Problem 1. Consider the sequence\[ a_n := n \cos \left( \frac{n\pi}{4} \right), \quad n \geq 1. \]

(a) Give the definition of subsequence of a sequence and find a monotone subsequence of \((a)_n\).
(b) Give the definition of lim sup and lim inf of a sequence and compute them for \((a)_n\).
(c) Give the definition of subsequential limits of a sequence and compute them for \((a)_n\).
(d) Is \((a)_n\) convergent? Justify your answer.

Problem 2. Answer the following questions:

(i) Give the definition of lim sup and lim inf of a sequence \((a)_n\) and explain why they always exist.
(ii) Let \((a)_n\) be the sequence given by \[ a_n = \left( -1 \right)^n \frac{n}{2n+1}. \] Prove that \(\limsup a_n = \frac{1}{2}\).
(iii) Does the sequence \((a)_n\) converge? Explain.

Problem 3. Answer the following questions.

(a) Define what it means for a series to be absolutely convergent.
(b) Given an example of a convergent but not absolutely convergent series, justifying your answer.
(c) Prove that if \(\sum_{n=1}^{\infty} a_n\) is a convergent series of positive numbers, then the series \(\sum_{n=1}^{\infty} a_n^2\) converges.

Problem 4. Prove that a sequence converges if and only is it is a Cauchy sequence.

Bonus Problem. Let \((a)_n\) be a sequence of real numbers such that the sequence of ratios \(\left( \frac{a_{n+1}}{a_n} \right)_n\) is constant. Prove that the series \(\sum_n a_n\) is a geometric series.