Homework #6

1. (a) Draw a graph of the piecewise linear interpolating polynomial for the data given in the following table:

x	0	1	3	4	6
$f(x)$	2	4	1	-2	-3

   (b) Write down the equation for the linear piece in the interval $[4, 6]$. Then use it to approximate $f'(4)$, $f'(6)$, and $\int_4^6 f(x)\,dx$.

   (c) Suppose instead we create the piecewise quadratic interpolating polynomial by using the parabola passing through the first three points in $x \in [0, 3]$ and the parabola passing through the last three points in $x \in [3, 6]$. Find the values of this piecewise quadratic interpolant at $x = 1.5$ and also at $x = 5.7$.

2. Use error bounds to find n ensuring that the piecewise linear interpolating polynomial $P(x)$ for data with nodes $x_j = j/n$, $j = 0, \ldots, n$ and values from the underlying function $f(x) = x^2 + 1$ satisfies

$$|f(x) - P(x)| \leq 10^{-7},$$

for all $x \in [0, 1]$.

3. Consider the interval $[a, b]$, for $a < b$.

   (a) Find the interpolation polynomial $q(x)$ for the data points $(a, -1), (b, 1)$. Under q, $[a, b]$ is mapped to what interval? Write down the expression for q^{-1} as well.

   (b) Find monic polynomial S_m of degree m, in terms of Chebyshev polynomial T_m and q, such that

$$\max_{x\in[a,b]} |r(x)| \geq \max_{x\in[a,b]} |S_m(x)|,$$

   for any monic polynomial r of degree m. What are the m roots of S_m, using the expression for the roots of T_m?

   (c) Given $n+1$ data points with nodes at the roots of S_{n+1} in $[a, b]$, with interpolation polynomial p, use your results to find C such that

$$|f(x) - p(x)| \leq C \frac{\max_{z\in[a,b]} |f^{(n+1)}(z)|}{(n+1)!}.$$

4. (a) Find the Lagrange form for the interpolation polynomial $p(x)$ using 3 optimal node locations, according to Chebyshev polynomials, for the function $f(x) = \cos(\pi x)$ in $[0, 1]$, and evaluate it at $x = 3/4$.

   (b) For arbitrary $x \in [0, 1]$, use your results from Problem 3c to bound $|f(3/4) - p(3/4)|$. Does the actual value of $|f(3/4) - p(3/4)|$ satisfy this bound?

5. Use error bounds to find how many nodes, chosen at optimal node locations, according to Chebyshev polynomials, are needed for the function $f(x) = e^x$ in $[-1, 1]$ to ensure the interpolation polynomial $p(x)$ satisfies $|f(x) - p(x)| \leq 10^{-7}$, for all $x \in [-1, 1]$. (You can plug in different n to arrive at your result, where $n + 1$ is the number of nodes)

6. Let $p(x)$ be a degree $n \geq 1$ monic polynomial satisyfing:

   - $\max_{x\in[-1,1]} p(x) = -\min_{x\in[-1,1]} p(x) = M$;
   - there exist $n+1$ distinct locations $-1 \leq x_0 < \ldots < x_n \leq 1$ such that $|p(x_i)| = M$, for all $i = 0, \ldots, n$, and $p(x_i), p(x_{i+1})$ have opposite signs, for $i = 0, \ldots, n-1$.

   Prove $M = 1/2^{n-1}$.

7. (Matlab) Suppose we have a function "hw6f.m" that takes as input x and outputs the value for a function $f(x)$. Write a Matlab program that inputs:

   - interval $[a, b]$;
   - m, the number of data points with evenly spaced nodes from $x_1 = a$ to $x_m = b$, and values from $f(x)$;
   - location z satisfying $x_2 < z < x_{m-1}$, where $h = (b - a)/(m - 1)$;

   and outputs the value of the interpolaton polynomial using only the four data points with nodes closest to z.

   (a) Write out or print out your program. You may find Matlab's "floor" or "ceil" commands to be helpful.

   (b) Apply your program to the case $m = 101$ and x_i evenly spaced, from $-\pi$ to π, and $y_i = \sin x_i$, and write out or print out your results when $z = -3, -1, 0.5, 2$.