Homework #5

1. Consider the data points $(-2,-1),(0,1),(-1,3)$.

   (a) Write down the Lagrange form for the interpolation polynomial $p_2(x)$ for these data points.

   (b) Find K, a, b, c such that $p_3(x) = p_2(x) + K(x-a)(x-b)(x-c)$ interpolates, in addition, the data point $(1,-1)$.

   (c) Evaluate $p_3(-0.7)$ and $p_3'(-0.7)$.

2. (a) Given $n+1$ data points with distinct nodes, prove there are an infinite number of different polynomials of degree $n+1$ that interpolate these data points. (Try adding a data point)

   (b) What about polynomials of degree $p > n+1$?

   (c) Write down two different polynomials of degree exactly 3 interpolating the data points
   $$(1,4),(2,-2),(3,1).$$

3. (a) Find the Lagrange form of the interpolation polynomial for the data points $(x_0 - h, f(x_0-h)),(x_0, f(x_0)),(x_0+h, f(x_0+h))$, where $h > 0$, and call it $p_2(x)$.

   (b) Simplify $p_2'(x_0)$ into the form $Af(x_0+h)+Bf(x_0)+Cf(x_0-h)$, for some A, B, C constants that depend on h.

   (c) Do the same for $p_2''(x_0)$.

4. Let $f(x) = e^x$ be the underlying function in the interval $[0.1, 0.6]$, and let $p(x)$ be the interpolation polynomial for the data points $(0.1, f(0.1)),(0.5, f(0.5)),(0.6, f(0.6))$.

   (a) Find $p(x)$ in Lagrange form.

   (b) Use calculus to find the maximum of $|f'''(x)|$ in the interval $[0.1, 0.6]$.

   (c) Use this result to find a constant A such that $|f(0.2) - p(0.2)| \leq A$.

   (d) Now use calculus to find the maximum of $|(x-0.1)(x-0.5)(x-0.6)|$ in the interval $[0.1, 0.6]$.

   (e) Use your results to find a constant B such that $|f(x) - p(x)| \leq B$, for all $x \in [0.1, 0.6]$.

5. Given $n+1$ data points with distinct nodes, consider a polynomial of least degree that interpolates them.

   (a) Prove this polynomial has degree $\leq n$.

   (b) What does this say about the uniqueness of this polynomial?

6. Consider data points $(x_i, f(x_i))$, $i = 0, \ldots, n$, and let $p(x)$ be the interpolation polynomial for the data. Suppose $Q(x)$ is the interpolation polynomial for the data points $(x_i, f(x_i))$, $i = 0, \ldots, n-1$, and $R(x)$ is the interpolation polynomial for the data points $(x_i, f(x_i))$, $i = 1, \ldots, n$. Prove

$$p(x) = \frac{(x - x_0)R(x) - (x - x_n)Q(x)}{x_n - x_0}.$$

7. (Matlab) Write a Matlab function that inputs

   - integer m;
   - two vectors of data points x and y, both with m components;
   - location z;

   and uses the Lagrange form to output the interpolation polynomial's value at z.

   (a) Write out or print out your program.

   (b) Apply your program to the case with data points $(x_i, f(x_i)), i = 0, \ldots, 20$, where the $f(x) = \frac{1}{1+25x^2}$, and x_i are equally spaced nodes satisfying

   $$-1 = x_0 < x_1 < \ldots < x_{20} = 1,$$

   and write out or print out your results for $z = 0.975$ (note the exact value is $f(0.975) = 0.040379$).

   (c) (not due) Do the same with the same $f(x)$, but equally spaced nodes

   $$-1 = x_0 < x_1 < \ldots < x_{40} = 1.$$