Homework #3

1. Consider the equation $xy^2 + \tan y = x$ that implicitly defines y as a function of x: $y = y(x)$. Approximate $y(1)$ with y_3 generated by Newton's method using initial guess $y_0 = 2.5$.

2. Perform two iterations of Newton's method on the following systems.

   (a) Starting with $(0, 1)$,
   $$\begin{cases} 4x_1^2 - x_2^2 &= 0 \\ 4x_1x_2^2 - x_1 &= 1. \end{cases}$$

   (b) Starting with $(1, 1)$,
   $$\begin{cases} xy^2 + x^2y + x^4 &= 3 \\ x^3y^5 - 2x^5y - x^2 &= -2. \end{cases}$$

3. (a) Use the secant method to generate the approximations x_2, x_3, x_4 of $2\sqrt{2}$ by finding the positive root of $x^2 - 8$ using $x_0 = 2, x_1 = 3$. Also write down the absolute error of your final approximation.

   (b) Consider the following variation of the bisection method: for each step, with bracketing interval $[a, b]$, approximation and interval cut location are both chosen at $b - \frac{f(b)(b-a)}{f(b)-f(a)}$. We call this variation the **method of false position**. Use this method to generate approximations c_0, c_1, c_2 of $2\sqrt{2}$ by finding the positive root of $x^2 - 8$ using starting interval $[a_0, b_0]$, with $a_0 = 2, b_0 = 3$. Also write down the absolute error of your final approximation.

4. Let $a < b$, and suppose $f \in C^2[a, b]$ satisfies $f'(x) > 0$ and $f''(x) > 0$ for all $x \in [a, b]$. Also suppose $f(a) < 0$ and $f(b) > 0$.

   (a) Prove there exists a unique root of f in (a, b), call it r.

   (b) Let the approximations c_n be generated from the method of false position using starting interval $[a, b]$. Prove $c_0 < r$ and $f(c_0) < 0$. Does this hold for all c_n?

   (c) Conclude that $c_{n+1} = c_n - \frac{f(c_n)(c_n-b)}{f(c_n)-f(b)}$, and $c_{n+1} > c_n$. Thus c_n is an increasing sequence bounded above by r, and has to converge to some $p \in [a, b]$. Why does p have to be r?

5. Suppose x_0 is chosen with absolute error 10^{-1} and we want to achieve absolute error $\leq 10^{-7}$.

   (a) Consider a sequence of approximations with $\frac{|e_{n+1}|}{|e_n|^{1.62}} = 1$. Find the first n with the desired absolute error. If this sequence of approximations requires 1 function evaluation for each of x_k, how many are needed to calculate $x_0, \ldots, x_n$, for your computed n?

(b) Instead, consider a sequence of approximations with $\frac{|e_{n+1}|}{|e_n|^2} = 1$. Find the first n with the desired absolute error. If this sequence of approximations requires 2 function evaluations for each of x_k, how many are needed to calculate $x_0, \ldots, x_n$, for your computed n?

(c) Which of the two has fewer function evaluations?

6. (a) Show the linear system $A\vec{x} = \vec{b}$ is equivalent to the fixed point problem $\vec{x} = D^{-1}[(E + F)\vec{x} + \vec{b}]$, where $A = D - E - F$ with D a diagonal matrix, E strictly lower triangular, and F strictly upper triangular.

(b) Find $\vec{x}^{(2)}$ of fixed point iterations, given $\vec{x}^{(0)} = [0, 0]^T$ when $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $\vec{b} = [-5, 4]^T$.

7. Consider fixed point function $F(x) = 1/x + x/2$ and consider the interval $[1.4, 1.45]$.

(a) Show $F(x) \in [1.4, 1.45]$ for $x \in [1.4, 1.45]$ by finding the maximum value, M, and minimum value, m, of F in the interval and showing $[m, M] \subseteq [1.4, 1.45]$.

(b) Find $\lambda < 1$ such that $|F'(x)| \leq \lambda$ for all $x \in [1.4, 1.45]$ by finding the maximum value of $|F'(x)|$ in the interval.

(c) Conclude that fixed point iterations converge for all $x_0 \in [1.4, 1.45]$.

8. (Matlab) Suppose we have a Matlab function, in "hw3f.m", that takes as input x and outputs the expression for $f(x)$. Write a Matlab function that inputs

- initial guesses x_0, x_1;
- tolerance tol;

and outputs the first $N \geq 1$ and x_N of secant method such that $|x_N - x_{N-1}| < tol$. Make sure you call "hw3f" when you need values of $f(x)$.

(a) Write out or print out your function.

(b) For $f(x) = \cos x - x$, write out or print out your results when $x_0 = 0$, $x_1 = \pi/2$, and $tol = 10^{-7}$.