Solutions to HW 3

1.1 (a) decreasing, bounded  (b) bounded
(c) increasing, unbounded  (d) bounded
(e) unbounded  (f) decreasing when n is large, bounded

0.8

\[ \sigma_{n+1} - \sigma_n \]
\[ = \frac{1}{n+1} (s_1 + \cdots + s_n + s_{n+1}) - \frac{1}{n} (s_1 + \cdots + s_n) \]
\[ = -\frac{1}{n(n+1)} (s_1 + \cdots + s_n) + \frac{s_{n+1}}{n+1} \]
\[ = \frac{1}{n(n+1)} \left\{ (s_{n+1} - s_1) + (s_{n+1} - s_2) + \cdots + (s_{n+1} - s_n) \right\} \geq 0. \]

So \( \{\sigma_n\} \) is increasing.
10.2 Let \((s_n)\) be a bounded decreasing sequence.

Since \(S = \{s_n : n \in \mathbb{N}\}\) is bounded, \(\exists a = \inf S \in \mathbb{R}\).

A \(\varepsilon > 0\), since \(a + \varepsilon\) is not a lower bound for \(S\),

\(\exists N \in \mathbb{N}\) s.t. \(s_N < a + \varepsilon\).

For \(n > N\), \(a \leq s_n \leq s_N < a + \varepsilon\).

Hence \(|s_n - a| = s_n - a < \varepsilon\) for \(n > N\),

which means \(\lim s_n = a\).

10.5 Since \((s_n)\) is decreasing, \(s_1\) is an upper bound

for \(S = \{s_n : n \in \mathbb{N}\}\). As \((s_n)\) is unbounded,

\(S\) has no lower bounds.

A \(M < 0\), since \(M\) can't be a lower bound,

\(\exists N \in \mathbb{N}\) s.t. \(s_N < M\).

Then for \(n > N\),

\(s_n \leq s_N < M\).

Hence \(\lim s_n = -\infty\).
\begin{align*}
10.10 \quad (a) \quad S_2 &= \frac{1}{3} (S_1 + 1) = \frac{2}{3}, \quad S_3 = \frac{1}{3} \left( \frac{2}{3} + 1 \right) = \frac{5}{9} \\
S_4 &= \frac{1}{3} \left( \frac{5}{9} + 1 \right) = \frac{14}{27} \\
(b) \quad \text{(basis)} \quad S_1 &= 1 > \frac{1}{2}.
\end{align*}

Suppose \( S_n > \frac{1}{2} \).

\[ S_{n+1} = \frac{1}{3} (S_n + 1) > \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}. \]

Hence \( S_n > \frac{1}{2} \), \( \forall n \in \mathbb{N} \).

(c) \quad S_{n+1} - S_n = \frac{1}{3} (S_n + 1) - S_n \\
= \frac{1}{3} \left( 1 - 2S_n \right) < 0 \quad \text{as} \quad S_n > \frac{1}{2}.

(d) \quad \text{Since} \quad (S_n) \quad \text{is decreasing and} \quad S_n > \frac{1}{2}, \\
by \quad \text{Theorem} \ 10.2, \quad (S_n) \quad \text{converges.} \\
\text{Let} \quad S = \lim_{n \to \infty} S_n. \quad \text{Taking limits on both sides of} \\
S_{n+1} = \frac{1}{3} (S_n + 1), \\
we \quad \text{have} \\
S = \frac{1}{3} (S + 1), \\
\frac{2}{3} S = \frac{1}{3}, \quad S = \frac{1}{2}. \]
10.11 \quad 0 < \frac{t_{n+1}}{t_n} = 1 - \frac{1}{4n^2} < 1.

Hence \((t_n)\) is decreasing.

Clearly \(0 < t_n < 1\). By Theorem 10.2, \(\lim t_n\) exists.

\[
\lim t_n = \frac{2}{\pi}.
\]

\[
t_{n+1} = \frac{4n^2 - 1}{4n^2} t_n = \frac{(2n-1)(2n+1)}{2n \cdot 2n} \cdots \frac{1 \cdot 3}{2 \cdot 2} = \prod_{k=1}^{n} \frac{(2k-1)(2k+1)}{4k^2}
\]

A proof: Let \(I_n = \int_0^\pi \sin^n x \, dx\)

\(I_0 = \pi, \quad I_1 = 2\)

\[
I_n = -\int_0^\pi \sin^{n-1} x (\cos x)' \, dx = -\sin^{n-1} x \cdot \cos x \bigg|_0^\pi + \int_0^\pi (n-1) \sin^{n-2} x \cdot \cos^2 x \, dx
\]

\[
= (n-1) \int_0^\pi \sin^{n-2} x (1 - \sin^2 x) \, dx = (n-1) I_{n-2} - (n-1) I_n
\]

So \(I_n = \frac{n-1}{n} I_{n-2}\).

\[
I_{2n+1} = \frac{2n}{2n+1} I_{2n-1} = \cdots = \frac{2n}{2n+1} \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot \frac{I_1}{2}
\]

\[
I_{2n} = \frac{2n-1}{2n} I_{2n-2} = \cdots = \frac{2n-1}{2n} \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot \pi
\]

\[
\frac{I_{2n}}{I_{2n+1}} = \frac{(2n-1)(2n+1)}{2n \cdot 2n} \frac{(2n-3)(2n-1)}{2(n-1) \cdot 2n-1} \cdots \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{\pi}{2} = t_{n+1} \cdot \frac{\pi}{2}.
\]

Since for \(x \in [0, \pi]\), \(\sin x \in [0, 1]\), \(\sin x \leq \sin^2 x \leq \sin^3 x\).

So \(I_{2n+1} \leq I_{2n} \leq I_{2n-1}\).

\[
1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n}} = \frac{2n+1}{2n}.
\]

Squeeze Lemma \(\Rightarrow\) \(\lim_{n \to \infty} \frac{I_{2n}}{I_{2n+1}} = 1\).

Hence \(\lim_{n \to \infty} \frac{I_{2n}}{I_{2n+1}} = 1\).

\[
\lim_{n \to \infty} t_n = \frac{2}{\pi}.
\]
(a) \( \forall \varepsilon > 0 \), pick \( N = \lceil \log_2 \frac{1}{\varepsilon} \rceil + 2 \).

Then for \( n > m > N \),

\[
|S_n - S_m| = |S_n - S_{n-1} + S_{n-1} - \cdots + S_m - S_m| \\
\leq |S_n - S_{n-1}| + |S_{n-1} - S_{n-2}| + \cdots + |S_{m+1} - S_m| \\
< 2^{-n+1} + 2^{-n+2} + \cdots + 2^{-m} \\
= 2^{-m} \left( 1 + \frac{1}{2} + \cdots + 2^{-n+1} \right) \\
< 2^{-m+1} < 2^{-N+1} = \varepsilon ,
\]

where we used a fact that for \( a \in (0,1) \),

\[
\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} < \frac{1}{1 - a}.
\]

Hence \( (S_n) \) is Cauchy.

(b) No, it's not true.

We may have the following counterexample:

Let \( S_1 = 1 \), \( S_2 = 1 + \frac{1}{2} \), \( \ldots \), \( S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \). \ldots

Then \( |S_m - S_n| = \frac{1}{n+1} < \frac{1}{n} \). But as a classical result, 

\( (S_n) \) diverges to \( \infty \) and hence is not Cauchy.

A simple proof: \( S_n = \int_1^2 1 + \int_2^3 \frac{1}{2} + \cdots + \int_n \frac{1}{n} \)

\[
> \int_1^2 \frac{1}{x} \, dx + \cdots + \int_n \frac{1}{x} \, dx \quad = \int_1^{n+1} \frac{1}{x} \, dx = \ln(n+1) \to \infty
\]
Problem 5  \( (a) \)

\[ |a_{n+1} - a_n| = \left| \ln(n+1) - \ln n \right| = \ln \frac{n+1}{n} \]

As \( \frac{n+1}{n} \to 1 \), \( |a_{n+1} - a_n| = \ln \frac{n+1}{n} \to 0 \).

Hence \( \forall \varepsilon > 0, \exists N \in \mathbb{N} \), \( \forall n > N \),

\[ |a_{n+1} - a_n| < \varepsilon . \]

\( (b) \)

\[ |a_{n+p} - a_n| = \ln \frac{n+p}{n} \to \infty \text{ as } p \to \infty . \]

Hence \( (a_n) \) cannot be Cauchy.

**Formal proof.** Fix \( \varepsilon = 1 \). \( \forall N \in \mathbb{N} \), let \( n = N \), \( m = N^2 \), say.

Then \( |a_m - a_n| = \ln \frac{N^2}{N} = 2 > 1 = \varepsilon . \)

Which means \( (a_n) \) is not Cauchy.

[To prove a sequence \((a_n)\) is not Cauchy, we need to negate the definition: we should show that \( \exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N} \),

\[ \exists n, m \geq N \] \( |a_n - a_m| > \varepsilon . \) \]