Homework #2

1.
   - The approximations will oscillate, $x_{i+2} = x_i$, for $i = 0, 1, \ldots$, so $x_i = x_0$ for all even i, and $x_i = x_1$ for all odd i.
   - Draw the graphical interpretation of Newton's method, including the tangent lines at approximations and their roots.
   - Newton's method will not converge, as the absolute errors of the approximations do not go to zero. This does not violate any theorem, because the function f is not continuously differentiable, and so not twice continuously differentiable.

2. (a) If $x_n \in [r - \alpha, r + \alpha]$, then $|x_{n+1} - r| < |x_n - r| \le \alpha$, so $x_{n+1} \in [r - \alpha, r + \alpha]$.
   (b) $|x_n - r| \le \rho^n |x_0 - r| \to 0$, so have convergence.
   (c) Largest valid α is $(2\rho r)/(1 + 2\rho)$.
   (d) For $x_0 = 1$, use $\alpha = |x_0 - r| = 0.41421$, and can have $2\rho r = (1 + 2\rho)\alpha$, which leads to $\rho = \alpha/(2r - 2\alpha) = 0.20711 < 1$, which implies convergence.

3. (a) $d(x) = (x - 3)^2 + (x^3 + 1)^2$.
   (b) $f(x) = d'(x) = 2(x - 3) + 6(x^3 + 1)x^2$, and use this with Newton's method using $x_0 = 2$ to get $x_3 = 0.97116$.

4. (a) $f(0) = 0$ and $f'(0) = 0$ and $f''(0) = -2$.
   (b) $|e_1|/|e_0| = 0.51239$, $|e_2|/|e_1| = 0.50638$, $|e_3|/|e_2| = 0.50324$, and $|e_1|/|e_0|^2 = 5.1239$, $|e_2|/|e_1|^2 = 9.8826$, $|e_3|/|e_2|^2 = 19.395$.
   (c) Have
   $$g(x) = \frac{x(1 - e^x)}{1 - e^x - xe^x},$$
   so by l'Hopital's rule, get $g(x) \to 0$ as $x \to r$.
   (d) $|e_1|/|e_0| = 0.025844$, $|e_2|/|e_1| = 6.4554 \cdot 10^{-4}$, $|e_3|/|e_2| = 4.1706 \cdot 10^{-7}$, and $|e_1|/|e_0|^2 = 0.25844$, $|e_2|/|e_1|^2 = 0.24978$, $|e_3|/|e_2|^2 = 0.24999$.
   (e) $|e_1|/|e_0| = 0.024787$, $|e_2|/|e_1| = 6.1954 \cdot 10^{-4}$, $|e_3|/|e_2| = 3.8393 \cdot 10^{-7}$, and $|e_1|/|e_0|^2 = 0.24787$, $|e_2|/|e_1|^2 = 0.24995$, $|e_3|/|e_2|^2 = 0.25002$.

5.
   - First show $f'(x) > 0$ and $f(x) < 0$ in $(a, r]$.
   - Use
   $$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} - r &= \frac{f''(\xi_n)}{2f'(x_n)}(x_n - r)^2 \end{aligned}$$
   to show if $x_n \in [a, r]$, then $x_{n+1} \in [x_n, r]$.

- This implies the sequence of approximations is increasing and bounded above by r, and so it has to converge to some $p \in [a, r]$.
- This p has to be a root of $f(x)$, so $p = r$.

6. (Matlab) See Matlab solutions.