. Given values $f(x_i, y_j) = x_i^2 + x_i(1 + y_j)$, for $i = 1, 2, 3$ and $j = 1, 2$, let $p(x, y)$ denote the interpolation polynomial for the data coming from the **tensor product method**. Find $p(2, 1/3)$.

7. (25 pts) Let $x_0 < x_1 < \ldots < x_{2n-1}$ be **nodes** and $t_0 < t_1 < \ldots < t_n$ be **knots**, satisfying $x_{2i}, x_{2i+1} \in [t_i, t_{i+1}]$, for all $i = 0, \ldots, n-1$. Consider a degree $\leq d$ **spline** S interpolating given data points $(x_i, f(x_i))$, for $i = 0, \ldots, 2n-1$, with knots at t_i, for $i = 0, \ldots, n$. Enforcing the number of unknowns to be always greater than or equal to the number of equations, find the **largest** k we can choose, when $d = 5$, so that $S \in C^k[t_0, t_n]$, for any $n \geq 1$?

8. (25 pts) Let G be a subspace (not necessarily finite dimensional) in an inner product space E. For $f \in E$ and $g \in G$, prove g is a **best approximation** to f in G if and only if $f - g \perp G$.

   Hint: Study $||f - g + \lambda h||^2$, for $\lambda > 0$, $h \in G$.