Homework #2

1. Let
$$f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ -\sqrt{-x}, & \text{if } x < 0. \end{cases}$$

   (a) Generate Newton's method's x_1, x_2, x_3 in terms of $x_0 > 0$. Then guess the value of x_n for n even and for n odd.

   (b) Draw the graphical interpretation of Newton's method in the previous part.

   (c) Will Newton's method converge for any $x_0 \neq 0$? Why does this not violate the theorem on the convergence of Newton's method (Theorem 1 on page 85) since the initial guess can be arbitrarily close to the root?

2. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of approximations, and suppose there is an interval of the form $[r - \alpha, r + \alpha]$, for some $\alpha > 0$, and a constant $0 \leq \rho < 1$ such that when $x_n \in [r - \alpha, r + \alpha]$, we get $|x_{n+1} - r| \leq \rho |x_n - r|$.

   (a) Given $x_0 \in [r - \alpha, r + \alpha]$, why is $x_n \in [r - \alpha, r + \alpha]$, for all $n \geq 0$?

   (b) For any $x_0 \in [r - \alpha, r + \alpha]$, prove x_n converges to r.

   (c) If $\{x_n\}_{n=0}^{\infty}$ comes from Newton's method, we know
$$x_{n+1} - r = \frac{f''(\xi_n)}{2f'(x_n)}(x_n - r)^2.$$
   For the case $f(x) = x^2 - 2$, for the root of interest $r = \sqrt{2}$, and given $\rho < 1$, find the largest $\alpha > 0$ satisfying: $|x_{n+1} - r| \leq \rho |x_n - r|$, whenever $x_n \in [r - \alpha, r + \alpha]$.

   (d) Conclude that the sequence of approximations generated by Newton's method, using initial guess $x_0 = 1$, will converge.

3. Consider the problem of finding the point on the graph of $y = x^3$ closest to the point $(3, -1)$.

   (a) Write down the expression for $d(x) =$ the square of the distance from (x, x^3) to $(3, -1)$.

   (b) Minimize $d(x)$ by finding the, in this case, unique critical point: approximating the solution of $f(x) = d'(x) = 0$ using Newton's method to generate x_3 when $x_0 = 2$.

4. Consider $f(x) = x(1 - e^x)$, with root $r = 0$.

   (a) Verify that $f(0) = 0$ and $f'(0) = 0$ and $f''(0) \neq 0$.

   (b) Using Newton's method with $x_0 = 0.1$, compute $|e_{n+1}/e_n|$ for $n = 0, 1, 2$ ($e_n = x_n - r$). Does this sequence look bounded? What about $|e_{n+1}/(e_n^2)|$ for $n = 0, 1, 2$?

(c) Simplify $g(x) = f(x)/f'(x)$ and verify $r = 0$ is still a root (find limit as $x \to 0$).

(d) Apply Newton's method with $x_0 = 0.1$ on $g(x)$, and study $|e_{n+1}/e_n|$ for $n = 0, 1$. Does this sequence look bounded? What about $|e_{n+1}/(e_n^2)|$?

(e) Apply the iterative method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)},$$

for $m = 2$, using $x_0 = 0.1$, and study $|e_{n+1}/e_n|$ for $n = 0, 1$. Does this sequence look bounded? What about $|e_{n+1}/(e_n^2)|$?

5. Let $f \in C^2[a, r]$, where r is a root of f and $a < r$. Furthermore, suppose $f''(x) < 0$ in $[a, r]$ and $f'(r) > 0$. Prove Newton's method's sequence of approximations converges to r for all initial guesses $x_0 \in [a, r]$.

6. (Matlab) Suppose we have two Matlab functions, in "hw2f.m" and "hw2fprime.m", that both take as input x and output expressions for $f(x)$ and $f'(x)$, respectively. Then write a Matlab function that inputs

- initial guess x_0;
- tolerance tol;

and outputs the first N, and x_N of Newton's method, such that $|f(x_N)| < tol$. Make sure you call "hw2f" and "hw2fprime" when you need values of $f(x)$ or $f'(x)$.

(a) Write out or print out your function.

(b) For $f(x) = x^2 - 8$, write out or print out your results when $x_0 = 2$ and $tol = 10^{-11}$. Do the same for $x_0 = 10$ and $tol = 10^{-11}$.

(c) For $f(x) = x^2$, write out or print out your results with when $x_0 = 1$ and $tol = 10^{-11}$.