Homework #1

* You may write your answers using 5 digit rounding, however, when computing in computer or calculator, keep all digits possible.

1. Will bisection method work, and will its sequence of approximations converge to a root, for the following functions, using the following starting intervals? Why or why not?

   (a) $f(x) = \cos x + e^x$ with $[0, \pi/2]$.

   (b) $f(x) = x^3 + x + 1$ with $[-1, 0]$.

   (c) $f(x) = 1/x$ with $[-1, 7]$.

   (d) $f(x) = \begin{cases} -x-1, & x \le 0 \\ x-1, & x > 0 \end{cases}$ with $[-2, 1/2]$.

2. (a) Use the bisection method to generate the approximations c_0, c_1, c_2, c_3 of $2\sqrt{2}$ by finding the positive root of $x^2 - 8$ using the starting interval $[2, 3]$.

   (b) Find the bound of the absolute error for the approximation c_3 and verify that the actual absolute error satisfies this bound.

3. Consider $f(x) = x(x-1)(x+2)$, which has roots at $x = 0, 1, -2$. Perform enough iterations to determine, definitively, which root the bisection method approximates when using the starting interval $[-3, 2]$.

4. Suppose $f(x)$ is a given continuous function in $[-1, 4]$ such that $f(-1)$ and $f(4)$ have different signs and consider the bisection method on $f(x)$ using starting interval $[-1, 4]$.

   (a) Bound the absolute error for the approximation c_{30} (Remember, we define $c_0 = (a_0 + b_0)/2$).

   (b) Use bisection method's bound on absolute error to determine which c_n are guaranteed to have absolute errors less than 10^{-11}.

5. (a) Given the first two approximations, c_0, c_1, generated from a bisection method, determine the starting interval $[a_0, b_0]$ that was used.

   (b) Apply this to find the starting interval when $c_0 = -0.2$ and $c_1 = 0.3$

6. Let f_1 be a continuous function with different signs at a, b, with $a < b$ and let $\{c_n\}_{n=0}^{\infty}$ be bisection method's sequence of approximations on f_1 using starting interval $[a, b]$. Let f_2 be a continuous function with different signs at a, b, with $a < b$ and let $\{d_n\}_{n=0}^{\infty}$ be bisection method's sequence of approximations on f_2 using starting interval $[a, b]$.

   (a) Prove (perhaps by induction) if $c_k = d_k$, for some k, then $c_i = d_i$ for all $i < k$.

   (b) Give an example of f_1, f_2, a, b, k such that $c_k = d_k$ but $c_j \neq d_j$, for some $j > k$.

7. Suppose we modify the bisection method into the following variation: for each step, with bracketing interval $[a, b]$, approximations are chosen at the location $(2a + b)/3$, but the interval is cut into two at the different location $(a + 3b)/4$.

   (a) For starting interval $[0, \pi/2]$, and a function f with opposite signs at the endpoints, explain why the absolute error of the approximation c_0 is $\leq \frac{\pi}{3}$. Then similarly bound the absolute errors of the approximations c_n.

   (b) Calculate the first 2 approximations c_0, c_1 for this variation when $f(x) = \cos x - x$ with starting interval $[0, \pi/2]$.

8. Answer True or False for the following questions (you do not need to show work but write down your explanation if you are unsure). In each, $f : \mathbb{R} \to \mathbb{R}$ denotes a continuous function in $[a, b]$, with different signs at a and b.

   (a) The bisection method on $f(x)$ using starting interval $[a, b]$ will always generate a sequence of approximations converging to a root of $f(x)$.

   (b) The bisection method only works when $f(x)$ has exactly one root in the starting interval $[a, b]$.

   (c) When there are multiple roots of $f(x)$ in the starting interval $[a, b]$, the bisection method will approximate the root closest to the midpoint $(a + b)/2$.

   (d) The sequence of approximations generated by the bisection method on $f(x)$ using starting interval $[a, b]$ is the same as the sequence generated by the bisection method on $-2f(x)$ using starting interval $[a, b]$.

   (e) The bisection method on $g(x)$ using starting interval $[a, b]$ may break down if g is a function that is continuous in $[a, b]$ but not defined outside of $[a, b]$.

9. (Matlab) Given $f(x)$, suppose "hw1f.m" is a function that takes as input the number x and outputs $f(x)$.

   (a) Write a Matlab function that inputs:

      - a, b: endpoints of starting interval $[a, b]$;
      - N: number of approximations;

      and outputs c_N using the bisection variation on f found in problem 7 (remember, $c_0 = (2a_0 + b_0)/3$). Use only basic programming and call "hw1f.m" when you need values for f. Write or print out your function and turn it in.

   (b) Apply your function to $f(x) = \sin x - 1/2$ with starting interval $[0, \pi/2]$ to calculate c_{20}. Write or print out the approximation and calculate the value of its absolute error (exact root is at $\arcsin(1/2) = \pi/6$).