Homework #5

1. (a) Use Lagrange form:

\[ p_2(x) = \frac{-x(x+1)}{2} + \frac{(x+2)(x+1)}{2} - 3(x+2)x \]

(b) Follows Newton form: use \(a = -2, b = 0, c = -1\), so the original three data points are still being interpolated. Then solve for \(K\) using \(p_3(1) = -1\) from the new data point to get \(K = 1\).

(c) Use your form for \(p_3(x)\), also take its derivative, and plug in to get \(p_3(-0.7)\) and \(p_3'(-0.7)\).

2. (a) First get the interpolation polynomial for the \(n+1\) data points, call it \(p(x)\). Newton’s form for adding a data point creates the polynomial

\[ p(x) + K(x - x_0) \cdots (x - x_n), \]

where \(x_0, \ldots, x_n\) are the nodes of the \(n+1\) data points. Then verify that for \(K \neq 0\), these are all degree \(n+1\) polynomials interpolate the \(n+1\) data points.

(b) Use Newton’s form and add more data points to see there are an infinite number of polynomials of degree \(p\) that interpolate the \(n+1\) data points.

(c) Use your results to actually write down two polynomials.

3. (a)

\[
\begin{align*}
p_2(x) &= f(x_0 - h)(x - x_0)(x - x_0 - h) + f(x_0)(x - x_0 + h)(x - x_0 - h) + \\
&\quad f(x_0 + h)(x - x_0 + h)(x - x_0) \\
&\quad \frac{h(-h)}{2h(h)}
\end{align*}
\]

(b) Take a derivative of \(p_2(x)\) and plug in \(x_0\) to get \(A = 1/(2h)\), \(B = 0\), and \(C = -1/(2h)\).

(c) Take two derivatives of \(p_2(x)\) and plug in \(x_0\) to get \(A = 1/h^2\), \(B = -2/h^2\), and \(C = 1/h^2\).

4. (a) Get

\[
p(x) = e^{0.1}\frac{(x - 0.5)(x - 0.6)}{(-0.4)(-0.5)} + e^{0.5}\frac{(x - 0.1)(x - 0.6)}{(0.4)(-0.1)} + e^{0.6}\frac{(x - 0.1)(x - 0.5)}{(0.5)(0.1)}
\]

and simplify.

(b) Look at critical points of \(f''''(x)\) (there are none) and endpoints and see which has the largest \(|f''''(x)|\) value. The answer for the value is \(e^{0.6}\).
(c) 
\[ |f(0.2) - p(0.2)| \leq \frac{e^{0.6}}{3!} (0.1)(0.3)(0.4). \]

(d) Look at critical points of \((x - 0.1)(x - 0.5)(x - 0.6)\) and endpoints and see which has the largest \(|(x - 0.1)(x - 0.5)(x - 0.6)|\) value. The answer for the value is approximately 0.0131.

(e) 
\[ |f(x) - p(x)| \leq \frac{e^{0.6}}{3!} 0.0131. \]
for all \(x \in [0.1, 0.6]\).

5. (a) We have existence of a polynomial of degree \(\leq n\) that interpolates the data points, so the polynomial of least degree has to be less than or equal to this, giving its degree as \(\leq n\).

(b) We have uniqueness of a polynomial of degree \(\leq n\) that interpolates the data points, so the polynomial of least degree has to be exactly that polynomial.

6. In order to show a polynomial is the interpolation polynomial for a set of data points, just make sure it has the correct degree and that it interpolates the data points.

7. (Matlab) See Matlab solutions.