Homework #8

1. Guess the error formula

\[ f(x) = p(x) + \frac{f^{(7)}(\xi(x))}{7!} (x - x_0)(x - x_1)^3(x - x_2)(x - x_3)^2, \]

which comes from doubling the node at \( x_3 \) and tripling it at \( x_1 \). For the proof of this form, study

\[ \phi(t) = f(t) - p(t) - [f(x) - p(x)] \frac{(t - x_0)(t - x_1)^3(t - x_2)(t - x_3)^2}{(x - x_0)(x - x_1)^3(x - x_2)(x - x_3)^2} \]

and use Rolle’s theorem.

2. (a) Calculus says \(|(1 - x_i)(1 - x_{i+1})|\) is maximized, in \([x_i, x_{i+1}]\), at the midpoint \( x = (x_i + x_{i+1})/2 \), so the square, \(|(1 - x_i)(1 - x_{i+1})|^2\), is also maximized at the midpoint. Thus \(|(1 - x_i)(1 - x_{i+1})|\leq h^4/16\), for any \( i \). In addition, find the exact \( i \) such that 1 \( \in [x_i, x_{i+1}] \) to maximize \(|\sin x|\) to fill out the error formula.

(b) For arbitrary \( x \in [0, 2\pi] \), use \(|\sin x| \leq 1\), and find \( h \), then \( n \), such that

\[ \frac{h^4}{384} \leq 10^{-10}. \]

3. (a) A degree \( \leq d \) polynomial has \( d + 1 \) unknowns, so \((d + 1)n\) unknowns total.

(b) Continuity and interpolation needs \( 2n \) equations.

(c) In total, need \( 2n + k(n - 1) \) equations.

(d) Largest is \( k = 4 \), where there are 4 more unknowns than equations.

4. Check the conditions for \( S \in C^1(−\infty, \infty) \), which are

\[ \lim_{x \to z^+} S^{(k)}(x) = \lim_{x \to z^-} S^{(k)}(x), \]

for junction points \( z = 1, 2 \) and continuity up to first derivatives: \( k = 0, 1 \).

5. (a) Enforce

\[ \lim_{x \to z^+} S^{(k)}(x) = \lim_{x \to z^-} S^{(k)}(x), \]

for junction points \( z = 1, 3 \) and continuity up to second derivatives: \( k = 0, 1, 2 \). Note, there are not enough equations to completely determine \( a, b, c, d, e \)

(b) Add the equations of interpolation \( S(0) = 26, S(1) = 7, S(4) = 25 \) and solve for \( a, b, c, d, e \).

6. (a) Enforce

\[ \lim_{x \to 1^+} S^{(k)}(x) = \lim_{x \to 1^-} S^{(k)}(x), \]

for continuity up to second derivatives: \( k = 0, 1, 2 \). Also include the natural boundary conditions \( S''(0) = 0 \) and \( S''(2) = 0 \). Solve for \( b, c, d \).
(b) Enforce
\[
\lim_{x \to 1^+} S^{(k)}(x) = \lim_{x \to 1^-} S^{(k)}(x),
\]
for continuity up to second derivatives: \( k = 0, 1, 2 \). Also include the clamped boundary conditions \( S'(0) = -1 \) and \( S'(2) = 6 \). Solve for \( a, b, c, d \).

7. Follow the proof for optimality of natural splines:

- Let \( g(x) = f(x) - S(x) \) and show
\[
\int_a^b g''(x)S''(x) \, dx \geq 0,
\]
this time using the clamped boundary conditions to see that
\[
g'(t_n)S''(t_n) - g'(t_0)S''(t_0) = 0.
\]

8. (Matlab)

(a) See Matlab solutions.

(b) At \( z = 0.02 \), we get the value 0.98915, and at \( z = 0.975 \), we get the value 0.040470.