Math 170B Midterm 2

March 2, 2018

- Please put your name, ID number, seat number, and sign and date.
- There are 4 problems worth a total of 100 points.
- **You must show your work to receive credit**.

Print Name: _____

Student ID: _____

Seat Number: _____

Signature and Date: _____

Problem	Score
1	/25
2	/25
3	/25
4	/25
Total	/100

Formulas and Definitions:

- $f \in C^k[a,b]$ means $f, f', \ldots, f^{(k)}$ all exist and are continuous in $[a,b]$
- A polynomial P has degree n means $P(x) = a_0 + a_1 x + \ldots + a_n x^n$, where $a_n \neq 0$
- Fundamental Theorem of Algebra: A nonzero polynomial of degree n has exactly n roots in the complex plane, counting multiplicity
- Generalized Rolle's Theorem: if $f \in C^n[a,b]$ and $f(x) = 0$ at $n+1$ distinct numbers in $[a,b]$, then there exists $c \in (a,b)$ such that $f^{(n)}(c) = 0$.
- Given data points (x_i, y_i), $i = 0, \ldots, n$, the nodes are defined to be the x_i, for $i = 0, \ldots, n$, and evenly spaced nodes have $x_{k+1} - x_k$ all the same value, for all $k = 0, \ldots, n-1$
- The Lagrange polynomial P interpolating data points $(x_i, f(x_i))$, $i = 0, \ldots, n$, with distinct nodes, is the unique deg $\leq n$ polynomial satisfying $P(x_i) = f(x_i)$
- Lagrange form:
$$P(x) = \sum_{i=0}^{n} \left(f(x_i) \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \right)$$
- Lagrange polynomial error formula: if $f \in C^{n+1}[a,b]$, and $(x_i, f(x_i))$, $i = 0, \ldots, n$, are data points with distinct nodes in $[a,b]$, and P is the Lagrange polynomial interpolating the data points, then for all $x \in [a,b]$, there exists $\xi(x) \in (a,b)$ such that
$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$
- For data points $(x_i, f(x_i))$, $i = 0, \ldots, n$, with nodes $x_0 < x_1 < \ldots < x_n$, the piecewise linear interpolant in $[x_i, x_{i+1}]$ is the line interpolating $(x_i, f(x_i), (x_{i+1}, f(x_{i+1})$
- Divided differences: zeroth divided difference is $f[x_i] = f(x_i)$ and kth divided difference is
$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+2}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$
- Newton form:
$$P(x) = f[x_0] + \sum_{k=1}^{n} \left(f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j) \right)$$

Do not remove this page. You can use below and back for scratch work.

1. (25 pts) Write a **Matlab** function with the header

   function [value] = polynomialpiece(m,x,y,k,z)

   that inputs:

   - number of data points m;
   - vectors x and y, both with m components, holding x- and y-coordinates, respectively, of data points, and where the components of x are **evenly spaced** and in **increasing order**;
   - an even number $k > 0$, and a location $z \in (x_{k/2}, x_{m+1-k/2})$ distinct from the nodes;

   and outputs, using **Lagrange form**, the **value** at z of the deg $\leq k-1$ Lagrange polynomial that interpolates only the k data points with nodes **closest** to z.

2. (25 pts) For each of the following problems, **circle** either **True** or **False**. You do **not** need to show your work.

   (a) Given $f \in C^\infty[-1,1]$ and **evenly spaced** nodes $-1 = x_0 < x_1 < \ldots < x_n = 1$, let P_n be the **Lagrange polynomial** interpolating the data points $(x_i, f(x_i))$. Then, for all $x \in [-1,1]$, we always have $\lim_{n\to\infty} |f(x) - P_n(x)| = 0$.

   True or False

   (b) Given any polynomial P of degree 4, and any 9 distinct nodes x_i, $i = 0, \ldots, 8$, then P has to be the **Lagrange polynomial** interpolating the data points $(x_i, P(x_i))$, $i = 0, \ldots, 8$.

   True or False

   (c) Given nodes $x_0 < x_1 < \ldots < x_{100}$, and values y_i and w_i satisfying $y_i = w_i$, for all $i \neq 30$, let P be the **piecewise linear** interpolant for data points (x_i, y_i), $i = 0, \ldots, 100$, and Q be the **piecewise linear** interpolant for data points (x_i, w_i), $i = 0, \ldots, 100$. Then, for all $x \in [x_0, x_1]$, we always have $P(x) = Q(x)$.

   True or False

   (d) Given nodes $-1 = x_0 < x_1 < \ldots < x_n = 1$, let $f \in C^2[-1,1]$, with $f''(x) < 0$ for all $x \in [-1,1]$, and let P be the **piecewise linear** interpolant for data points $(x_i, f(x_i))$. Then, for all $x \in [-1,1]$, we always have $f(x) \geq P(x)$.

   True or False

   (e) Given distinct nodes $x_i \in [-1,1]$, $i = 0, \ldots, 5$ and $f \in C^\infty[-1,1]$ such that $P(x) = x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6$ is the **Lagrange polynomial** interpolating the data points $(x_i, f(x_i))$, $i = 0, \ldots, 5$. Then we always have $f[x_0, x_1, \ldots, x_5] = 6$.

   True or False

   (f) Given (x_i, y_i), $i = 0, \ldots, 8$ data points with distinct nodes, let P be the **Lagrange polynomial** interpolating these data points. If $\deg P = 4$, then, in the **divided difference table** for Newton form, all kth divided differences must be zero, for $5 \leq k \leq 8$.

   True or False

   (g) $P(x) = 2x^3 - x^2 + 2x + 4$ is the **Lagrange polynomial** interpolating the data points $(-1,-1), (0,4), (1,9), (2,20)$.

   True or False

   (h) Given data points (x_i, y_i), $i = 0, \ldots, n$, with nodes $x_0 < x_1 < \ldots < x_n$, and satisfying $y_i < 0$, for $i = 0, \ldots, n$, let P be the **piecewise linear** interpolant for the data points. Then, for all $x \in [x_0, x_n]$, we always have $P(x) < 0$.

   True or False

3. (25 pts) Let $(x_i, f(x_i))$, $i = 0, \ldots, 3$, be data points, where $x_i = i + 2$, for $i = 0, \ldots, 3$. Given the divided differences

$$f[x_0] = 1, \quad f[x_0, x_1] = 2, \quad f[x_0, x_1, x_2] = -7, \quad f[x_0, x_1, x_2, x_3] = 9,$$

add the data point $(0, 3)$, find a **Newton form** for the Lagrange polynomial interpolating all 5 data points.

4. (25 pts) Let $f \in C^n[a,b]$ and let $x_i \in [a,b]$, $i = 0, \ldots, n$, be distinct nodes. **Prove** there exists $\xi \in (a,b)$ such that

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$