Statistical properties for holomorphic endomorphisms of projective spaces

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Outline

Endomorphisms, Fatou set and Julia set

- Let $\mathbb{P}^k$ be the complex projective space of dimension k.
- Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of degree $d \geqslant 2$.
- The iterate of order n of f is of degree d^n and given by

$$f^n := f \circ \cdots \circ f \quad (n \text{ times}).$$

- If $V \subset \mathbb{P}^k$ is a subvariety of dimension p then (counting multiplicity)

$$\deg f^n(V) = d^{np} \deg V \qquad \text{and} \qquad \deg f^{-n}(V) = d^{n(k-p)} \deg V.$$

- In particular, for $p = 0$ and $V = \{z\}$ is a point

$$\#f^n(z) = 1 \qquad \text{and} \qquad \#f^{-n}(z) = d^{nk}.$$

- Generic polynomial maps $f : \mathbb{C}^k \to \mathbb{C}^k$ of degree d can be extended to a holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$.

Endomorphisms, Fatou set and Julia set

- In dimension $k = 1$, a famous example is $f : \mathbb{C}$ (or $\mathbb{P}^1$) $\to \mathbb{C}$ (or $\mathbb{P}^1$)

$$f(z) = z^2 + c \quad (c \text{ is a constant})$$

$$f^4(z) = (((z^2 + c)^2 + c)^2 + c)^2 + c.$$

- The phase space $\mathbb{P}^1$ is divided into two disjoint completely invariant sets: the Fatou set (open) and the Julia set (compact)

$$\mathbb{P}^1 = F \sqcup J \qquad f^{-1}(F) = f(F) = F \qquad f^{-1}(J) = f(J) = J.$$

- The dynamics in F is tame and stable, and the dynamics in J is chaotic.

Problem (a main problem)

Understand the dynamics (namely, the orbits of points) in J.

Julia sets

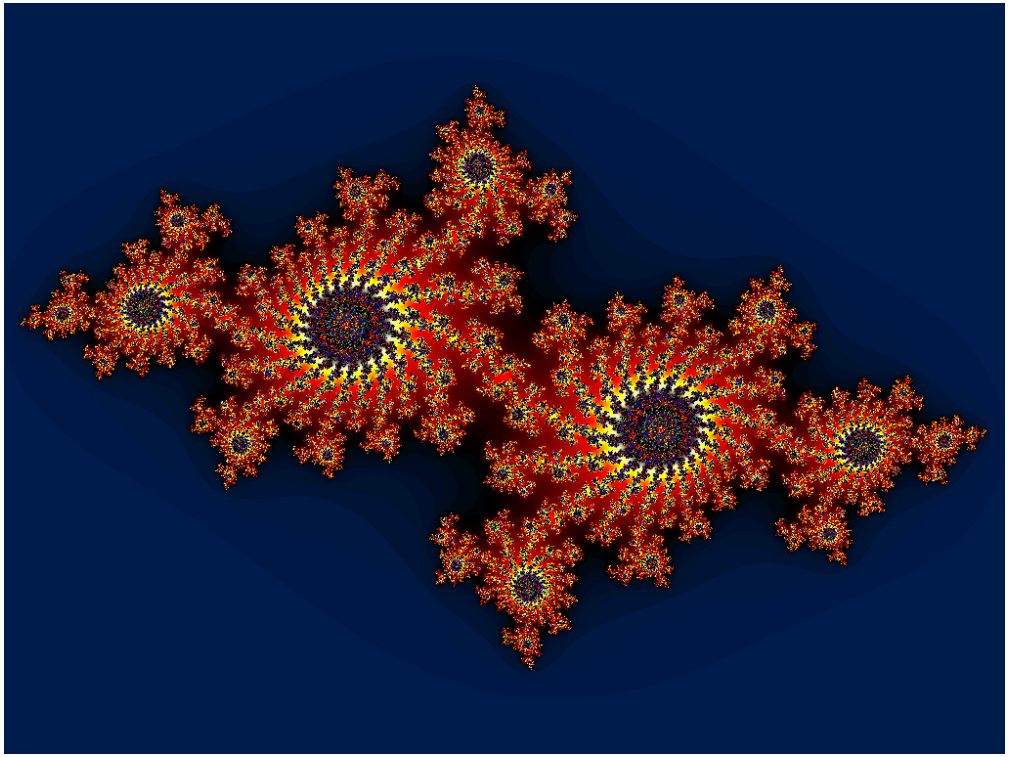


Figure: Julia set of $z^2 + c$ with $c = 0.687 + 0.312i$. Source : mcgoodwin.net.

Julia sets

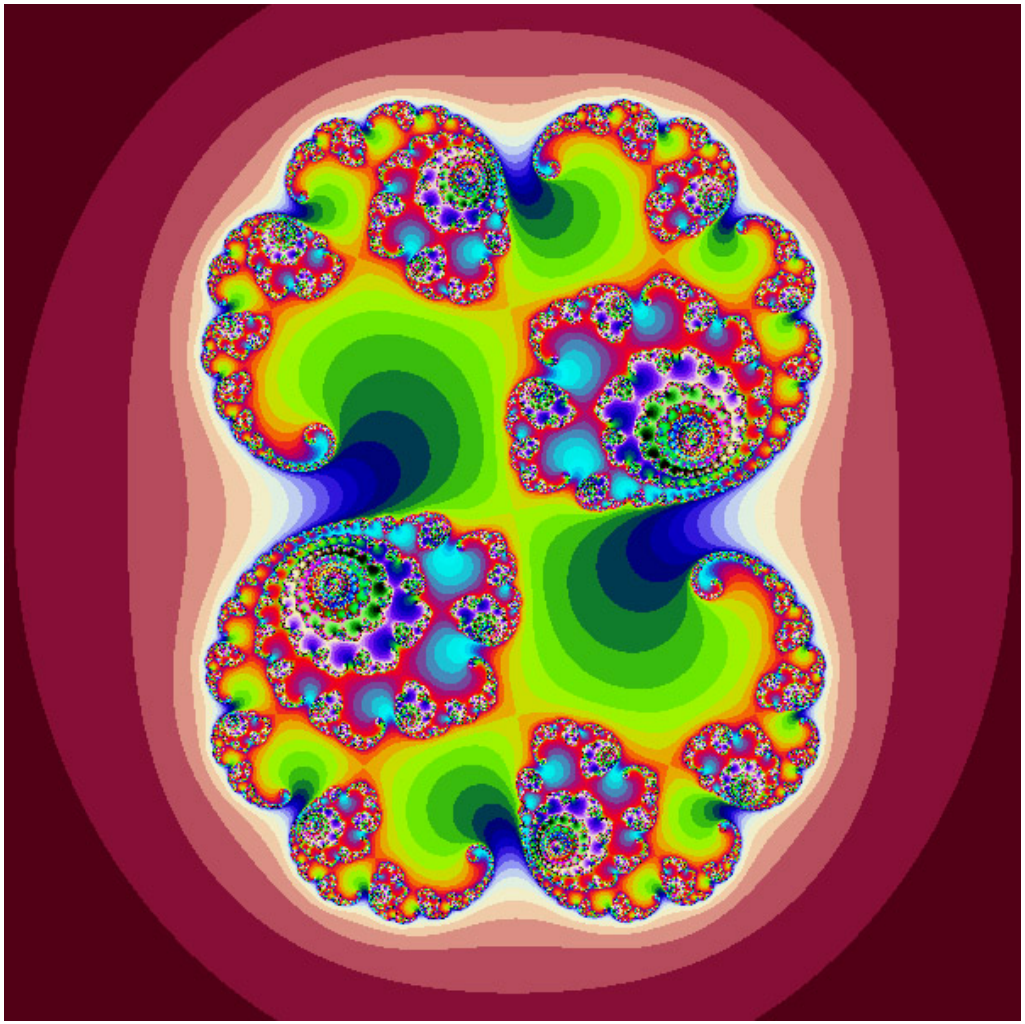


Figure: Julia set of $z^2 + c$ with $c = 0.285 + 0.01i$. Source : WikiMedia.

Any dimension

- In dimension k, we have several Julia sets

$$P^k = J_0 \supset J_1 \supset \cdots \supset J_k.$$

- We only consider the small Julia set J_k where the dynamics is most chaotic.
- For simplicity, we assume f is generic in a sense close to "the absence of periodic critical point": sub-exponential growth of multiplicity

$$\text{multiplicity}(f^n, z) \leqslant A^n \quad \text{for all } z \in \mathbb{P}^k,\ A > 1 \text{ and } n \text{ big.}$$

- There is no way to describe all orbits $(z, f(z), f^2(z), \ldots)$ of f on J_k.

Probabilistic point of view (for canonical invariant probability measures on J_k)

Study the sequence of random variables

$$u, u \circ f, u \circ f^2, \ldots$$

where $u : \mathbb{P}^k \to \mathbb{R}$ is an observable (function of suitable regularity).

Expected outcomes

Remarks

- *With respect to invariant probability measures, the random variables* $u \circ f^n$ *are identically distributed: for all* n, m *and interval* $[a, b]$

$$\text{measure of } \{u \circ f^n \in [a, b]\} = \text{measure of } \{u \circ f^m \in [a, b]\}.$$

- *These random variables are clearly not independent (i.d. but not i.i.d.).*

- We expect that the dependence (correlation) between $u \circ f^n$ and $u \circ f^m$ is weak when $|n - m|$ is large.
- We expect that properties of i.i.d. random variables still hold for the sequence $u \circ f^n$: the law of large numbers (ergodicity, mixing, K-mixing, exponential mixing), central limit theorem, Berry-Esseen theorem, local central limit theorem, almost sure central limit theorem, large deviation principle, law of iterated logarithms, almost sure invariance principle.

Outline

Backward orbits of points

- If $z \in \mathbb{P}^k$, then $\#f^{-n}(z) = d^{kn}$ (counting multiplicity).
- If δ_z is the Dirac mass at z, then

$$d^{-kn}(f^n)^*(\delta_z) = d^{-kn} \sum_{w \in f^{-n}(z)} \delta_w$$

is a probability measure.

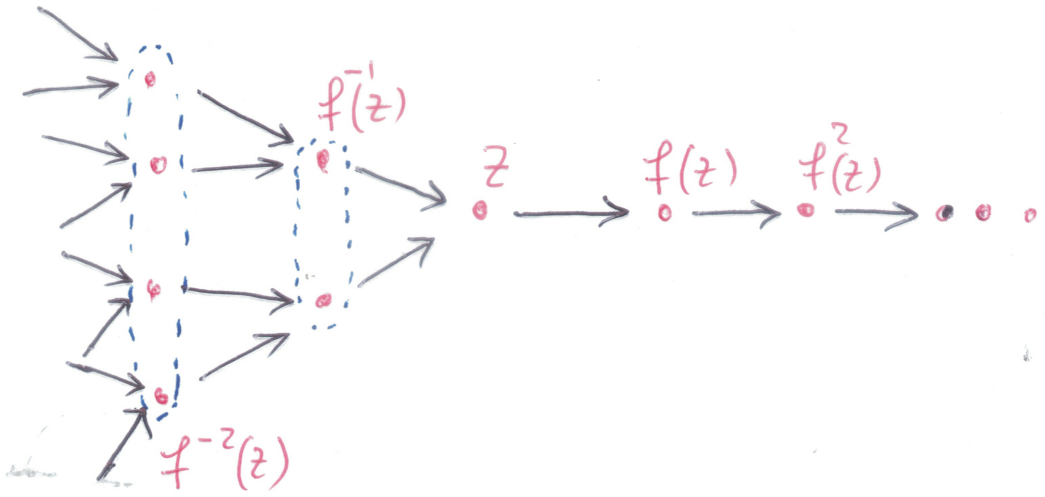


Figure: Backward orbit of a point.

Equidistribution of points: case without weight

Theorem (Fornaess-Sibony, Briend-Duval, D.-Sibony)

There is an invariant probability measure μ with support J_k such that

$$\lim_{n\to\infty} d^{-kn}(f^n)^*(\delta_z) = \mu \quad \textit{for every } z \in \mathbb{P}^k.$$

Moreover, the convergence is exponentially fast.

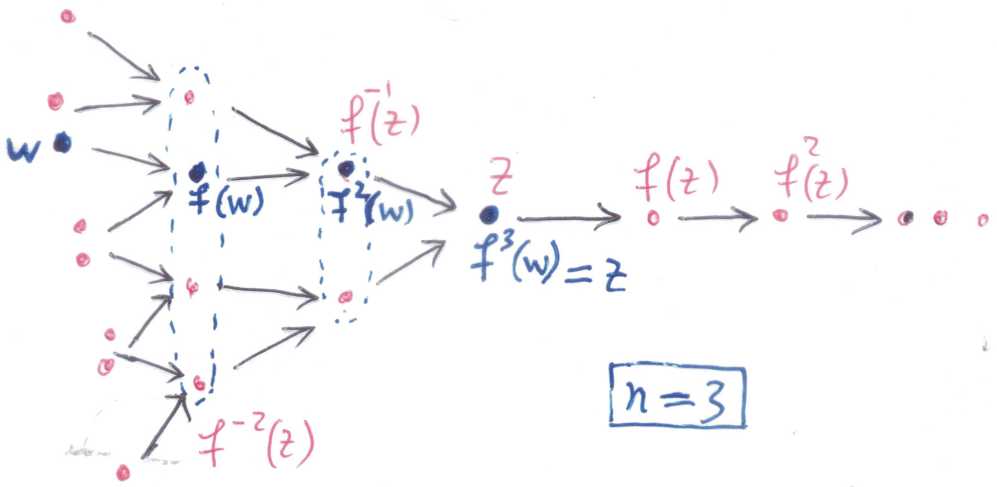
Remarks

- *To see the speed of convergence we consider Hölder continuous test functions.*
- *The measure μ satisfies several interesting properties. In particular, it is the measure of maximal entropy $k \log d$.*
- *Recall that we assume that f is generic. Otherwise, the statement is more complicated.*
- *The small Julia set J_k could be a Cantor set or equal to $\mathbb{P}^k$ but not pluripolar.*

Case with weight

- Consider a weight $\phi : \mathbb{P}^k \to \mathbb{R}$ and the measures

$$\mu_{\phi,z,n} := \sum_{w \in f^{-n}(z)} e^{\phi(w)+\cdots+\phi(f^{n-1}(w))} \delta_w \quad \text{(with accumulated weight).}$$



- Assume $\max \phi - \min \phi < \log d$ and a weak regularity: for some $p > 2$

$$\forall x, y \in \mathbb{P}^k : \quad |\phi(x) - \phi(y)| \lesssim (1 + |\log \operatorname{dist}(x, y)|)^{-p}.$$

- Example: any Hölder continuous function satisfies the last condition.

Case with weight

Theorem (Urbanski-Zdunik, Bianchi-D.)

There are an invariant probability measure μ_ϕ with support J_k, a number $\lambda > 0$ and a continuous function $\rho : \mathbb{P}^k \to \mathbb{R}_+$ such that for $m_\phi := \rho^{-1}\mu_\phi$

$$\lim_{n\to\infty} \lambda^{-n}\mu_{\phi,z,n} = \rho(z)m_\phi \quad \textit{for every } z \in \mathbb{P}^k.$$

Moreover, if ϕ satisfies a suitable regularity (e.g. Hölder continuity), then the convergence is exponentially fast.

Remarks

- *The points in $f^{-n}(z)$, with weights, are equidistributed with respect to m_ϕ when $n \to \infty$.*
- *The measure μ_ϕ maximises the pressure involving ϕ (similar to the entropy).*

Duality and Perron-Frobenius operator: case without weight

- Let $g : \mathbb{P}^k \to \mathbb{R}$ be a test continuous function. Then

$$\langle (f^n)^*(\delta_z), g\rangle = \Big\langle \sum_{w\in f^{-n}(z)} \delta_w, g \Big\rangle = \sum_{w\in f^{-n}(z)} g(w).$$

- Define

$$(f^n)_*(g)(z) := \sum_{w\in f^{-n}(z)} g(w).$$

- We have

$$\langle (f^n)^*(\delta_z), g\rangle = (f^n)_*(g)(z) = \langle \delta_z, (f^n)_* g\rangle.$$

- So $(f^n)_*$ acting on functions is dual to $(f^n)^*$ acting on measures.
- In this setting, $(f^n)_*$ is the Perron-Frobenius operator.
- Equidistribution:

$$\text{convergence of } d^{-kn}(f^n)^*(\delta_z) \iff \text{convergence of } d^{-kn}(f^n)_*(g).$$

- Notice that $(f^n)^* = (f^*) \circ \cdots \circ (f^*)$ and $(f^n)_* = (f_*) \circ \cdots \circ (f_*)$.

Duality and Perron-Frobenius operator: case with weight

- The operators acting on measures

$$\delta_z \longrightarrow \sum_{w \in f^{-n}(z)} e^{\phi(w)+\cdots+\phi(f^{n-1}(w))} \delta_w.$$

- The Perron-Frobenius operators acting on functions

$$g \longrightarrow \mathcal{L}_\phi^n(g) \quad \text{with} \quad \mathcal{L}_\phi^n(g)(z) := \sum_{w \in f^{-n}(z)} e^{\phi(w)+\cdots+\phi(f^{n-1}(w))} g(w).$$

- Equidistribution $\Longleftrightarrow$ convergence of $\lambda^{-n}\mathcal{L}_\phi^n(g)$ for suitable $\lambda > 0$.
- Notice that $\mathcal{L}_\phi^n = \mathcal{L}_\phi \circ \cdots \circ \mathcal{L}_\phi$.

Linear dynamics: ideal situation (to be applied for $\mathcal{L} := \mathcal{L}_\phi$)

- Consider a linear continuous operator $\mathcal{L} : E \to E$ on a Banach space E.
- Assume that $\lambda > 0$ is an isolated eigenvalue of multiplicity 1. So there is a vector $v \in E \setminus \{0\}$ such that $\mathcal{L}(v) = \lambda v$.
- Assume that the spectrum of $\mathcal{L}$ is contained in

$$\mathbb{D}(0, r) \cup \{\lambda\} \quad \text{for some } r < \lambda \quad \text{(spectral gap)}.$$

- Then $\lambda^{-n}\mathcal{L}^n(g) \to c_g v$ exponentially fast for some constant c_g (modulo v, we have $\lambda^{-n}\mathcal{L}^n \to 0$ exponentially fast because $r < \lambda$).

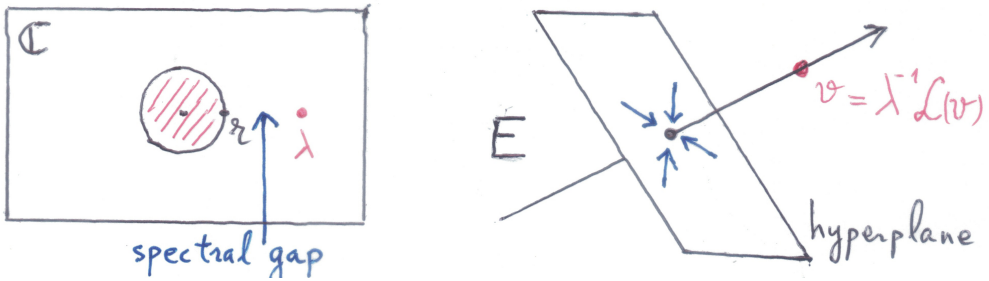


Figure: Spectral gap and linear dynamics.

- Stability (for small $t \in \mathbb{C}$): similar properties for small perturbations $\mathcal{L}_t : E \to E$ with $\lambda_t \simeq \lambda$, $r_t \simeq r$ and $v_t \simeq v$.

Outline

Statistical properties: a rough strategy

Problem

Study the identically distributed random variables $u, u \circ f, u \circ f^2 \dots$ *in the probability space* $(\mathbb{P}^k, \mu_\phi)$.

- Find a good Banach space E and consider the perturbations $\mathcal{L}_{\phi+tu}$ of $\mathcal{L}_\phi$ with $t \in \mathbb{C}$ small.
- We have for $g \in E$

$$\mathcal{L}^n_{\phi+tu} g(z) = \sum_{w \in f^{-n}(z)} e^{\phi(w)+\dots+\phi(f^{n-1}(w))+t(u(w)+\dots+u(f^{n-1}(w)))} g(w)$$

- Using Taylor's expansion in t and its coefficients gives statistical properties of u (standard arguments but need to solve some technical problems, Nagaev, Guivarc'h, Liverani, Gouëzel...).

Problem

To find a good Banach space E, *i.e. to find a good norm for* $\mathcal{L}_\phi$ *and its small perturbations* $\mathcal{L}_{\phi+tu}$. *In particular, we need* $\mathcal{L}_{\phi+tu} : E \to E$.

Spectral gap and consequences

Theorem (Bianchi-D., for every $0 < \gamma \leqslant 1$, new even in 1D)

There is a Banach space $(\mathrm{E}, \|\cdot\|)$ such that

1. $\|\cdot\|_{\mathcal{C}^0} \lesssim \|\cdot\| \lesssim \|\cdot\|_{\mathcal{C}^\gamma}$ *(Hölder norm);*
2. *Assume that $\|\phi\| < \infty$ (e.g. $\|\phi\|_{\mathcal{C}^\gamma} < \infty$). Then $\mathcal{L}_\phi : \mathrm{E} \to \mathrm{E}$ has an isolated maximal eigenvalue λ of multiplicity 1 and a spectral gap.*
3. *If $\|u\| < \infty$, then $\mathcal{L}_{\phi + tu}$, with $|t|$ small enough, satisfies a similar property.*

Corollary

Assume that $\|\phi\| < \infty$ and $\|u\| < \infty$. Then the sequence $u, u \circ f, u \circ f^2 \ldots$ satisfies strong ergodic theorems (mixing, K-mixing, exponential mixing, mixing of all order), the central limit theorem, Berry-Esseen theorem, local central limit theorem, almost sure central limit theorem, large deviation principle, almost sure invariance principle and law of iterated logarithms.

Remark (related results)

Fornaess-Sibony, D-Nguyen-Sibony, Dupont, Szostakiewicz-Urbanski-Zdunik. In 1D: Ruelle, Smirnov, Makarov, Denker, Przytycki, Urbanski, Haydn ...

Ergodic theorem or Law of Large Numbers

Corollary (there are stronger versions such as mixing, K-mixing...)

For μ_ϕ-almost every point z we have

$$\lim_{n\to\infty} \frac{1}{n}\big(u(z) + u(f(z)) + \cdots + u(f^{n-1}(z))\big) = \int u d\mu_\phi \ \textit{(mean value)}$$

or equivalently, by duality

$$\lim_{n\to\infty} \frac{1}{n}\big(\delta_z + \delta_{f(z)} + \cdots + \delta_{f^{n-1}(z)}\big) = \mu_\phi.$$

So the orbit of z is asymptotically equidistributed with respect to μ_ϕ.

- By adding to u a suitable constant, we can assume for simplicity that

$$\int u d\mu_\phi = 0.$$

- So we have for μ_ϕ-almost every point z we have

$$\lim_{n\to\infty} \frac{1}{n}\big(u(z) + u(f(z)) + \cdots + u(f^{n-1}(z))\big) = 0.$$

Central Limit Theorem

Corollary (there are more precise versions with convergence rate)

There is a number $\sigma \geqslant 0$ such that we have the following convergence in law

$$\frac{1}{\sqrt{n}}\big(u(z) + u(f(z)) + \cdots + u(f^{n-1}(z))\big) \longrightarrow \text{the Gaussian } N(0, \sigma^2).$$

Moreover, $\sigma = 0 \iff N(0, \sigma^2) = \delta_0 \iff u$ *is a coboundary, that is*

$$u = v \circ f - v \quad \text{for some continuous } v \text{ on the small Julia set.}$$

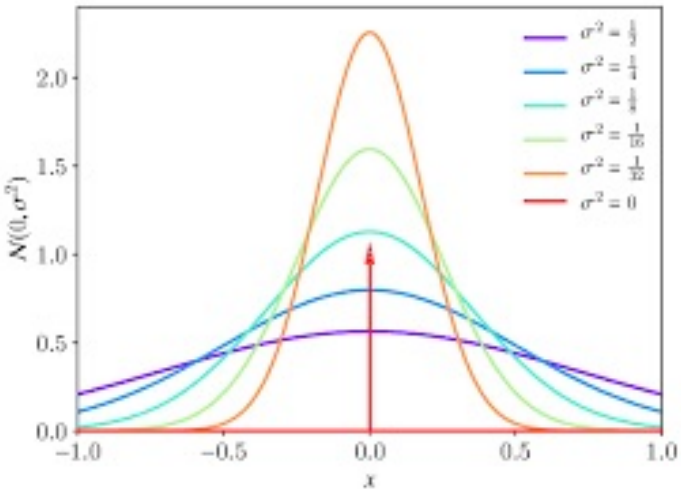


Figure: Gaussian distributions. Source: internet.

Law of iterated logarithms

Corollary

For μ_ϕ-almost every z, we have

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n\log\log n}}\big(u(z)+u(f(z))+\cdots+u(f^{n-1}(z))\big)=1$$

and

$$\liminf_{n\to\infty} \frac{1}{\sqrt{2n\log\log n}}\big(u(z)+u(f(z))+\cdots+u(f^{n-1}(z))\big)=-1.$$

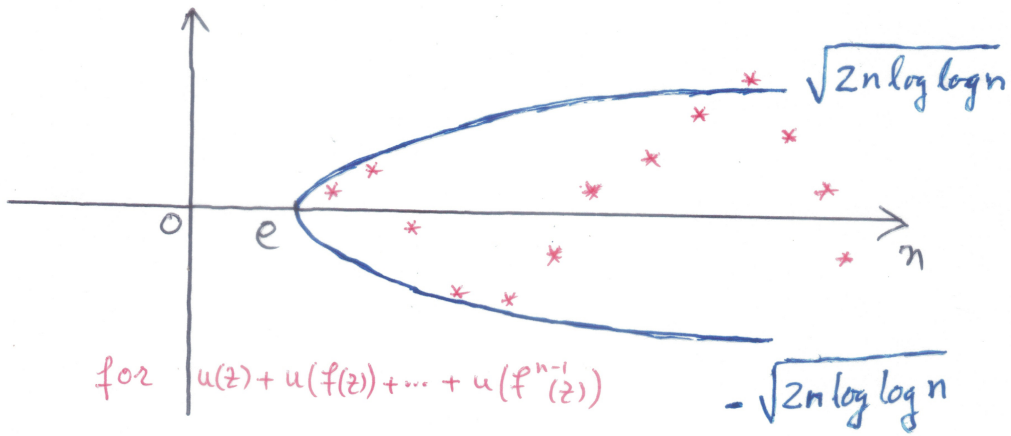


Figure: Law of iterated logarithms.

Large Deviation Principle

Corollary (Large Deviation Theorem, exponential concentration)

For every $\epsilon > 0$, there is a constant $c(\epsilon) > 0$ such that

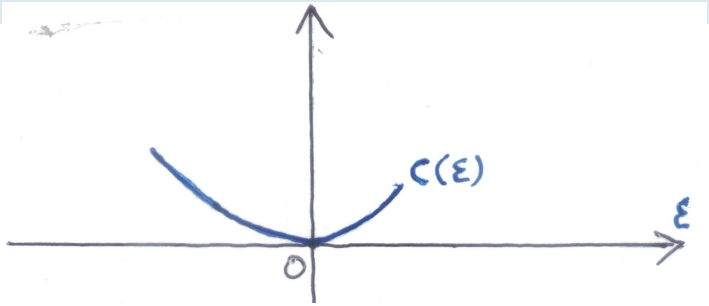
$$\mu_\phi\Big\{z : \Big|\frac{1}{n}\big(u(z) + u(f(z)) + \cdots + u(f^{n-1}(z))\big)\Big| > \epsilon\Big\} \lesssim e^{-c(\epsilon)n}.$$

This is a consequence of the following.

Corollary (Large Deviation Principle)

There is a strictly convex function $c(\epsilon)$ defined in a neighbourhood of $0 \in \mathbb{R}$ such that $c(0) = 0$ and

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\phi\Big\{z : \frac{1}{n}\big(u(z) + u(f(z)) + \cdots + u(f^{n-1}(z))\big) > \epsilon\Big\} = -c(\epsilon).$$



Outline

Main difficulties: we want $\mathcal{L}_\phi : E \to E$, $\mathcal{L}_\phi$ is a perturbation of f_*

- Smooth functions are good for perturbations but not invariant by Perron-Frobenius operators.

Example (in dimension $k = 1$ for $f(z) = z^2$)

$$f_*(u)(z) = \sum_{w \in f^{-1}(z)} u(w) = u(\sqrt{z}) + u(-\sqrt{z}).$$

So u γ-Hölder $\not\Longrightarrow$ $f_*(u)$ γ-Hölder (only $\frac{1}{2}\gamma$-Hölder).

- Functions spanned by psh ones are good for f_* but not for perturbations.
- More generally, complex analysis objects like psh functions, positive closed currents are rigid for perturbations.
- Recall that we also need a spectral gap.
- We spent more than 1 year to find good norms and there are more than 10 candidates which are not good.

Construction of norms and semi-norms (almost norms)

$$\|g\|_{\log^p} := \sup_{x,y} |g(x) - g(y)|(1 + |\log \operatorname{dist}(x,y)|)^p \quad \text{(exp weaker than Hölder).}$$

$$\|g\|_* := \inf \|T^+\| + \|T^-\|$$

with $T^\pm$ positive closed $(1,1)$-currents such that $dd^c g = T^+ - T^-$.

$$\|g\|_p \simeq \|g\|_* + \|g\|_{\log^p}.$$

For $(1,1)$-current R and $0 < \alpha < 1$

$$\|R\|_{p,\alpha} := \min\left\{c : \ \exists S \text{ such that } |R| \leqslant c \sum_{n=0}^{\infty} \alpha^n d^{-(k-1)n} (f^n)_*(S)\right\}$$

with S positive closed $(1,1)$-current of mass 1 and potential $\|u_S\|_p \leqslant 1$.

$$\|g\|_{\langle p,\alpha\rangle} := \|i\partial g \wedge \overline{\partial} g\|_{p,\alpha} \qquad \text{dynamical Sobolev norm.}$$

$$\|g\|_{\langle p,\alpha\rangle,\gamma} := \inf\Big\{c \geqslant 0 \colon \forall\, 0 < \epsilon \leqslant 1\ \exists\, g_\epsilon^{(1)}, g_\epsilon^{(2)} \colon \quad g = g_\epsilon^{(1)} + g_\epsilon^{(2)},$$
$$\|g_\epsilon^{(1)}\|_{\langle p,\alpha\rangle} \leqslant c(1/\epsilon)^{1/\gamma}, \|g_\epsilon^{(2)}\|_\infty \leqslant c\epsilon\Big\} \text{ (cost of regularization)}$$

Technical problems and a key point

- Recall that

$$\mathcal{L}_\phi^n(g)(z) := \sum_{w \in f^{-n}(z)} e^{\phi(w)+\cdots+\phi(f^{n-1}(w))} g(w).$$

- We need to evaluate

$$i\partial\mathcal{L}_\phi^n(g) \wedge \overline{\partial}\mathcal{L}_\phi^n(g)$$

  with the above norms in order to get a spectral gap.
- There are other technical problems.

Lemma

If u is psh and g is such that

$$i\partial g \wedge \overline{\partial} g \leqslant dd^c u$$

then the modulus of continuity of g is controlled by the one of u.

- We can avoid local analysis by using global positive closed currents and quasi-psh functions.

Thank you and stay safe !

Figure: Barbers and Social Distancing. Source : internet.