Homework #8

1. Given $f \in C^\infty[a,b]$ and nodes $x_0 < x_1 < x_2 < x_3$ in $[a,b]$, let p be the interpolation polynomial for the table of data:

x	x_0	x_1	x_2	x_3
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$
$f'(x)$		$f'(x_1)$		$f'(x_3)$
$f''(x)$		$f''(x_1)$		

Find j, m, and k_i, $i = 0, \ldots, 3$ such that

$$f(x) = p(x) + \frac{f^{(j)}(\xi_x)}{m!} \prod_{i=0}^{3} (x - x_i)^{k_i},$$

for some $\xi_x \in (a,b)$.

2. Consider $f(x) = \sin x$ and evenly spaced nodes $0 = x_0 < x_1 < \ldots < x_n = 2\pi$. Let $P(x)$ be the piecewise cubic interpolant given values and first derivatives of f at the nodes.

   (a) In the case $n = 100$, use calculus and the error formula

   $$|f(1) - P(1)| \leq \frac{\max_{z \in [x_i, x_{i+1}]} |f''''(z)|}{4!} |(1 - x_i)^2 (1 - x_{i+1})^2|,$$

   where $1 \in [x_i, x_{i+1}]$, to bound the absolute error $|f(1) - P(1)|$.

   (b) For arbitrary $x \in [0, 2\pi]$, use error bounds to determine n ensuring that $|f(x) - P(x)| \leq 10^{-10}$.

3. Consider a deg $\leq d$ piecewise polynomial spline S interpolating data points with knots and nodes at $t_0 < t_1 < \ldots < t_n$ and values at the nodes from a function f.

   (a) If we write each piece in a standard polynomial form $(a_0 + a_1 x + \ldots + a_d x^d)$, how many unknown coefficients are there in total?

   (b) How many equations are needed in total to make $S \in C[t_0, t_n]$ interpolate the data points?

   (c) Thus, how many equations are needed in total to make $S \in C^k[t_0, t_n]$ interpolate the data points?

   (d) Choosing $d = 5$, what is the largest k we can choose so that the number of unknowns is $\geq$ the number of equations, for all n? For this choice of k, how many more unknowns are there than equations?

4. Determine whether the following is a quadratic spline:

$$S(x) = \begin{cases} x, & x \in (-\infty, 1], \\ -\frac{1}{2}(2-x)^2 + \frac{3}{2}, & x \in [1, 2] \\ \frac{3}{2}, & x \in [2, \infty]. \end{cases}$$

5. (a) Find all values of a, b, c, d, e such that $S(x)$ is a cubic spline, where

$$S(x) = \begin{cases} a(x-2)^2 + b(x-1)^3, & x \in (-\infty, 1], \\ c(x-2)^2, & x \in [1, 3] \\ d(x-2)^2 + e(x-3)^3, & x \in [3, \infty]. \end{cases}$$

(b) Further determine the values such that the cubic spline interpolates $(0, 26), (1, 7), (4, 25)$.

6. (a) Let

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & \text{if } 0 \le x \le 1, \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if } 1 \le x \le 2, \end{cases}$$

be a natural cubic spline on $[0, 2]$. Find b, c, d.

(b)

$$S(x) = \begin{cases} S_0(x) = a + bx + cx^2 + dx^3, & \text{if } 0 \le x \le 1, \\ S_1(x) = 1 + (x-1) + (x-1)^2 + (x-1)^3, & \text{if } 1 \le x \le 2, \end{cases}$$

be a clamped cubic spline on $[0, 2]$, satisfying the boundary conditions $S'(0) = -1$ and $S'(2) = 6$. Find a, b, c, d.

7. Given knots $a = t_0 < t_1 < \ldots < t_n = b$, and data points (t_i, y_i), for $i = 0, \ldots, n$, prove the clamped cubic spline S satisfying $S'(a) = \alpha$, $S'(b) = \beta$ satisfies

$$\int_a^b [S''(x)]^2 \, dx \le \int_a^b [f''(x)]^2 \, dx,$$

for all $f \in C^2[a, b]$ interpolating the data points and satisfying $f'(a) = \alpha$, $f'(b) = \beta$.

8. (Matlab) Write a Matlab program that inputs:

- number of nodes m;
- the m component vectors x of nodes, y of values at the nodes;
- location z;

and outputs the value, at z, of the natural cubic spline interpolating the data, by solving (using Matlab's backslash command) the linear system generated by (10) on page 352 of the book.

(a) Write out or print out your program.

(b) Apply your program to the case with evenly spaced nodes at $-1 = x_0 < x_1 < \ldots < x_n = 1$ and values and derivatives from $f(x) = \frac{1}{1+25x^2}$. Write out or print out your results, at $z = 0.02$ and $z = 0.975$, when $n = 20$.