Statistical properties for holomorphic endomorphisms of projective spaces

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Conference in Several Complex Variables, 18-21 August, 2020
organised by Shiferaw Berhanu and Ming Xiao
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Let $\mathbb{P}^k$ be the complex projective space of dimension $k$.

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of degree $d \geq 2$.

The iterate of order $n$ of $f$ is of degree $d^n$ and given by

$$f^n := f \circ \cdots \circ f \quad (n \text{ times}).$$

If $V \subset \mathbb{P}^k$ is a subvariety of dimension $p$ then (counting multiplicity)

$$\deg f^n(V) = d^{np} \deg V \quad \text{and} \quad \deg f^{-n}(V) = d^{n(k-p)} \deg V.$$

In particular, for $p = 0$ and $V = \{z\}$ is a point

$$\#f^n(z) = 1 \quad \text{and} \quad \#f^{-n}(z) = d^{nk}.$$

Generic polynomial maps $f : \mathbb{C}^k \to \mathbb{C}^k$ of degree $d$ can be extended to a holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$. 
Endomorphisms, Fatou set and Julia set

- In dimension $k = 1$, a famous example is $f : \mathbb{C} \ (\text{or } \mathbb{P}^1) \rightarrow \mathbb{C} \ (\text{or } \mathbb{P}^1)$

\[ f(z) = z^2 + c \quad (c \text{ is a constant}) \]

\[ f^4(z) = (((z^2 + c)^2 + c)^2 + c)^2 + c. \]

- The phase space $\mathbb{P}^1$ is divided into two disjoint completely invariant sets: the Fatou set (open) and the Julia set (compact)

\[ \mathbb{P}^1 = F \sqcup J \quad f^{-1}(F) = f(F) = F \quad f^{-1}(J) = f(J) = J. \]

- The dynamics in $F$ is tame and stable, and the dynamics in $J$ is chaotic.

**Problem (a main problem)**

*Understand the dynamics (namely, the orbits of points) in $J$.***
Figure: Julia set of $z^2 + c$ with $c = 0.687 + 0.312i$. Source: mcgoodwin.net.
Figure: Julia set of $z^2 + c$ with $c = 0.285 + 0.01i$. Source: WikiMedia.
Any dimension

- In dimension $k$, we have several Julia sets
  \[ P^k = J_0 \supset J_1 \supset \cdots \supset J_k. \]

- We only consider the small Julia set $J_k$ where the dynamics is most chaotic.

- For simplicity, we assume $f$ is generic in a sense close to "the absence of periodic critical point": sub-exponential growth of multiplicity
  \[ \text{multiplicity}(f^n, z) \leq A^n \text{ for all } z \in P^k, A > 1 \text{ and } n \text{ big.} \]

- There is no way to describe all orbits $(z, f(z), f^2(z), \ldots)$ of $f$ on $J_k$.

Probabilistic point of view (for canonical invariant probability measures on $J_k$)

*Study the sequence of random variables*

\[ u, u \circ f, u \circ f^2, \ldots \]

*where $u : P^k \to \mathbb{R}$ is an observable (function of suitable regularity).*
Expected outcomes

Remarks

- **With respect to invariant probability measures, the random variables $u \circ f^n$ are identically distributed: for all $n, m$ and interval $[a, b]$**
  \[
  \text{measure of } \{u \circ f^n \in [a, b]\} = \text{measure of } \{u \circ f^m \in [a, b]\}.
  \]

- **These random variables are clearly not independent (i.d. but not i.i.d.).**

- We expect that the dependence (correlation) between $u \circ f^n$ and $u \circ f^m$ is weak when $|n - m|$ is large.

- We expect that properties of i.i.d. random variables still hold for the sequence $u \circ f^n$: the law of large numbers (ergodicity, mixing, K-mixing, exponential mixing), central limit theorem, Berry-Esseen theorem, local central limit theorem, almost sure central limit theorem, large deviation principle, law of iterated logarithms, almost sure invariance principle.
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If $z \in \mathbb{P}^k$, then $\#f^{-n}(z) = d^{kn}$ (counting multiplicity).

If $\delta_z$ is the Dirac mass at $z$, then

$$d^{-kn}(f^n)^*(\delta_z) = d^{-kn} \sum_{w \in f^{-n}(z)} \delta_w$$

is a probability measure.

**Figure:** Backward orbit of a point.
Equidistribution of points: case without weight

Theorem (Fornaess-Sibony, Briend-Duval, D.-Sibony)

There is an invariant probability measure $\mu$ with support $J_k$ such that

$$\lim_{n \to \infty} d^{-kn}(f^n)^* (\delta_z) = \mu \quad \text{for every} \; z \in \mathbb{P}^k.$$

Moreover, the convergence is exponentially fast.

Remarks

- To see the speed of convergence we consider Hölder continuous test functions.
- The measure $\mu$ satisfies several interesting properties. In particular, it is the measure of maximal entropy $k \log d$.
- Recall that we assume that $f$ is generic. Otherwise, the statement is more complicated.
- The small Julia set $J_k$ could be a Cantor set or equal to $\mathbb{P}^k$ but not pluripolar.
Consider a weight $\phi : \mathbb{P}^k \to \mathbb{R}$ and the measures

$$\mu_{\phi, z, n} := \sum_{w \in f^{-n}(z)} e^{\phi(w) + \cdots + \phi(f^{n-1}(w))} \delta_w$$

(with accumulated weight).

Assume $\max \phi - \min \phi < \log d$ and a weak regularity: for some $p > 2$

$$\forall x, y \in \mathbb{P}^k : |\phi(x) - \phi(y)| \lesssim (1 + |\log \text{dist}(x, y)|)^{-p}.$$

Example: any Hölder continuous function satisfies the last condition.
Case with weight

**Theorem (Urbanski-Zdunik, Bianchi-D.)**

There are an invariant probability measure $\mu_\phi$ with support $J_k$, a number $\lambda > 0$ and a continuous function $\rho : \mathbb{P}^k \to \mathbb{R}_+$ such that for $m_\phi := \rho^{-1}\mu_\phi$

$$\lim_{n \to \infty} \lambda^{-n} \mu_{\phi,z,n} = \rho(z)m_\phi \quad \text{for every } z \in \mathbb{P}^k.$$

Moreover, if $\phi$ satisfies a suitable regularity (e.g. Hölder continuity), then the convergence is exponentially fast.

**Remarks**

- *The points in $f^{-n}(z)$, with weights, are equidistributed with respect to $m_\phi$ when $n \to \infty$.*
- *The measure $\mu_\phi$ maximises the pressure involving $\phi$ (similar to the entropy).*
Let $g : \mathbb{P}^k \to \mathbb{R}$ be a test continuous function. Then

$$\langle (f^n)^*(\delta_z), g \rangle = \langle \sum_{w \in f^{-n}(z)} \delta_w, g \rangle = \sum_{w \in f^{-n}(z)} g(w).$$

Define

$$(f^n)_*(g)(z) := \sum_{w \in f^{-n}(z)} g(w).$$

We have

$$\langle (f^n)^*(\delta_z), g \rangle = (f^n)_*(g)(z) = \langle \delta_z, (f^n)_*g \rangle.$$  

So $(f^n)_*$ acting on functions is dual to $(f^n)^*$ acting on measures.

In this setting, $(f^n)_*$ is the Perron-Frobenius operator.

Equidistribution:

$$\text{convergence of } d^{-kn}(f^n)^*(\delta_z) \iff \text{convergence of } d^{-kn}(f^n)_*(g).$$

Notice that $(f^n)^* = (f^*) \circ \cdots \circ (f^*)$ and $(f^n)_* = (f_*) \circ \cdots \circ (f_*)$. 
The operators acting on measures
\[ \delta_z \rightarrow \sum_{w \in f^{-n}(z)} e^{\Phi(w) + \cdots + \Phi(f^{n-1}(w))} \delta_w. \]

The Perron-Frobenius operators acting on functions
\[ g \rightarrow L^n_{\Phi}(g) \quad \text{with} \quad L^n_{\Phi}(g)(z) := \sum_{w \in f^{-n}(z)} e^{\Phi(w) + \cdots + \Phi(f^{n-1}(w))} g(w). \]

Equidistribution \iff convergence of \( \lambda^{-n}L^n_{\Phi}(g) \) for suitable \( \lambda > 0 \).

Notice that \( L^n_{\Phi} = L_{\Phi} \circ \cdots \circ L_{\Phi} \).
Consider a linear continuous operator $L : E \to E$ on a Banach space $E$.

Assume that $\lambda > 0$ is an isolated eigenvalue of multiplicity 1. So there is a vector $v \in E \setminus \{0\}$ such that $L(v) = \lambda v$.

Assume that the spectrum of $L$ is contained in $\mathbb{D}(0, r) \cup \{\lambda\}$ for some $r < \lambda$ (spectral gap).

Then $\lambda^{-n} L^n(g) \to c_g v$ exponentially fast for some constant $c_g$ (modulo $v$, we have $\lambda^{-n} L^n \to 0$ exponentially fast because $r < \lambda$).

Stability (for small $t \in \mathbb{C}$): similar properties for small perturbations $L_t : E \to E$ with $\lambda_t \simeq \lambda$, $r_t \simeq r$ and $v_t \simeq v$. 

Figure: Spectral gap and linear dynamics.
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Problem

Study the identically distributed random variables $u, u \circ f, u \circ f^2 \ldots$ in the probability space $(\mathbb{P}^k, \mu_\phi)$.

- Find a good Banach space $E$ and consider the perturbations $\mathcal{L}_{\phi+tu}$ of $\mathcal{L}_\phi$ with $t \in \mathbb{C}$ small.

- We have for $g \in E$

$$\mathcal{L}_{\phi+tu}^n g(z) = \sum_{w \in f^{-n}(z)} e^{\phi(w)+\ldots+\phi(f^{n-1}(w)) + t(u(w)+\ldots+u(f^{n-1}(w)))} g(w)$$

- Using Taylor’s expansion in $t$ and its coefficients gives statistical properties of $u$ (standard arguments but need to solve some technical problems, Nagaev, Guivarc’h, Liverani, Gouëzel...).

Problem

To find a good Banach space $E$, i.e. to find a good norm for $\mathcal{L}_\phi$ and its small perturbations $\mathcal{L}_{\phi+tu}$. In particular, we need $\mathcal{L}_{\phi+tu} : E \rightarrow E$. 
Spectral gap and consequences

**Theorem (Bianchi-D., for every $0 < \gamma \leq 1$, new even in 1D)**

There is a Banach space $(E, \| \cdot \|)$ such that

1. $\| \cdot \|_{C^0} \lesssim \| \cdot \| \lesssim \| \cdot \|_{C^\gamma}$ (Hölder norm);
2. Assume that $\|\phi\| < \infty$ (e.g. $\|\phi\|_{C^\gamma} < \infty$). Then $\mathcal{L}_\phi : E \to E$ has an isolated maximal eigenvalue $\lambda$ of multiplicity 1 and a spectral gap.
3. If $\|u\| < \infty$, then $\mathcal{L}_\phi + tu$, with $|t|$ small enough, satisfies a similar property.

**Corollary**

Assume that $\|\phi\| < \infty$ and $\|u\| < \infty$. Then the sequence $u, u \circ f, u \circ f^2 \ldots$ satisfies strong ergodic theorems (mixing, K-mixing, exponential mixing, mixing of all order), the central limit theorem, Berry-Esseen theorem, local central limit theorem, almost sure central limit theorem, large deviation principle, almost sure invariance principle and law of iterated logarithms.

**Remark (related results)**

Fornaess-Sibony, D-Nguyen-Sibony, Dupont, Szostakiewicz-Urbanski-Zdunik. In 1D: Ruelle, Smirnov, Makarov, Denker, Przytycki, Urbanski, Haydn ...
Ergodic theorem or Law of Large Numbers

Corollary (there are stronger versions such as mixing, K-mixing...)

For $\mu_\phi$-almost every point $z$ we have

$$\lim_{n \to \infty} \frac{1}{n} \left( u(z) + u(f(z)) + \cdots + u(f^{n-1}(z)) \right) = \int u \, d\mu_\phi \quad (\text{mean value})$$

or equivalently, by duality

$$\lim_{n \to \infty} \frac{1}{n} \left( \delta_z + \delta_{f(z)} + \cdots + \delta_{f^{n-1}(z)} \right) = \mu_\phi.$$

So the orbit of $z$ is asymptotically equidistributed with respect to $\mu_\phi$.

- By adding to $u$ a suitable constant, we can assume for simplicity that

  $$\int u \, d\mu_\phi = 0.$$

- So we have for $\mu_\phi$-almost every point $z$ we have

  $$\lim_{n \to \infty} \frac{1}{n} \left( u(z) + u(f(z)) + \cdots + u(f^{n-1}(z)) \right) = 0.$$
Corollary (there are more precise versions with convergence rate)

There is a number $\sigma \geq 0$ such that we have the following convergence in law

$$\frac{1}{\sqrt{n}} (u(z) + u(f(z)) + \cdots + u(f^{n-1}(z))) \rightarrow \text{the Gaussian } N(0, \sigma^2).$$

Moreover, $\sigma = 0 \iff N(0, \sigma^2) = \delta_0 \iff u \text{ is a coboundary, that is}$

$$u = v \circ f - v \text{ for some continuous } v \text{ on the small Julia set.}$$

**Figure:** Gaussian distributions. Source: internet.
Corollary

For $\mu_\phi$-almost every $z$, we have

$$\limsup_{n \to \infty} \frac{1}{\sqrt{2n \log \log n}} \left( u(z) + u(f(z)) + \cdots + u(f^{n-1}(z)) \right) = 1$$

and

$$\liminf_{n \to \infty} \frac{1}{\sqrt{2n \log \log n}} \left( u(z) + u(f(z)) + \cdots + u(f^{n-1}(z)) \right) = -1.$$
Corollary (Large Deviation Principle, exponential concentration)

For every $\epsilon > 0$, there is a constant $c(\epsilon) > 0$ such that

$$
\mu_{\phi} \left\{ z : \left| \frac{1}{n} (u(z) + u(f(z)) + \cdots + u(f^{n-1}(z))) \right| > \epsilon \right\} \lesssim e^{-c(\epsilon)n}.
$$

This is a consequence of the following.

Corollary (Large Deviation Principle)

There is a strictly convex function $c(\epsilon)$ defined in a neighbourhood of $0 \in \mathbb{R}$ such that $c(0) = 0$ and

$$
\lim_{n \to \infty} \frac{1}{n} \log \mu_{\phi} \left\{ z : \frac{1}{n} (u(z) + u(f(z)) + \cdots + u(f^{n-1}(z))) > \epsilon \right\} = -c(\epsilon).
$$
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Main difficulties: we want $\mathcal{L}_\phi : E \to E$, $\mathcal{L}_\phi$ is a perturbation of $f_*$

- Smooth functions are good for perturbations but not invariant by Perron-Frobenius operators.

Example (in dimension $k = 1$ for $f(z) = z^2$)

$$f_*(u)(z) = \sum_{w \in f^{-1}(z)} u(w) = u(\sqrt{z}) + u(-\sqrt{z}).$$

So $u$ $\gamma$-Hölder $\not\Rightarrow f_*(u)$ $\gamma$-Hölder (only $\frac{1}{2}\gamma$-Hölder).

- Functions spanned by psh ones are good for $f_*$ but not for perturbations.
- More generally, complex analysis objects like psh functions, positive closed currents are rigid for perturbations.
- Recall that we also need a spectral gap.
- We spent more than 1 year to find good norms and there are more than 10 candidates which are not good.
Construction of norms and semi-norms (almost norms)

\[ \|g\|_{\log^p} := \sup_{x, y} |g(x) - g(y)| (1 + |\log \text{dist}(x, y)|)^p \quad (\text{exp weaker than Hölder}). \]

\[ \|g\|_* := \inf \|T^+\| + \|T^-\| \]

with \( T^\pm \) positive closed \((1, 1)\)-currents such that \( \text{dd}^c g = T^+ - T^- \).

\[ \|g\|_p \simeq \|g\|_* + \|g\|_{\log^p} . \]

For \((1, 1)\)-current \( R \) and \( 0 < \alpha < 1 \)

\[ \|R\|_{p, \alpha} := \min \left\{ c : \exists S \text{ such that } |R| \leq c \sum_{n=0}^{\infty} \alpha^n d^{-(k-1)n} (f^n)_*(S) \right\} \]

with \( S \) positive closed \((1, 1)\)-current of mass 1 and potential \( \|u_S\|_p \leq 1 \).

\[ \|g\|_{\langle \alpha \rangle} := \|i\partial g \wedge \bar{\partial} g\|_{p, \alpha} \quad \text{dynamical Sobolev norm.} \]

\[ \|g\|_{(p, \alpha), \gamma} := \inf \left\{ c > 0 : \forall 0 < \epsilon < 1 \ \exists g^{(1)}_\epsilon, g^{(2)}_\epsilon : \ g = g^{(1)}_\epsilon + g^{(2)}_\epsilon , \right. \]

\[ \left. \|g^{(1)}_\epsilon\|_{\langle \alpha \rangle} \leq c (1/\epsilon)^{1/\gamma}, \|g^{(2)}_\epsilon\|_{\infty} \leq c \epsilon \right\} (\text{cost of regularization}) \]
Technical problems and a key point

Recall that

\[ \mathcal{L}_\phi^n(g)(z) := \sum_{w \in f^{-n}(z)} e^{\phi(w) + \cdots + \phi(f^{n-1}(w))} g(w). \]

We need to evaluate

\[ i\partial \mathcal{L}_\phi^n(g) \wedge \bar{\partial} \mathcal{L}_\phi^n(g) \]

with the above norms in order to get a spectral gap.

There are other technical problems.

Lemma

If \( u \) is psh and \( g \) is such that

\[ i\partial g \wedge \bar{\partial} g \leq \dd c u \]

then the modulus of continuity of \( g \) is controlled by the one of \( u \).

We can avoid local analysis by using global positive closed currents and quasi-psh functions.
Statistical properties for endomorphisms

F. Bianchi, T.-C. Dinh

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Invariant measures

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Figure: Barbers and Social Distancing. Source: internet.