Homework #4

1. Let $F \in C[a,b]$ and suppose F maps $[a,b]$ to $[a,b]$. Use the Intermediate Value Theorem on $f(x) = F(x) - x$ to show F has at least one fixed point in $[a,b]$.

2. (a) Let $F \in C[a,b]$ be a contractive mapping and let s, t be fixed points of F. Prove, using the definition of contractive mapping, that $s = t$.

   (b) Alternatively, let $F \in C^1[a,b]$ and $F'(x) \neq 1$ for all $x \in [a,b]$. Prove, using the Mean Value Theorem, that $s = t$.

3. Let $f(x) = x^3 - 2x + 1$. To solve $f(x) = 0$, consider the three fixed point functions

$$F_1(x) = \frac{1}{2}(x^3 + 1), \quad F_2(x) = \frac{2}{x} - \frac{1}{x^2}, \quad F_3(x) = \sqrt{2 - \frac{1}{x}}.$$

   (a) Verify $r = 1$ is a root of f, and also a fixed point for each of the fixed point functions.

   (b) Compute $F_1'(1), F_2'(1), F_3'(1)$ and comment whether fixed point iterations will converge for all initial guesses x_0 sufficiently close to r. For the ones that converge, find the order of convergence.

4. Let $f(x) \in C^\infty[a,b]$ with root at $x = r$ and suppose $f'(r) \neq 0$. Define the fixed point function

$$F(x) = x - \frac{(f(x)/f'(x))}{(f(x)/f'(x))'}.$$

   (a) Show r is a fixed point of F.

   (b) Show $F'(r) = 0$. What does this imply about the order of convergence of fixed point iterations on F?

5. Consider the polynomial $p(x) = 2x^3 - 12x^2 + 22x - 12$.

   (a) Verify $x = 1, 2, 3$ are roots of p.

   (b) Suppose a method is used and approximates a root of 2.1. Use polynomial division to find $q(x)$ and constant R such that $p(x) = (x - 2.1)q(x) + R$.

   (c) Use the quadratic formula to find the roots of $q(x)$.

6. In each part, for given data points

$$(-1, 2), (1, 3), (2, -2),$$

   write down (but do not solve) a linear system, enforcing $p(-1) = 2, p(1) = 3, p(2) = -2$, for the unknown coefficients a, b, c:

   (a) $p(x) = ax^2 + bx + c$.

(b) $p(x) = a\frac{(x-1)(x-2)}{6} + b\frac{(x+1)(x-2)}{-2} + c\frac{(x+1)(x-1)}{3}$ (Lagrange form).

(c) $p(x) = a + b(x+1) + c(x+1)(x-1)$ (Newton form).

7. (Matlab) Let $f(x)$ be a function with values given by the Matlab function "hw4f.m". First, write a bisection method function on $f(x)$, in "hw4afn.m", that inputs

   - a, b, endpoints of a starting interval that contains at least one integer;

   and outputs the integer when there is only one left in the (shrinking) bisection method interval.

   (a) Print out or write out your function.

   (b) The next step will be done partly by hand and partly in the command line of Matlab. Suppose we know that $f(x) = x^5 + 3x^4 - 8x^3 - 12x^2 + 16x$ has integer roots. In the command line of Matlab, use the bisection method function to find an integer root using either $[-6, 3]$ or $[-6, 1.5]$ as starting interval (whichever has opposite signs of f). By hand, perform deflation on f to remove this root. Call the result your new f, and repeat the process until you have found all 5 integer roots. Write down your results.