QUANTUM K-THEORY OF TORIC STACKS

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Abstract. In this paper, we define the quantum product on the full orbifold K-group of a smooth Deligne–Mumford stack. Its degree-zero piece is the full orbifold K-ring introduced by Jarvis–Kaufmann–Kimura [40] and Adem–Ruan–Zhang [4]. We obtain the explicit formulas of K-theoretic small I-functions of toric stacks. In the case of weight projective spaces, we give an explicit description of the relations in the small orbifold quantum K-ring, which generalize the relations obtained by Goldin–Harada–Holm–Kimura [34] for the full orbifold K-ring.

Contents

1. Introduction

1.1. **Overview.** Quantum K-theory was introduced by Givental [21] and Lee [46] as the K-theoretic generalization of Gromov-Witten theory. Let X be a smooth projective variety over $\mathbb{C}$ and let $\overline{M}_{g,n}(X,d)$ be the moduli space of degree d, genus-g, n-pointed stable maps into X. Quantum K-invariants of X are defined as holomorphic Euler characteristics of natural K-theory classes over $\overline{M}_{g,n}(X,d)$. Recently, there has been increased interest in studying quantum K-invariants due to their connections to 3d gauge theories [37, 38, 41–43, 58] and representation theory [5, 6, 45, 47, 48, 51].

It is shown in [21, 46] that the WDVV equation and most of Kontsevich-Manin axioms hold in quantum K-theory. However, there are two major differences between quantum K-theory and Gromov-Witten theory. First, there is no degree constraint or *divisor equations* in quantum K-theory. As a result, the quantum K-invariants do not vanish for large degrees even if X is Fano. A priori, a product of two K-theory classes in the quantum K-ring of a Fano variety could be a formal power series in the Novikov variables with infinitely many nonzero terms. Finiteness of quantum K-theory was first proved for the

Grassmannians [14], and the finiteness result was generalized to cominuscule homogeneous spaces in [12, 13] and any general homogeneous space $X = G/P$ in [8]. The second major difference between K-theoretic invariants and cohomological invariants is that the former is more sensitive to the stacky structure of the moduli spaces. The Kawasaki–Hirzebruch–Riemann–Roch formula [44, 54] expresses the holomorphic Euler characteristics of coherent sheaves on a smooth orbifold (or Deligne–Mumford stack) $\mathcal{M}$ as intersection numbers over its *inertia stack* $I\mathcal{M}$. By applying the (virtual) Kawasaki–Hirzebruch–Riemann–Roch formula [55], Givental–Tonita [33] expressed quantum K-invariants in terms of intersection numbers over the inertia stack $I\,\overline{M}_{g,n}(X,d)$ of the moduli space of stable maps. They gave a sophisticated description, called the *adelic characterization*, of the image of the K-theoretic *J-function* in terms of the Lagrangian cones of certain twisted Gromov-Witten invariants of the same target space.

Using the adelic characterization, Givental–Tonita [33] showed that certain explicit q-hypergeometric series, called the K-theoretic *I-function*, lies on the Lagrangian cone of the quantum K-theory of Fano complete intersections in projective spaces. Such a statement is called a *mirror theorem*. To establish mirror theorems for quantum K-theory of general target spaces, Givental studied a variant of quantum K-invariants in a series of papers [22–32]. These invariants are called *permutation-equivariant* quantum K-invariants whose definition takes into account the S_n-action on $\overline{M}_{g,n}(X,\beta)$ defined by permuting the marked points. K-theoretic mirror theorems have been proved in [24, 25, 33] for toric varieties and in [57] for general GIT quotients without stacky structure. Similar results have also been established for quantum K-theory with *level structure* in [50] for toric varieties and in [61] for general "stacky" GIT quotients. Note that mirror theorems for quantum K-theory with level structure recover those for ordinary quantum K-theory.

1.2. **Background and motivation.** We now describe part of our motivation for studying orbifold quantum K-theory, i.e., quantum K-theory of smooth Deligne–Mumford stacks. Consider the Fermat quintic three-fold X defined by the homogeneous polynomial

$$W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5.$$

In [17], Chiodo-Ruan proved a Landau–Ginzburg/Calabi–Yau (LG/CY) correspondence relating the Gromov-Witten (GW) theory of the quintic three-fold to the Fan–Jarvis–Ruan–Witten (FJRW) theory of the singularity defined by W. Let I_{GW} (resp. I_{FJRW}) denote the I-function of the GW theory of the quintic three-fold (resp. the FJRW theory of the Fermat quintic singularity W). In particular, Chiodo–Ruan showed that there is a Picard–Fuchs equation of degree four such that the summands of I_{GW} and I_{FJRW} form its fundamental solutions at two different points: "large complex structure point" and "Gepner point." They explicitly computed a symplectic transformation mapping I_{FJRW} to the analytic continuation of I_{GW}. It is natural to expect similar results to hold in K-theory.

Let $P = \mathcal{O}_{\mathbb{P}^4}(-1)$. Then the K-theoretic I-function of the Fermat quintic three-fold X is given by

$$I_X^K(Q,q) = 1 + \sum_{d=1}^{\infty} Q^d \frac{\prod_{k=1}^{5d}(1-P^5q^k)}{\prod_{k=1}^{d}(1-Pq^k)^5},$$

where Q is the Novikov variable and q is a formal parameter. Instead of satisfying a differential equation of degree 4, it satisfies a *finite-difference* equation (or q-difference equation) of degree 25 given by

$$\left[(1-q^{Q\partial_Q})^5 - Q\prod_{k=1}^{5}(1-q^{5Q\partial_Q+k})\right] I_X^K(Q,q)=0. \tag{1}$$

The above finite-difference equation is the K-theoretic counterpart of the Picard–Fuchs equation satisfied by the cohomological I-function of X. We refer the reader to [25, 59] for more details. Since the K-theoretic I-function takes value in the K-group $K(X)$, it only has four summands and therefore can not span the solution space of (1) at $Q=0$. Wen [59] constructed the extra solutions of (1) at $Q=0$ via Adams' method. The first challenge in obtaining a K-theoretic LG/CY correspondence is to understand the enumerative geometric meaning of the extra solutions. In [38], Gu, Du and the author investigated this question by studying different phases of 3d $\mathcal{N}=2$ Chern–Simons-matter theories. It was observed in [38] that the extra states should correspond to generating series of *Verlinde invariants* for the abelian group $\mathbb{C}^*$ of specific *level*. However, such invariants in the framework of the K-theoretic LG/CY correspondence have not been constructed in mathematics yet.

The second challenge in establishing the K-theoretic LG/CY correspondence is to define K-theoretic enumerative invariants in the presence of orbifold structures. Note that both FJRW theory and GW theory can be recovered by a mathematical theory of the gauged linear sigma model (GLSM) developed in [20]. A GLSM has different phases. Phases correspond to the GW theory and the FJRW theory are called geometric phases and affine phases, respectively. The affine phases naturally involve orbifold structures. To the best of our knowledge, the theory of K-theoretic enumerative invariants of GLSMs has not been developed yet. The starting point of this project is to define and study the quantum K-theory for geometric phases with non-trivial orbifold structures. We leave the study of other phases for future research and refer the reader to [7] for a recent development in this direction.

Let l,n be positive integers and let $\mathbb{P}(l,\dots,l)$ be the weighted projective space defined by the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ with weight l (see Section 5). In [38], Gu, Du and the author computed the Witten index of the 3d gauge theory for $\mathbb{P}(l,\dots,l)$, and it is equal to $l^2(n+1)$. This has surprising implications in mathematics because the Witten index is the dimension of the *state space* of the quantum K-theory of $\mathbb{P}(l,\dots,l)$. Recall that the state space of the Gromov-Witten theory of $\mathbb{P}(l,\dots,l)$ is defined to be the *Chen–Ruan* (orbifold) cohomology[1]

$$H^\bullet_{\mathrm{CR}}(\mathbb{P}(l,\dots,l)) := H^\bullet(I\mathbb{P}(l,\dots,l),\mathbb{C}),$$

where $I\mathbb{P}(l,\dots,l)=\coprod_{k=0}^{l-1}\mathbb{P}(l,\dots,l)$ is the inertia stack of $\mathbb{P}(l,\dots,l)$. Since the cohomology of $\mathbb{P}(l,\dots,l)$ is the same as that of its coarse moduli space $\mathbb{P}^n$, we conclude that the dimension of the state space of the Gromov-Witten of $\mathbb{P}(l,\dots,l)$ is $l(n+1)$. The above

[1] The grading of the Chen–Ruan cohomology of X is different from the usual grading on the cohomology of IX. We refer the reader to [2, 15] for more details on the grading.

analysis shows that if $l > 1$, then $l(n+1) < l^2(n+1)$, and therefore there are extra states in quantum K-theory of the orbifold target space $\mathbb{P}(l,\dots,l)$. It turns out that the K-group with the correct dimension has been independently defined by Jarvis–Kaufmann–Kimura [40] and Adem–Ruan–Zhang [4], which we will review in the next subsection.

1.3. **Orbifold quantum K-invariants and quantum K-ring.** Let X be a smooth projective Deligne–Mumford (DM) stack over $\mathbb{C}$. Let IX be its cyclotomic inertia stack. There are three candidates for the state space of the quantum K-theory of X: (i) $K(\underline{IX})$, where $\underline{IX}$ is the coarse moduli space of IX. (ii) $K(\bar{I}X)$, where $\bar{I}X$ is the rigidified inertia stack of X (c.f. [2, §3.1]). (iii) $\mathrm{K}_{\mathrm{orb}}(X) := K(IX)$. Note that these K-groups in general have different dimensions. In fact, there are morphisms

$$IX \xrightarrow{\varpi} \bar{I}X \xrightarrow{\pi} \underline{IX}$$

where ϖ is a rigidification map and π is the map to the coarse moduli space. One can identify the first two K-groups, $K(\underline{IX})$ and $K(\bar{I}X)$, as subspaces of the third K-group $K(IX)$ via the pullback maps $(\pi\circ\varpi)^*$ and ϖ^*. In this paper, we will choose the third K-group $\mathrm{K}_{\mathrm{orb}}(X)$ as the state space. It is referred to as the *full orbifold K-group* in [40]. One of the reasons for this choice is that the dimension of $\mathrm{K}_{\mathrm{orb}}(X)$ matches with the Witten index of the corresponding 3d gauge theory [38].

Let $\overline{M}_{g,n}(X,\beta)$ be the moduli stack of n-pointed genus-g orbifold stable maps to X of curve class β with sections to all gerbes (see [1, §4.5]). For $1 \le i \le n$, there are an evaluation map

$$\mathrm{ev}_i : \overline{M}_{g,n}(X,\beta) \to IX$$

at the i-th marking and a locally constant function $\mathbf{r}_i : \overline{M}_{g,n}(X,\beta) \to \mathbb{Z}$ defined by the index of the gerbe at the i-th marking. Let $\tilde{L}_i$ be the line bundle on $\overline{M}_{g,n}(X,\beta)$ formed by the cotangent spaces of the *orbifold* source curves at the i-th marking. We refer to $\tilde{L}_i$ as the i-th orbifold cotangent line bundle. Given a line bundle L and a positive integer r, we define the r-th *Bott's cannibalistic class* of L by $\theta^r(L) := \sum_{i=0}^{r-1} L^i$. Let $\alpha_1,\dots,\alpha_k,\mathbf{t} \in K^\circ(IX)$ be algebraic K-theory classes. Consider the natural S_n-action on $\overline{M}_{g,k+n}(X,\beta)$ defined by permuting the last n marked points. We define the permutation-equivariant quantum K-invariant with the insertions $\alpha_1,\dots,\alpha_k$, and n repeated insertion $\mathbf{t}$ by

$$\begin{aligned}&\big\langle \theta^{\mathbf{r}}(\tilde{L})\cdot\alpha_1,\dots,\theta^{\mathbf{r}}(\tilde{L})\cdot\alpha_k,\theta^{\mathbf{r}}(\tilde{L})\cdot\mathbf{t},\dots,\theta^{\mathbf{r}}(\tilde{L})\cdot\mathbf{t}\big\rangle^{S_n}_{g,k+n,\beta}\\ &\qquad := p_*\left(\mathcal{O}^{\mathrm{vir}}_{\overline{M}_{g,k+n}(X,\beta)}\cdot\prod_{i=1}^{k}\Big(\mathrm{ev}_i^*(\alpha_i)\cdot\theta^{\mathbf{r}_i}(\tilde{L}_i)\Big)\cdot\prod_{j=k+1}^{n+k}\Big(\mathrm{ev}_j^*(\mathbf{t})\cdot\theta^{\mathbf{r}_j}(\tilde{L}_j)\Big)\right),\end{aligned}$$

where $\mathcal{O}^{\mathrm{vir}}_{\overline{M}_{g,k+n}(X,\beta)} \in K_\circ(\overline{M}_{g,k+n}(X,\beta))$ is the virtual structure sheaf and p_* is the proper pushforward along the projection

$$p : \big[\overline{M}_{g,k+n}(X,\beta)/S_n\big] \to \operatorname{Spec}\mathbb{C}.$$

We refer the reader to Section 2.5 for an alternate definition. Quantum K-invariants with *topological* K-theory class insertions are defined in Definition 2.5.

Let us recall the definition of the orbifold quantum K-ring using *genus-zero* quantum K-invariants. Let $\{\Phi_a\}$ be a $\mathbb{C}$-basis for $\mathrm{K}_{\mathrm{orb}}(X)$. We define the genus-zero potential of quantum K-invariants by

$$F_0(\mathbf{t}) = \sum_{\beta \in \mathrm{Eff}(X)} \sum_{n \geq 0} Q^\beta \langle \theta^{\mathbf{r}}(\tilde{L}) \cdot \mathbf{t}, \ldots, \theta^{\mathbf{r}}(\tilde{L}) \cdot \mathbf{t} \rangle_{0,n,\beta}^{S_n},$$

where $\mathbf{t} = \sum_a t^a \Phi_a$ and $\mathrm{Eff}(X)$ is the semigroup of effective curve classes on X. The quantized pairing G on $\mathrm{K}_{\mathrm{orb}}(X)$ is defined by

$$G(\Phi_a, \Phi_b) := \partial_0 \partial_a \partial_b F_0,$$

where $\partial_a = \partial / \partial t^a$. The degree-zero piece of G is the *Mukai pairing* on $\mathrm{K}_{\mathrm{orb}}(X)$ defined by $(\alpha, \beta) := \chi(IX, \alpha \cdot \iota^* \beta)$, where ι is the involution on IX reversing the banding. Let $\Lambda = \mathbb{C}[[Q]]$ be the Q-adic completion of the semigroup ring on the effective curve classes on X. Let $\Lambda_+ \subseteq \Lambda$ be the maximal ideal generated by Q of positive degrees. Set $\mathrm{K}_{\mathrm{orb}}(X)_\Lambda := \mathrm{K}_{\mathrm{orb}}(X) \widehat{\otimes} \Lambda$, where "$\widehat{\ }$" means the completion in the Λ_+-adic topology. We assume the Mukai pairing is non-degenerate. Then the quantum product $\bullet_{\mathbf{t}}$ on $\mathrm{K}_{\mathrm{orb}}(X)_\Lambda$ is defined by

$$G(\Phi_a \bullet_{\mathbf{t}} \Phi_b, \Phi_c) = \partial_a \partial_b \partial_c F_0(\mathbf{t}).$$

We refer to the algebra $(\mathrm{K}_{\mathrm{orb}}(X)_\Lambda, \bullet_{\mathbf{t}})$ as the *full orbifold quantum K-ring* at $\mathbf{t}$. The restriction $(\mathrm{K}_{\mathrm{orb}}(X)_\Lambda, \bullet_0)$ at $\mathbf{t} = 0$ is called the *small* orbifold quantum K-ring of X.

1.4. **Summary of main results.** Let W be an affine variety with a right action of a reductive group G. Let θ be a character of G such that the θ-stable locus $W^s(\theta)$ is smooth, nonempty, and coincides with the θ-semistable locus $W^{ss}(\theta)$. We do not assume that G acts freely on $W^s(\theta)$ and consider the "stacky" GIT quotient

$$X = [W^{ss}(\theta)/G].$$

The K-theoretic small I-function of X is defined using the $(0+)$-stable *quasimap graph space* (see Section 3.3). It is proved in [61] that the K-theoretic small I-function lies in the range of the permutation-equivariant K-theoretic big J-function (see Proposition 3.13).

When X is a toric Deligne–Mumford stack (or simply toric stack), we follow the strategies in [16] and use the *stacky loop spaces* to compute the K-theoretic small I-function. Consider the action of $K = (\mathbb{C}^*)^r$ on $W = \oplus_{i=1}^m \mathbb{C}_{\rho_i}$ with weights $\rho_1, \ldots, \rho_m \in \mathbb{L}^\vee := \mathrm{Hom}(K, \mathbb{C}^*)$. Let $\theta \in \mathbb{L}^\vee \otimes \mathbb{R}$ be a stability condition and let $X = [W^{ss}(\theta)/K]$ be the corresponding toric stack. Any character $\rho \in \mathbb{L}^\vee$ determines a line bundle $L_\rho := [(W^{ss} \times \mathbb{C}_\rho)/K]$ on X. Consider the (ineffective) action of $T = (\mathbb{C}^*)^m$ on X. Let D_j be the toric divisor defined by

$$D_j := [\{(z_1, \ldots, z_m) \in W^{ss}(\theta) \mid z_j = 0\}/K], \quad 1 \leq j \leq m.$$

Let U_j be the T-equivariant line bundle $\mathcal{O}_X(-D_j)$. Note that U_j is a T-equivariant lift of L_{ρ_j}.

For any effective class β, we define $g_\beta := (e^{2\pi\sqrt{-1}\beta(L_{\pi_i})})_i \in K$. Let $(W^{ss}(\theta))^{g_\beta}$ be the subset of $W^{ss}(\theta)$ fixed by g_β and let $X^{g_\beta} = [(W^{ss}(\theta))^{g_\beta}/K]$ be the component of IX corresponding to g_β.

Theorem 1.1 (Theorem 4.8)**.** *The K-theoretic small I-function of the toric stack X is given by*

$$I(Q,q)=1+\sum_{\beta>0}Q^{\beta}\prod_{j=1}^{m}\frac{\prod_{\nu:\langle\nu\rangle=\langle\beta(L_{\rho_j})\rangle,\nu\leq 0}(1-U_jq^{\nu})}{\prod_{\nu:\langle\nu\rangle=\langle\beta(L_{\rho_j})\rangle,\nu\leq\beta(L_{\rho_j})}(1-U_jq^{\nu})}\mathbf{1}_{g_{\beta}^{-1}},$$

where $\langle\nu\rangle$ denotes the fractional part of ν and $\mathbf{1}_{g_{\beta}^{-1}}$ denotes the K-theory class of the structure sheaf in $K_T(X^{g_{\beta}^{-1}})$.

From this, we deduce the following result.

Corollary 1.2. *The K-theoretic small I-function of the toric stack X satisfies the finite-difference equations*

$$\prod_{j=1}^{m}\frac{\prod_{\ell=-\infty}^{m_{ij}-1}(1-q^{-\ell}U_j(Pq^{Q\partial_Q}))}{\prod_{\ell=-\infty}^{-1}(1-q^{-\ell}U_j(Pq^{Q\partial_Q}))}I(Q,q)=Q_iI(Q,q),\quad i=1,\dots,r.$$

Remark 1.3. Note that the K-theoretic I-function involves fractional powers of q. To obtain such a result, it is crucial to define K-theoretic invariants using moduli spaces of stable maps or quasimaps with *trivialized* gerbe markings.

Let $J_0(Q,q)$ be the *small* J-function of X (see Definition 3.4). The following proposition gives a sufficient condition for the K-theoretic mirror map to be trivial.

Proposition 1.4 (Corollary 4.9)**.** *If $\beta(\det\mathbb{T}_{[W/K]})=\sum_{j=1}^{m}\beta(L_{\rho_j})>1$ for all effective $\beta\neq 0$ and $\beta(L_{\rho_j})\geq 0$ for all j and effective β, then $J_0(Q,q)=(1-q)I(Q,q)$.*

In the last section, we follow Iritani–Milanov–Tonita [39] and use the q-difference module structure in quantum K-theory to compute the small quantum K-ring of weighted projective spaces. Let $n\in\mathbb{Z}_{>0}$ and let $w_0,\dots,w_n$ be a sequence of positive integers. Set $V=\mathbb{C}^{n+1}$. Let $\mathbb{P}^{\mathbf{w}}=\mathbb{P}(w_0,\dots,w_n)$ be the weighted projective space defined by

$$[(V-\{0\})/\mathbb{C}^*],$$

where $\mathbb{C}^*$ acts with weights $-w_0,\dots,-w_n$. The components of the inertia stack $I\mathbb{P}^{\mathbf{w}}$ are indexed by elements of the set

$$F=\left\{\frac{j}{w_i}\,\middle|\,0\leq j<w_i,0\leq i\leq n\right\}.$$

For $f\in F$, let $\mathbb{P}(V^f):=[(V^{\exp(2\pi\sqrt{-1}f)}-\{0\})/\mathbb{C}^*]$ be the component in $I\mathbb{P}^{\mathbf{w}}$ corresponding to f. Recall that the full orbifold K-group is

$$\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})=\bigoplus_{f\in F}K(\mathbb{P}(V^f)).$$

Let $\mathbf{1}_f$ denote the K-theory class of the structure sheaf of $\mathbb{P}(V^f)$ in $K(\mathbb{P}(V^f))$. In particular, $\mathbf{1}_0$ is the unit class of $\mathbb{P}^{\mathbf{w}}$, which is sometimes denoted by 1. Let $P\in\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})$ be the image of $\mathcal{O}_{\mathbb{P}^{\mathbf{w}}}(-1)\in K(\mathbb{P}^{\mathbf{w}})$ under the inclusion $K(\mathbb{P}^{\mathbf{w}})\subseteq\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})$.

Set $\Lambda = \mathbb{C}[[Q^{1/\mathrm{lcm}(w_0,\dots,w_n)}]]$. Let $Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda := (\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda, \bullet_0)$ denote the small orbifold quantum K-ring of $\mathbb{P}^{\mathbf{w}}$. For simplicity, we denote by $\cdot$ the small quantum product. Let $f_1, \dots f_k$ be the elements of F arranged in *increasing* order. Set $f_{k+1} = 1$ and $\mathbf{1}_1 = \mathbf{1}_0$. For $f \in F$, we define $I_f := \{i \mid w_i \cdot f \in \mathbb{Z}\}$. We have the following description of $Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda$.

Theorem 1.5 (see Corollaries 5.8 and 5.6)**.** *The small orbifold quantum K-ring $Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda$ is the free Λ-module generated as a Λ-algebra by the classes*

$$\mathbf{1}_{f_1}, \quad \mathbf{1}_{f_2}, \quad \dots, \quad \mathbf{1}_{f_k}, \quad P, \quad P^{-1}$$

with identity element $\mathbf{1}_{f_1} = \mathbf{1}_0$ and relations generated by

$$Q^{f_{i+1}-f_i}\mathbf{1}_{f_{i+1}} = \prod_{b \in I_{f_i}} (1 - P^{w_b})\mathbf{1}_{f_i}$$

for $1 \le i \le k$. In particular,

$$\prod_{j=0}^{n} (1 - P^{w_j})^{w_j} = Q\mathbf{1}_0. \tag{2}$$

We also have the following relations in $Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda$:

$$\mathbf{1}_{f_i} \cdot \mathbf{1}_{f_j} = Q^{f_{i,j} - \langle f_i + f_j \rangle} \prod_{a=0}^{n} (1 - P^{w_a})^{\langle -f_i w_a \rangle + \langle -f_j w_a \rangle - \langle -(f_i+f_j) w_a \rangle} \mathbf{1}_{f_{i,j}},$$

where $f_{i,j}$ is the smallest element in $F \cup \{f_{k+1}\}$ such that $f_{i,j} \ge \langle f_i + f_j \rangle$.

If we invert Q, then the small orbifold quantum K-ring is generated by P, P^{-1}, and we have a ring isomorphism

$$Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda \cong \frac{\Lambda[P, P^{-1}]}{\left\langle \prod_{j=0}^{n} (1 - P^{w_j})^{w_j} - Q\mathbf{1}_0 \right\rangle}.$$

Remark 1.6. We learned the above presentation of $Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda$ from Wei Gu (W Gu 2021, personal communication, May 11). He obtained the ring relation (2) using the twisted effective superpotential of the 3d gauge theory for $\mathbb{P}^{\mathbf{w}}$. The ring relation (2) also appears in [35].

1.5. **Review of previous related work.** Jarvis–Kaufmann–Kimura [40] defined a product called *orbifold product* on $\mathrm{K}_{\mathrm{orb}}(X)$ using an orbifold obstruction bundle over the double inertia stack of X. They refer to $\mathrm{K}_{\mathrm{orb}}(X)$ equipped with the orbifold product as the *full orbifold K-theory.* A related construction was discovered by Adem–Ruan–Zhang [4] in the topological category. The small orbifold quantum product $\bullet_0$ defined in this paper is a *deformation* of Jarvis–Kaufmann–Kimura's orbifold product. More precisely, we recover the orbifold product by specializing the small quantum product $\bullet_0$ at $Q = 0$ (see Section 2.8). The full orbifold K-theory of $\mathbb{P}^{\mathbf{w}}$ has been computed by Goldin–Harada–Holm–Kimura [34]. Theorem 1.5 is a generalization of their result.

Two different versions of orbifold quantum K-theory have been studied by Tonita–Tseng [56] and González–Woodward [35]. Tonita–Tseng's quantum K-invariants are special cases of the invariants studied in this paper; see Remark 2.8 for more details. They chose $K(\bar{I}X)$ as the state space and used the moduli stack $K_{g,n}(X,\beta)$ of stable maps *without* sections at the gerbe markings to define quantum K-invariants. González–Woodward's quantum K-theory is also different from ours because they chose $K(\underline{IX})$ as the state space. They used K-theoretic *affine gauged* Gromov–Witten invariants to define a *quantum Kirwan map* [35, Theorem 1.1]. By using the quantum Kirwan map, they gave an explicit presentation of their version of the small quantum K-ring of weighted projective space [35, Example 1.3], which is different from that in Theorem 1.5; see Remark 5.7 for more details. Note that their quantum K-ring of weighted projective space does not specialize to the full orbifold K-ring computed by Goldin–Harada–Holm–Kimura [34].

The quantum orbifold cohomology of weighted projective spaces has been computed by Coates–Corti–Lee–Tseng [19]. Theorem 1.5 is a generalization of their result in K-theory.

1.6. **Plan of the paper.** This paper is organized as follows. In Section 2, we recall the basic notation in K-theory and define permutation-equivariant quantum K-invariants for smooth Deligne–Mumford stacks. Using the genus-zero invariants, we define the full orbifold quantum K-ring. In Section 3, we recall the definitions of the K-theoretic J- and I-functions and describe the finite-difference module structure of the range of the K-theoretic big J-function. In Section 4, we compute the K-theoretic small I-functions of toric Deligne–Mumford stacks. In Section 5, we compute the small orbifold quantum K-ring of weighted projective spaces.

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2. Orbifold quantum K-theory

2.1. **Basic notation in K-theory.** Let X be a Deligne–Mumford (DM) stack. We denote by $K_\circ(X)$ the Grothendieck group of coherent sheaves on X and by $K^\circ(X)$ the Grothendieck group of locally free sheaves on X. If X has a $\mathbb{C}^*$-action, we denote by $K_\circ^{\mathbb{C}^*}(X)$ and $K^\circ_{\mathbb{C}^*}(X)$ the equivariant K-groups. Let $K(X)$ denote the Grothendieck group of topological complex vector bundles on X with complex coefficients. Given a coherent sheaf (or vector bundle) F, we denote by $[F]$ or simply F its associated K-theory class. Tensor product makes $K_\circ(X)$ a $K^\circ(X)$-module:

$$K^\circ(X) \otimes K_\circ(X) \to K_\circ X, \quad \big([E],[F]\big) \mapsto [E]\cdot[F] := [E \otimes_{\mathcal{O}_X} F].$$

We denote by $\mathbf{1}_X := [\mathcal{O}_X]$ or simply 1 the class of the structure sheaf of X. Throughout the paper, we consider Grothendieck groups with complex coefficients $K_\circ(X)_\mathbb{C} := K_\circ(X)\otimes\mathbb{C}$, $K^\circ(X)_\mathbb{C} := K^\circ(X)\otimes\mathbb{C}$. Let $\mathrm{pt} := \operatorname{Spec}\mathbb{C}$ be a the point and let $BG = [\mathrm{pt}/G]$ be the classifying stack of a reductive group G. Then $K^\circ(BG) \cong R(G)$, the representation ring of G.

When considering $\mathbb{C}^*$-actions, we will use q to denote the equivariant parameter (or weight). More precisely, the parameter q corresponds to the standard representation $\mathbb{C}_{\mathrm{std}}$ of $\mathbb{C}^*$. Recall that $K^\circ_{\mathbb{C}^*}(\mathrm{pt}) = K^\circ(B\mathbb{C}^*) \cong \mathbb{Z}[q, q^{-1}]$.

Let E be a vector bundle on X. We define the *K-theoretic Euler class* of E by

$$\lambda_{-1}(E^\vee) := \sum_i (-1)^i \wedge^i E^\vee \in K^\circ(X),$$

where $E^\vee$ denotes the dual of E and $\wedge^i E^\vee$ denotes the i-th exterior power of $E^\vee$. If X has a $\mathbb{C}^*$-action and E is a $\mathbb{C}^*$-equivariant vector bundle, we use the same formula to define its $\mathbb{C}^*$-equivariant K-theoretic Euler class $\lambda_{-1}^{\mathbb{C}^*}(E^\vee) \in K^\circ_{\mathbb{C}^*}(X)$.

Let L be a line bundle and let r be an integer. We denote by L^r the r-th tensor power of L. For $r \in \mathbb{Z}_{>0}$, the r-th Bott's cannibalistic class of L is defined by $\theta^r(L) := \sum_{i=0}^{r-1} L^i$.

For a flat morphism $f : X \to Y$, we have the flat pullback $f^* : K_\circ(Y) \to K_\circ(X)$. For a proper morphism $g : X \to Y$, we have the proper pushforward $f_* : K_\circ(X) \to K_\circ(Y)$ defined by

$$[F] \mapsto \sum_n (-1)^n [R^n f_* F].$$

In the case when X is proper and Y is a point, the proper pushforward can be identified with the holomorphic Euler characteristic

$$\chi(F) := \sum_{i \geq 0} \dim_{\mathbb{C}} H^i(X, F)$$

for $F \in K_\circ(X)$.

We recall the analogue of the holomorphic Euler characteristic in topology K-theory. In this paper, we will only consider target spaces that are quotient stacks coming from the geometric invariant theory (GIT). To be more precise, we make the following

Assumption 2.1. We assume the target space X can be written as a "stacky" GIT quotient

$$X = [W^{ss}(\theta)/G],$$

where

- W is an affine variety with a right action of a reductive group G,
- θ is a character of G and $W^{ss}(\theta)$ is the θ-semistable locus.

We further assume that W has at worst local complete intersection singularities and the θ-stable locus $W^s(\theta)$ is smooth, nonempty, and coincides with $W^{ss}(\theta)$.

Take $W \subseteq \mathbb{C}^n$. Let H be a maximal compact subgroup of G. Let $\mathfrak{h}$ be the Lie algebra of H and let $(\, , \,)$ be a Hermitian inner product on $\mathbb{C}^n$, preserved by H. We have a Hamiltonian action of H on W with moment map $\mu : W \to \mathfrak{h}^*$ for the action of H on W, given by $\mu(w)(A) = (w, Aw)$ for $w \in W$ and $A \in \mathfrak{h}$. Let $d\theta$ be the restriction to $\mathfrak{h}$ of the derivative of θ at the identity in G. Note that the coadjoint orbit of $-d\theta$ in $\mathfrak{h}^*$ is trivial. The symplectic orbifold quotient of W at $-d\theta$ is defined as $[\mu^{-1}(-d\theta)/H]$. By a generalization of the celebrated Kempf–Ness theorem, we have an equivalence of orbifolds:

$$[\mu^{-1}(-d\theta)/H] = [W^{ss}(\theta)/G]$$

(see [20, Corollary 3.1.4]).

Set $Z := \mu^{-1}(-d\theta)$. Then the topological K-group $K([Z/H])$ is isomorphic to the *equivariant* K-group $K_H(Z)$ studied by Atiyah–Segal [9, 52], i.e. the Grothedieck group of H-equivariant topological complex vector bundles on Z with complex coefficients. Suppose that X is proper. Then Z is compact. Let $f : Z \to \mathrm{pt}$ be the unique proper H-equivariant map to a point. We recall the construction of the pushforward or equivariant *Gysin* map

$$f_* : K_H(Z) \to K_H(\mathrm{pt}) = R(H) \otimes_{\mathbb{Z}} \mathbb{C}$$

in topological K-theory. In general, for a locally compact topological space U with a continuous H-action, Segal [52] introduced the H-equivariant topological K-cohomology $K^c_H(U)$ with *compact supports* in U. The functor $U \mapsto K^c_H(U)$ is *covariant* for open embeddings. Recall that $W \subseteq \mathbb{C}^n$ and $Z = \mu^{-1}(-d\theta) \subseteq W$. Let N be the normal bundle of Z in $\mathbb{C}^n$. Then the pushforward f_* is given by the composition of the Thom isomorphism

$$K_H(Z) \cong K^c_H(N),$$

the natural extension homomorphism $K^c_H(N) \to K^c_H(\mathbb{C}^n)$ obtained by identifying the normal bundle N as a (H-equivariant) tubular neighbourhood of Z in $\mathbb{C}^n$, and the Thom isomorphism $K^c_H(\mathbb{C}^n) \cong K_H(\mathrm{pt})$.

We define the following Gysin map in topological K-theory

$$\chi^{\mathrm{top}} : K(X) \cong K_H(Z) \to K(\mathrm{pt}) \cong \mathbb{C}, \quad F \mapsto (f_* F)^H,$$

where $(f_*F)^H$ denotes the H-invariant part of the virtual H-representation f_*F. The number $\chi^{\mathrm{top}}(F)$ is sometimes referred to as the *topological index* of F and does not depend on the orbifold quotient presentation $[Z/H]$ of X. If F is an algebraic K-theory class in $K_\circ(X)$, then its holomorphic Euler characteristic coincides with the topological index of the corresponding topological K-theory class (see [53, §5]).

2.2. **The full orbifold K-group.** Let X be a smooth projective DM stack over $\mathbb{C}$, satisfying Assumption 2.1 . Let $IX = \coprod_r I_r X$ be its cyclotomic inertia stack (c.f. [2, §3.1]).

Definition 2.2. The full orbifold K-group of X is defined by

$$\mathrm{K}_{\mathrm{orb}}(X) := K(IX) = \bigoplus_r K(I_r X),$$

where $K(IX)$ (resp. $K(I_rX)$) is the Grothendieck group of topological complex vector bundles on IX (resp. I_rX) with complex coefficients.

Recall that objects in the category underlying IX are pairs (x, g) with x an object in X and $g \in \mathrm{Aut}(x)$. Let $IX = \coprod_{c \in \mathcal{I}} X_c$ be the connected component decomposition of IX for some index set $\mathcal{I}$. Then we have

$$\mathrm{K}_{\mathrm{orb}}(X) = \bigoplus_{c \in \mathcal{I}} K(X_c).$$

Let V be a vector bundle on X_c. At a point $(p, g) \in X_c$, the fiber of V admits an action of g, and therefore decomposes into a direct sum of eigenspaces of the g-action. Let r be the order of g and let $\zeta_r = \exp(2\pi\sqrt{-1}/r)$ be a primitive r-th root of unity. Then V has a canonical decomposition

$$V \cong \bigoplus_{0\le l<r} V_r^{(l)},$$

where $V_r^{(l)}$ is the subbundle of V on which g acts with eigenvalue ζ_r^l. Let $K(I_rX)^{(l)}$ be the Grothendieck group of topological complex vector bundles on I_rX which are eigen-bundles with eigenvalue ζ_r^l. Then we have decompositions

$$K(I_rX) \cong \bigoplus_{l=0}^{r-1} K(I_rX)^{(l)} \tag{3}$$

and

$$\mathrm{K}_{\mathrm{orb}}(X) \cong \bigoplus_{r\ge 1}\bigoplus_{l=0}^{r-1} K(I_rX)^{(l)}.$$

The *Mukai pairing* on $\mathrm{K}_{\mathrm{orb}}(X)$ is defined by

$$(\alpha,\beta) = \chi(IX, \alpha\cdot\iota^*\beta), \tag{4}$$

where ι is the involution on IX reversing the banding. For convenience, we make the following

Assumption 2.3. The Mukai pairing (4) on $\mathrm{K}_{\mathrm{orb}}(X)$ is non-degenerate.

2.3. **The symplectic loop space formalism.** Let X be a smooth projective DM stack satisfying Assumption 2.1. Let $\mathrm{Eff}(X) \subseteq H_2(X,\mathbb{Q})$ denote the semigroup of effective curve classes on X. We fix the ground coefficient ring to be a λ-algebra Λ, i.e. an algebra over $\mathbb{Q}$ equipped with abstract Adams operations $\Psi^k, k = 1, 2, \ldots$. Here $\Psi^k : \Lambda \to \Lambda$ are ring homomorphisms satisfying $\Psi^r\Psi^s = \Psi^{rs}$ and $\Psi^1 = \mathrm{id}$. We assume that Λ is over $\mathbb{C}$ and includes the Novikov variables $Q^\beta, \beta \in \mathrm{Eff}(X)$ and the torus-equivariant K-ring of a point if we consider a torus-action on X. We also assume that Λ is equipped with a maximal ideal Λ_+ such that $\Psi^i(\Lambda_+) \subseteq (\Lambda_+)^2$ for $i > 1$ and Λ is complete with respect to the Λ_+-adic topology. For example, one can choose

$$\Lambda = \mathbb{C}[[N_1, N_2, \ldots]][[Q]][\lambda_1^{\pm 1}, \ldots, \lambda_N^{\pm 1}],$$

where N_i are the Newton polynomials (in infinitely or finitely many variables), and λ_i denote the torus-equivariant parameters. The Adams operations Ψ^r act on N_m and Q via $\Psi^r(N_m) = N_{rm}$ and $\Psi^r(Q^\beta) = Q^{r\beta}$, respectively, and they act trivially on the torus-equivariant parameters. One can take Λ_+ to be the maximal ideal generated by N_i, λ_i and Novikov variables of positive degrees. When studying the small quantum K-ring of weighted projective space in Section 5, we will choose the ground ring to be

$$\Lambda = \mathbb{C}[[Q]],$$

which is the Q-adic completion of the semigroup ring on effective curve classes on X.

We generalize the *symplectic loop space formalism* in *permutation-equivariant* quantum K-theory [27, 56].

Definition 2.4. Define

$$\mathcal{K}_r := [K(I_rX) \otimes \mathbb{C}(q^{1/r})]\widehat{\otimes}\Lambda,$$

where $\mathbb{C}(q^{1/r})$ is the field of rational functions in $q^{1/r}$ and "$\widehat{\ }$" means the completion in the Λ_+-adic topology. The K-theoretic loop space is defined by

$$\mathcal{K} := \bigoplus_r \mathcal{K}_r.$$

By definition, an element $f(q)$ in $\mathcal{K}$ is a tuple $(f_r(q^{1/r}))_r$ such that for any $r \geq 1$, *modulo any power of* Λ_+, $f_r(q)$ is a rational function in q with coefficients in $K(I_rX) \otimes \Lambda$. From now on, we simplify refer to elements in $\mathcal{K}$ as rational functions.

By viewing elements in $\mathbb{C}(q^{1/r})\widehat{\otimes}\Lambda, r \geq 1$, as coefficients, we extend the Mukai pairing to $\mathcal{K}$ via linearity. There is a natural Λ-valued symplectic form Ω on $\mathcal{K}$ defined by

$$\Omega(f, g) := [\mathrm{Res}_{q=0} + \mathrm{Res}_{q=\infty}](f(q), g(q^{-1}))\frac{dq}{q}, \quad \text{where } f, q \in \mathcal{K}.$$

Consider the following subspaces of $\mathcal{K}_r$:

$$\mathcal{K}_{r,+} = (K(I_rX) \otimes \mathbb{C}[q^{1/r}, q^{-1/r}])\widehat{\otimes}\Lambda \quad \text{and} \quad \mathcal{K}_{r,-} = \{f \in \mathcal{K}_r | f(0) \neq \infty, f(\infty) = 0\}.$$

In other words, $\mathcal{K}_{r,+}$ is the space of $K(I_rX) \otimes \Lambda$-valued Laurent polynomials in $q^{1/r}$ (in the Λ_+-adic sense) and $\mathcal{K}_{r,-}$ consists of *proper* rational functions in $q^{1/r}$ whose limits at 0 exist. With respect to Ω, there is a *Lagrangian polarization* $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, where

$$\mathcal{K}_+ = \bigoplus_r \mathcal{K}_{r,+} \quad \text{and} \quad \mathcal{K}_- = \bigoplus_r \mathcal{K}_{r,-}.$$

For any $f(q) \in \mathcal{K}$, we write $f(q) = [f(q)]_+ + [f(q)]_-$, where $[f(q)]_+ \in \mathcal{K}_+$ and $[f(q)]_- \in \mathcal{K}_-$.

2.4. **Quantum K-invariants.** In a series of works [22–32], Givental developed the theory of *permutation-equivariant* quantum K-invariants, which takes into account the S_n-action on the moduli spaces of stable maps by permuting the markings. We introduce the basic definitions in our setting. As before, X is a smooth projective DM stack satisfying Assumption 2.1. Let $\overline{M}_{g,n}(X, \beta)$ be the moduli stack of n-pointed genus-g orbifold stable maps to X of curve class β with *sections to all gerbes* (see [1, §4.5]). For $1 \leq i \leq n$, there are an evaluation map

$$\mathrm{ev}_i : \overline{M}_{g,n}(X, \beta) \to IX$$

at the i-th marking and a locally constant function $\mathbf{r}_i : \overline{M}_{g,n}(X, \beta) \to \mathbb{Z}$ defined by the index of the gerbe at the i-th marking. Let $\mathbf{r}$ be the locally constant function on IX that takes value r on the component I_rX. Then we have $\mathbf{r}_i = (\mathrm{ev}_i)^*(\mathbf{r})$, for $1 \leq i \leq n$.

Let $\tilde{L}_i$ denote the *orbifold* cotangent line bundle at the i-th marking. Let L_i denote the cotangent line bundle at the i-th marking of *coarse* curves. We have $\tilde{L}_i^{\otimes \mathbf{r}_i} \cong L_i$. For this reason, we write $\tilde{L}_i = L_i^{1/\mathbf{r}_i}$. By the general constructions in [46, 49], the perfect obstruction theory of $\overline{M}_{g,n}(X, \beta)$ induces a virtual structure sheaf

$$\mathcal{O}^{\mathrm{vir}}_{\overline{M}_{g,n}(X,\beta)} \in K_\circ(\overline{M}_{g,n}(X, \beta)).$$

Consider the S_n-action on $\overline{M}_{g,n}(X,\beta)$ defined by permuting the n markings and the S_n-action on $(IX)^n$ via permutation. Then the total evaluation map

$$\mathrm{ev} := \prod_{i=1}^{n} \mathrm{ev}_i : \overline{M}_{g,n}(X,\beta) \to (IX)^n$$

is S_n-equivariant and proper. Let $\mathrm{pr}_i : (IX)^n \to IX$ be the i-th projection.

Definition 2.5. For any Laurent polynomial $\mathbf{t}(q) = \sum_{r\geq 1}\sum_{j\in\mathbb{Z}} \mathbf{t}_{r,j} q^{j/r} \in \mathcal{K}_+$, we define the permutation-equivariant quantum K-invariant by

$$\big\langle \mathbf{t}(L),\ldots,\mathbf{t}(L)\big\rangle^{S_n}_{g,n,\beta} :=$$
$$\chi^{\mathrm{top}}\left([(IX)^n/S_n], \sum_{\substack{r_1,\ldots,r_n\geq 1,\\ j_1,\ldots,j_n}} \left(\prod_{i=1}^{n} \mathrm{pr}_i^*\mathbf{t}_{r_i,j_i}\right)\cdot \mathrm{ev}_*\left(\mathcal{O}^{\mathrm{vir}}_{\overline{M}_{g,n}(X,\beta)} \cdot \prod_{i=1}^{n} \tilde{L}_i^{j_i}\right)\right).$$

Remark 2.6. If $\mathbf{t}_{r,j}$ are algebraic K-theory classes in $K^\circ(IX)$, then we can remain in the algebraic setting and give the following equivalent definition of the permutation-equivariant quantum K-invariant:

$$\big\langle \mathbf{t}(L),\ldots,\mathbf{t}(L)\big\rangle^{S_n}_{g,n,\beta} = p_*\big(\mathcal{O}^{\mathrm{vir}}_{\overline{M}_{g,n}(X,\beta)} \cdot \prod_{i=1}^{n} \mathrm{ev}_i^*(\mathbf{t}(L_i))\big), \tag{5}$$

where p_* is the proper pushforward along the projection

$$p : \big[\overline{M}_{g,n}(X,\beta)/S_n\big] \to \operatorname{Spec}\mathbb{C}.$$

and $\mathrm{ev}_i^*(\mathbf{t}(L_i)) := \sum_{r,j} \mathrm{ev}_i^*(\mathbf{t}_{r,j}) L_i^{j/r} = \sum_{r,j} \mathrm{ev}_i^*(\mathbf{t}_{r,j}) \tilde{L}_i^{j}$.

In the literature of (permutation-equivariant) quantum K-theory, the definition of quantum K-invariants is usually given by (5), even if $\mathbf{t}_{r,j}$ are topological K-theory classes. In this case, the proper pushforward p_* should be understood as the composition of ev_* : $K_\circ([\overline{M}_{g,n}(X,\beta)/S_n]) \to K_\circ([(IX)^n/S_n])$, the map $K_\circ([(IX)^n/S_n]) \to K([(IX)^n/S_n])$ from the algebraic K-group to the topological K-group, and the topological index map $\chi^{\mathrm{top}} : K([(IX)^n/S_n]) \to \mathbb{C}$. In the rest of the paper, we will adopt this convention and abuse the notation for proper pushforwards in K-theory.

Remark 2.7. Let $K_{g,n}(X,\beta)$ be the moduli stack of n-pointed genus-g orbifold stable maps to X of curve class β studied in [2, 3], where the sections at the markings are absent. Let $\mathcal{G}_i$ be the i-th marking in the universal curve of $K_{g,n}(X,\beta)$, which is a gerbe over $K_{g,n}(X,\beta)$. Then we have

$$\overline{M}_{g,n}(X,\beta) = \mathcal{G}_1 \underset{K_{g,n}(X,\beta)}{\times} \cdots \underset{K_{g,n}(X,\beta)}{\times} \mathcal{G}_n.$$

Let $\rho : \overline{M}_{g,n}(X,\beta) \to K_{g,n}(X,\beta)$ be the projection. Then $\rho^*\mathcal{O}^{\mathrm{vir}}_{K_{g,n}(X,\beta)} = \mathcal{O}^{\mathrm{vir}}_{\overline{M}_{g,n}(X,\beta)}$ and $\rho_*\big(\prod_{i=1}^n \mathrm{ev}_i^*(\mathbf{t}(L_i))\big)$ is an S_n-equivariant K-theory class on $K_{g,n}(X,\beta)$. Hence by the projection formula, we have

$$\big\langle \mathbf{t}(L),\dots,\mathbf{t}(L)\big\rangle_{g,n,\beta}^{S_n} = q_*\Big(\mathcal{O}^{\mathrm{vir}}_{K_{g,n}(X,\beta)}\cdot \rho_*\big(\prod_{i=1}^{n}\mathrm{ev}_i^*(\mathbf{t}(L_i))\big)\Big),$$

where q_* is the proper pushforward along the projection

$$q:\big[K_{g,n}(X,\beta)/S_n\big]\to \operatorname{Spec}\mathbb{C}.$$

For any $j\in\mathbb{Z}$, we define $\bar{j}_r\in\{0,\dots,r-1\}$ to be the remainder of j divided by r. Let $\mathbf{t}_{r,j}^{(\bar{j}_r)}\in \mathrm{K}_{\mathrm{orb}}(X)^{(\bar{j}_r)}$ be the component of $\mathbf{t}_{r,j}$ under the decomposition (3). Then we have

$$\rho_*\big(\prod_{i=1}^{n}\mathrm{ev}_i^*(\mathbf{t}(L_i))\big) = \sum_{r,j}\rho_*\big(\prod_{i=1}^{n}\mathrm{ev}_i^*(\mathbf{t}_{r,j}^{(\bar{j}_r)})\tilde{L}_i^{j}\big).$$

This is because any generator of the automorphism group at the i-th marking acts on $\mathrm{ev}_i^*(\mathbf{t}_{r,j}^{(l)})$ with eigenvalue ζ^l and on $\tilde{L}_i^j$ with weight ζ^{-j}, where ζ is a primitive r-th root of unity. The pushforward of the class $\mathrm{ev}_i^*(\mathbf{t}_{r,j}^{(l)})\tilde{L}_i^j$ along $\mathcal{G}_i\to K_{g,n}(X,\beta)$ is zero unless $l\equiv j \bmod r$. In sum, we have

$$\big\langle \mathbf{t}(L),\dots,\mathbf{t}(L)\big\rangle_{g,n,\beta}^{S_n} = \big\langle \underline{\mathbf{t}}(L),\dots,\underline{\mathbf{t}}(L)\big\rangle_{g,n,\beta}^{S_n},$$

where $\underline{\mathbf{t}}(q):=\sum_{r\geq 1}\sum_{j\in\mathbb{Z}}\mathbf{t}_{r,j}^{(\bar{j}_r)}q^{j/r}$.

Remark 2.8. In [56], Tonita–Tseng studied *non*-permutation-equivariant quantum K-invariants for DM stacks. They chose the moduli stack $K_{g,n}(X,\beta)$ instead of $\overline{M}_{g,n}(X,\beta)$. Since the orbifold cotangent line bundles $\tilde{L}_i$ do not exist over $K_{g,n}(X,\beta)$, Tonita–Tseng only considered the coarse cotangent line bundles L_i when defining descendant invariants. Hence, our definition is a generalization of Tonita–Tseng's orbifold quantum K-invariants.

Definition 2.5 can be generalized to the case when there are different insertions. Suppose that we are given several Laurent polynomials $\mathbf{t}_a=\sum_{r,m}(\mathbf{t}_a)_{r,m}q^{m/r}$ with $a=1,\dots s$. Let $(k_1,\dots,k_s)$ be a partition of n. We define the permutation-equivariant quantum K-invariant with symmetry group $S_{k_1}\times\cdots\times S_{k_s}$ by

$$\big\langle \mathbf{t}_1,\dots,\mathbf{t}_1,\mathbf{t}_2,\dots,\mathbf{t}_2,\dots,\mathbf{t}_s,\dots,\mathbf{t}_s\big\rangle_{g,n,\beta}^{S_{k_1}\times\cdots\times S_{k_s}} := \pi_*\big(\mathcal{O}^{\mathrm{vir}}_{\overline{M}_{g,n}(X,\beta)}\cdot\prod_{a=1}^{s}\prod_{i=1}^{k_a}\big(\sum_{r,m}\mathrm{ev}_i^*((\mathbf{t}_a)_{r,m})\tilde{L}_i^{m}\big)\big),$$

where π_* is the proper pushforward along the projection

$$\pi:\big[\overline{M}_{g,n}(X,\beta)/S_{k_1}\times\cdots\times S_{k_s}\big]\to\operatorname{Spec}\mathbb{C}.$$

We have the following *permutation-equivariant multinomial formula*:

$$\Big\langle \sum_{i=1}^{m} \mathbf{t}_i, \ldots, \sum_{i=1}^{m} \mathbf{t}_i \Big\rangle_{g,n,\beta}^{S_n} = \sum_{k_1+k_2+\cdots+k_m=n} \Big\langle \mathbf{t}_1, \ldots, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_2, \ldots, \mathbf{t}_m, \ldots, \mathbf{t}_m \Big\rangle_{g,n,\beta}^{S_{k_1}\times S_{k_2}\times\cdots S_{k_m}},$$

where $\mathbf{t}_1, \ldots \mathbf{t}_m \in \mathcal{K}_+$.

2.5. **Genus-0 theory.** Let $\mathbf{t} \in \mathrm{K}_{\mathrm{orb}}(X)$. Using the decomposition (3), we can uniquely write $\mathbf{t} = \sum_{r=1}^{\infty} \sum_{j=0}^{r-1} \mathbf{t}_{r,j}$, where $\mathbf{t}_{r,j} \in K(I_r X)^{(j)}$. We define an injective linear map

$$\mathrm{tr}_q : \mathrm{K}_{\mathrm{orb}}(X) \to \mathcal{K}_+,$$
$$\mathbf{t} \mapsto \sum_{r=1}^{\infty} \sum_{j=0}^{r-1} \mathbf{t}_{r,j} q^{j/r}.$$

Remark 2.9. Recall that for any line bundle L and $r \in \mathbb{Z}_{>0}$, the r-th Bott's cannibalistic class of L is defined by $\theta^r(L) = \sum_{i=0}^{r-1} L^i$. Let $\alpha_1, \ldots, \alpha_k, \mathbf{t} \in \mathrm{K}_{\mathrm{orb}}(X)$. The discussion in Remark 2.7 implies the following

$$\Big\langle \theta^{\mathbf{r}}(\tilde{L}) \cdot \alpha_1, \ldots, \theta^{\mathbf{r}}(\tilde{L}) \cdot \alpha_k, \theta^{\mathbf{r}}(\tilde{L}) \cdot \mathbf{t}, \ldots, \theta^{\mathbf{r}}(\tilde{L}) \cdot \mathbf{t} \Big\rangle_{g,k+n,\beta}^{S_n} = \Big\langle \mathrm{tr}_q(\alpha_1), \ldots, \mathrm{tr}_q(\alpha_k), \mathrm{tr}_q(\mathbf{t}), \ldots, \mathrm{tr}_q(\mathbf{t}) \Big\rangle_{g,k+n,\beta}^{S_n}.$$

Let $\{\Phi_a\}$ be a $\mathbb{C}$-basis for $\mathrm{K}_{\mathrm{orb}}(X)$. We define the genus-zero potential of quantum K-invariants by

$$F_0(\mathbf{t}) = \sum_{\beta \in \mathrm{Eff}(X)} \sum_{n \geq 0} Q^\beta \langle \mathrm{tr}_q(\mathbf{t}), \ldots, \mathrm{tr}_q(\mathbf{t}) \rangle_{g,n,\beta}^{S_n},$$

where $\mathbf{t} = \sum_a t^a \Phi_a$. For any $\alpha_1, \ldots, \alpha_k, \mathbf{t} \in \mathrm{K}_{\mathrm{orb}}(X)$, we use double brackets to denote the generating series of genus-0 invariants:

$$\langle\!\langle \alpha_1 \tilde{L}_1^{m_1}, \ldots, \alpha_k \tilde{L}_k^{m_k} \rangle\!\rangle_{\mathbf{t}}^{S_\infty} := \sum_{n,\beta} Q^\beta \langle \alpha_1 \tilde{L}_1^{m_1}, \ldots, \alpha_k \tilde{L}_k^{m_k}, \mathrm{tr}_q(\mathbf{t}), \ldots, \mathrm{tr}_q(\mathbf{t}) \rangle_{0,k+n,\beta}^{S_n}.$$

We restrict the above summation to the stable cases (or introduce by hand terms that correspond to tuples $(k+n, \beta)$ which are unstable).

We define the *quantized pairing* G by

$$G(\Phi_a, \Phi_b) := \partial_0 \partial_a \partial_b F_0,$$

where $\partial_a = \partial / \partial t^a$. More explicitly, we have

$$G(\alpha, \beta) = (\alpha, \beta) + \langle\!\langle \mathrm{tr}_q(\alpha), \mathrm{tr}_q(\beta) \rangle\!\rangle_{\mathbf{t}}^{S_\infty}, \quad \alpha, \beta \in \mathrm{K}_{\mathrm{orb}}(X).$$

Let $\{\Phi^a\}$ be the dual basis of $\{\Phi_a\}$ with respect to the non-degenerate Mukai pairing (see Assumption 2.3). Set $G_{ab} := G(\Phi_a, \Phi_b)$. Then the inverse is given by

$$G^{ab} = (\Phi^a, \Phi^b) - \langle\!\langle \mathrm{tr}_q(\Phi^a), \mathrm{tr}_q(\Phi^b) \rangle\!\rangle_{\mathbf{t}}^{S_\infty} + \sum_c \langle\!\langle \mathrm{tr}_q(\Phi^a), \mathrm{tr}_q(\Phi^c) \rangle\!\rangle_{\mathbf{t}}^{S_\infty} \langle\!\langle \mathrm{tr}_q(\Phi_c), \mathrm{tr}_q(\Phi^b) \rangle\!\rangle_{\mathbf{t}}^{S_\infty} - \cdots$$

The orbifold quantum K-theory defined in this paper shares the same properties as the quantum K-theory with manifold target spaces. We collect some facts here, whose proofs are given in [60].

Let $\mathcal{U} \subseteq \overline{M}_{0,n+1}(X,\beta)$ be the open and closed substack for which the last marking is untwisted. Then the forgetful morphism $\pi : \mathcal{U} \to \overline{M}_{0,n}(X,\beta)$ forgetting the last marking is naturally identified with the universal curve over $\overline{M}_{0,n}(X,\beta)$ (see [2, §8.1]). Let $\tilde{L}_i$ and $\tilde{L}'_i$ denote the orbifold cotangent line bundles on $\overline{M}_{0,n+1}(X,\beta)$ and $\overline{M}_{0,n}(X,\beta)$, respectively.

Proposition 2.10 (String equation)**.** *We have*

$$\pi_*\left(\mathcal{O}^{\mathrm{vir}}\left(\prod_{i=1}^{n}\frac{1}{1-q^{1/\mathbf{r}_i}\tilde{L}_i}\right)\right) = \left(1+\sum_{i=1}^{n}\frac{q_i}{1-q_i}\right)\left(\mathcal{O}^{\mathrm{vir}}\left(\prod_{i=1}^{n}\frac{1}{1-q^{1/\mathbf{r}_i}\tilde{L}'_1}\right)\right),$$

where q_i are formal variables.

Proposition 2.11 (Topological recursion relations)**.** *Let $\alpha,\beta,\gamma \in \mathrm{K}_{\mathrm{orb}}(X)$. We have*

$$\langle\!\langle \alpha(L_1-1)^{a+1}\tilde{L}_1^d, \beta\tilde{L}_2^b, \gamma\tilde{L}_3^c\rangle\!\rangle_{\mathbf{t}}^{S_\infty} = \sum_{\mu,\nu}\langle\!\langle \alpha L_1(L_1-1)^a\tilde{L}_1^b, \mathrm{tr}_q(\Phi_\mu)\rangle\!\rangle_{\mathbf{t}}^{S_\infty} G^{\mu\nu}\langle\!\langle \mathrm{tr}_q(\Phi_\nu), \beta\tilde{L}_2^b, \gamma\tilde{L}_3^c\rangle\!\rangle_{\mathbf{t}}^{S_\infty}$$

for $a,b,c,d \geq 0$.

Proposition 2.12 (WDVV equation)**.** *Let $\mathbf{t}_i(q) \in \mathcal{K}_+, i = 1,2,3,4$. We have*

$$\sum_{ab}\langle\!\langle \mathbf{t}_1(L),\mathbf{t}_2(L),\mathrm{tr}_q(\Phi_a)\rangle\!\rangle_{\mathbf{t}}^{S_\infty} G^{ab}\langle\!\langle \mathrm{tr}_q(\Phi_b),\mathbf{t}_3(L),\mathbf{t}_4(L)\rangle\!\rangle_{\mathbf{t}}^{S_\infty} = \sum_{ab}\langle\!\langle \mathbf{t}_1(L),\mathbf{t}_3(L),\mathrm{tr}_q(\Phi_a)\rangle\!\rangle_{\mathbf{t}}^{S_\infty} G^{ab}\langle\!\langle \mathrm{tr}_q(\Phi_b),\mathbf{t}_2(L),\mathbf{t}_4(L)\rangle\!\rangle_{\mathbf{t}}^{S_\infty}.$$

2.6. The full orbifold quantum K-ring.

Definition 2.13. Under Assumption 2.3, we define the quantum product $\bullet_{\mathbf{t}}$ at $\mathbf{t} \in \mathrm{K}_{\mathrm{orb}}(X)$ by

$$G(\Phi_a \bullet_{\mathbf{t}} \Phi_b, \Phi_c) = \partial_a\partial_b\partial_c F_0(\mathbf{t}).$$

Remark 2.14. Let $\alpha,\beta \in \mathrm{K}_{\mathrm{orb}}(X)$. As observed in [46, Remark 10] and [14, Remark 5.3], an equivalent definition of the quantum product at $\mathbf{t}$ is given by

$$\alpha \bullet_{\mathbf{t}} \beta = \sum_{\beta\in\mathrm{Eff}(X)}\sum_{n\geq 0} Q^\beta (\check{\mathrm{ev}}_3)_*\Bigg(\left(\mathcal{O}^{\mathrm{vir}}_{[\overline{M}_{0,3+n}(X,\beta)/S_n]} - \mathcal{O}^{\mathrm{vir}}_{[D/S_n]}\right) \cdot \mathrm{ev}_1^*(\mathrm{tr}_q(\alpha)) \cdot \mathrm{ev}_2^*(\mathrm{tr}_q(\beta)) \cdot \prod_{l=4}^{n+3}\mathrm{ev}_l^*(\mathrm{tr}_q(\mathbf{t}))\Bigg),$$

where $\check{\mathrm{ev}}_3 : [\overline{M}_{0,3+n}(X,\beta)/S_n] \to IX$ is the evaluation map at the third marking composed with the involution on IX, and D is the virtual divisor on $\overline{M}_{0,3+n}(X,\beta)$ which parametrizes stable maps such that the first two markings are separated from the third marking by a node in the domain curves. This definition of the quantum product does not rely on Assumption 2.3.

By definition, we have $\alpha \bullet_{\mathbf{t}} \beta \in \mathrm{K}_{\mathrm{orb}}(X)_\Lambda := \mathrm{K}_{\mathrm{orb}}(X)\widehat{\otimes}\Lambda$. By extending the quantum product $\bullet_{\mathbf{t}}$ bilinearly over Λ, we define the *full orbifold quantum K-ring* $(\mathrm{K}_{\mathrm{orb}}(X)_\Lambda, \bullet_{\mathbf{t}})$ at $\mathbf{t}$. It satisfies the following properties

(a) the quantum product is commutative and associative;
(b) the pairing G is multiplicatively invariant, i.e. $G(\Phi_a \bullet_{\mathbf{t}} \Phi_b, \Phi_c) = G(\Phi_a, \Phi_b \bullet_{\mathbf{t}} \Phi_c)$;
(c) $\mathbf{1}_X$ is the identity of the quantum product;
(d) the classicial limit $Q \to 0$ of the quantum product is the product of the *full orbifold K-ring* defined in [40].

The commutativity of the quantum product follows from the definition and the associativity is due to the K-theoretic WDVV equation stated in Proposition (2.12). Property (b) follows from the definition. Property (c) follows from the string equation stated in Proposition 2.10. Property (d) will be discussed in detail in Section 2.8.

We will refer to the restriction $(\mathrm{K}_{\mathrm{orb}}(X)_\Lambda, \bullet_0)$ at $\mathbf{t} = 0$ as the *small* orbifold quantum K-ring of X.

2.7. **Smooth Deligne–Mumford stacks with good torus actions.** If X is not proper but has a good torus-action, one may still define a good theory of *torus-equivariant* quantum K-invariants. Let $X = [W^{ss}(\theta)/G]$ be a stacky GIT quotient stack. We assume there is an algebraic torus $T \cong (\mathbb{C}^*)^m$ acting on W and this action commutes with the G-action.

Let
$$R_T := K_T(\mathrm{pt}) = \mathbb{C}[\lambda_1^{\pm 1}, \ldots, \lambda_m^{\pm 1}]$$
be the equivariant K-group of a point and let $S_T := \mathbb{C}(\lambda_1, \ldots, \lambda_m) = K_{T,\mathrm{loc}}(\mathrm{pt})$ be the localized equivariant K-group of a point.

Assumption 2.15. We assume the following conditions:

(a) The T-fixed locus X^T is proper.
(b) The T-equivariant topological K-group $K_T(IX)$ is a free module over R_T and one has an isomorphism of R_T-modules $K_T(IX) \cong K(IX) \otimes_{\mathbb{C}} R_T$.

We define the T-equivariant full orbifold K-group by
$$\mathrm{K}_{\mathrm{orb},T}(X) = K_T(IX).$$
By letting the ground λ-algebra Λ contain the equivariant parameters for T (see the discussion in Section 2.4), we can define the K-theoretic loop space, denoted by $\mathcal{K}_T$, in the T-equivariant setting.

It follows from Conditions (a) of Assumption 2.15 that the T-fixed loci in the moduli spaces of stable maps are proper. Let $\mathbf{t}(q) \in \mathcal{K}_T$. We define the T-equivariant permutation-equivariant quantum K-invariant via the virtual K-theoretic localization formula:
$$\left\langle \mathbf{t}(L), \ldots, \mathbf{t}(L) \right\rangle_{g,n,\beta}^{S_n} := p_*^T \left(\frac{\mathcal{O}^{\mathrm{vir}}_{\overline{M}_{g,n}(X,\beta)^T} \cdot \iota^* \left(\prod_{i=1}^n \mathrm{ev}_i^*(\mathbf{t}(L_i)) \right)}{\lambda_{-1}^T((N_\iota^{\mathrm{vir}})^\vee)} \right),$$

where $\iota : \overline{M}_{g,n}(X,\beta)^T \hookrightarrow \overline{M}_{g,n}(X,\beta)$ is the inclusion of the fixed point loci, $\lambda_{-1}^T((N_\iota^{\mathrm{vir}})^\vee)$ is the T-equivariant K-theoretic Euler class of the virtual normal bundle, and p_*^T is the proper pushforward along the projection

$$p^T : [\overline{M}_{g,n}(X,\beta)^T/S_n] \to [(\operatorname{Spec}\mathbb{C})/T].$$

Using the formula in Remark 2.14, we can define the quantum product $\bullet_{\mathbf{t}}$ on $\mathrm{K}_{\mathrm{orb},T}(X)$ at a point $\mathbf{t} \in \mathrm{K}_{\mathrm{orb},T}(X)$. Note that $\mathrm{ev}_3 : \overline{M}_{0,3+n}(X,\beta) \to IX$ is proper, and therefore the pushforward along $\check{\mathrm{ev}}_3$ is well-defined. It follows that

$$\alpha \bullet_{\mathbf{t}} \beta \in \mathrm{K}_{\mathrm{orb},T}(X)\widehat{\otimes}_{R_T}\Lambda$$

for $\alpha, \beta \in \mathrm{K}_{\mathrm{orb},T}(X)$. In particular, the non-equivariant limit of $\bullet_{\mathbf{t}}$ exists, and this limit defines the non-equivariant full orbifold quantum K-ring $(\mathrm{K}_{\mathrm{orb}}(X)_\Lambda, \bullet_{\mathbf{t}})$.

2.8. **Relation to Jarvis–Kaufmann–Kimura.** In [40], Jarvis–Kaufmann–Kimura introduced the full orbifold K-group $\mathrm{K}_{\mathrm{orb}}(X)$ and used the *orbifold obstruction bundle* over the *double inertia stack* of X to define an *orbifold product* on $\mathrm{K}_{\mathrm{orb}}(X)$. It is shown in the proofs of Theorem 9.5 and Theorem 9.10 of [40] that the double inertia stack of X is isomorphic to the moduli stack $\overline{M}_{0,3}(X,0)$ of degree-zero, genus-zero, 3-pointed stable maps into X, and the orbifold product can be defined using the standard obstruction bundle on $\overline{M}_{0,3}(X,0)$ coming from the orbifold Gromov-Witten theory. More precisely, let $\pi : \mathcal{C} \to \overline{M}_{0,3}(X,0)$ be the universal curve and let $f : \mathcal{C} \to X$ be the universal stable map. Then $\mathcal{R} := R^1\pi_*(f^*TX)$ is the obstruction (vector) bundle and we have

$$\mathcal{O}^{\mathrm{vir}}_{\overline{M}_{0,3}(X,0)} = \lambda_{-1}\left(\mathcal{R}^\vee\right).$$

The orbifold product $\bullet_{\mathrm{orb}}$ in [40, Definition 9.3] is equivalent to

$$\alpha \bullet_{\mathrm{orb}} \beta := (\check{\mathrm{ev}}_3)_*\left(\mathrm{ev}_1^*(\mathrm{tr}_q(\alpha)) \cdot \mathrm{ev}_1^*(\mathrm{tr}_q(\beta)) \cdot \lambda_{-1}\left(\mathcal{R}^\vee\right)\right), \quad \alpha, \beta \in \mathrm{K}_{\mathrm{orb}}(X).$$

Hence we recover the above orbifold product by specializing the small quantum product $\bullet_0$ defined in Section 2.6 at $Q = 0$.

3. Generating functions for genus-zero quantum K-invariants

3.1. **Quantum connection and fundamental solution.** Let X be a smooth projective DM stack satisfying Assumption 2.1. Suppose $\dim \mathrm{K}_{\mathrm{orb}}(X) = M$. Let $\{\Phi_a\}_{0 \le a \le M-1}$ be a $\mathbb{C}$-basis for $\mathrm{K}_{\mathrm{orb}}(X)$ such that $\Phi_0 = \mathbf{1}_X$. Let $\mathbf{t} = \sum_a t^a \Phi_a \in \mathrm{K}_{\mathrm{orb}}(X)$ be a generic point. The quantum connection is defined by

$$\nabla_a^q := (1-q)\partial_a + \Phi_a \bullet_{\mathbf{t}}, \quad 0 \le a \le M-1,$$

where $\partial_a := \partial/\partial t^a$. As a formal consequence of WDVV equation, the connection ∇_a^q is flat (see [21, 46]). We recall the explicit construction of its fundamental solution.

Definition 3.1. Let $\gamma, \mathbf{t} \in \mathrm{K}_{\mathrm{orb}}(X)$. The *$S$-operator* is defined by[2]

$$\mathrm{S}_{\mathbf{t}}(q)(\gamma) := \gamma + \sum_{(k,\beta)\neq(0,0)} Q^{\beta}(\check{\mathrm{ev}}_1)_* \left(\frac{\mathcal{O}^{\mathrm{vir}}_{[\overline{M}_{0,2+k}(X,\beta)/S_k]}}{1 - q^{1/\mathbf{r}_1}\tilde{L}_1} \cdot \mathrm{ev}_2^*(\mathrm{tr}_q(\gamma)) \cdot \prod_{i=3}^{k+2} \mathrm{ev}_i^*(\mathrm{tr}_q(\mathbf{t})) \right),$$

where $\check{\mathrm{ev}}_1 : [\overline{M}_{0,2+k}(X,\beta)/S_k] \to IX$ is the evaluation map at the first marking composed with the involution on IX.

By definition, we have $\mathrm{S}_{\mathbf{t}}(q) \in \oplus_r \left(\mathrm{End}(K(I_rX)) \otimes \mathbb{C}(q^{1/r})\right) \widehat{\otimes} \Lambda$. The S-operator can also be written as follows:

$$\mathrm{S}_{\mathbf{t}}(q)(\gamma) = \sum_a \Phi_a \left\langle\!\!\left\langle \frac{\Phi^a}{1 - q^{1/\mathbf{r}_1}\tilde{L}_1}, \mathrm{tr}_q(\gamma) \right\rangle\!\!\right\rangle_{\mathbf{t}}^{S_\infty},$$

where the unstable terms are defined as in Definition 3.1. By definition, the S-operator has the asymptotic expansion

$$\mathrm{S}_{\mathbf{t}}(q)(\gamma) = \gamma + r_\gamma(Q,q),$$

where $r_\gamma(Q,q) \in \mathcal{K}_-$.

We define the *L-operator* by

$$\mathrm{L}_{\mathbf{t}}(q)(\gamma) := \sum_{a,b} \Phi_a G^{ab} \left\langle\!\!\left\langle \frac{\gamma}{1 - q^{-1/\mathbf{r}_1}\tilde{L}_1}, \mathrm{tr}_q(\Phi_b) \right\rangle\!\!\right\rangle_{\mathbf{t}}^{S_\infty}.$$

Note that it has a similar asymptotic expansion to the S-operator:

$$\mathrm{L}_{\mathbf{t}}(q)(\gamma) = \gamma + r'_\gamma(Q,q),$$

where $r'_\gamma(Q,q) \in \mathcal{K}_-$. The following proposition is a direct generalization of the results in [21, 46] and it follows from the WDVV equation and the string equation.

Proposition 3.2. *The endomorphism-valued functions* $\mathrm{L}_{\mathbf{t}} = \mathrm{L}_{\mathbf{t}}(q)$ *and* $\mathrm{S}_{\mathbf{t}} = \mathrm{S}_{\mathbf{t}}(q)$ *satisfy the differential equations*

$$(1-q)\partial_a \mathrm{L}_{\mathbf{t}} + \Phi_a \bullet_{\mathbf{t}} \mathrm{L}_{\mathbf{t}} = 0,$$
$$(1-q)\partial_a \mathrm{S}_{\mathbf{t}} = \mathrm{S}_{\mathbf{t}}(\Phi_a \bullet_{\mathbf{t}}).$$

Equivalently, we have $\nabla^q_a \circ \mathrm{L}_{\mathbf{t}} = \mathrm{L}_{\mathbf{t}} \circ (1-q)\partial_a$ *and* $\mathrm{S}_{\mathbf{t}} \circ \nabla^q_a = (1-q)\partial_a \circ \mathrm{S}_{\mathbf{t}}$.

The following proposition is a direct generalization of [39, Proposition 2.3], whose proof again relies on the WDVV equation and the string equation.

Proposition 3.3. *We have the following:*

(1) $\mathrm{S}_{\mathbf{t}}(q) = (\mathrm{L}_{\mathbf{t}}(q))^{-1}$.
(2) $\left(\mathrm{S}_{\mathbf{t}}(q^{-1})(\Phi_a), \mathrm{S}_{\mathbf{t}}(q)(\Phi_b)\right) = G_{ab}$.
(3) $G(\mathrm{L}_{\mathbf{t}}(q^{-1})(\Phi_a), \mathrm{L}_{\mathbf{t}}(q)(\Phi_b)) = (\Phi_a, \Phi_b)$.

[2] The S-operator defined in this paper is the inverse of that in [39].

3.2. **The J-function and the finite-difference module structure.** Let

$$\check{\mathrm{ev}}_1 : \overline{M}_{0,1+k}(X,\beta) \to IX$$

be the evaluation map at the first marking composed with the involution on IX. It descends to a map $[\overline{M}_{0,1+k}(X,\beta)/S_k] \to IX$ which we also denote by $\check{\mathrm{ev}}_1$.

Definition 3.4. Let $\mathbf{t}(q) \in \mathcal{K}_+$. The permutation-equivariant K-theoretic big J-function is

$$J(\mathbf{t}(q),Q) := 1 - q + \mathbf{t}(q) + \sum_{(k,\beta)\neq(0,0),(1,0)} Q^\beta (\check{\mathrm{ev}}_1)_* \left(\frac{\mathcal{O}^{\mathrm{vir}}_{[\overline{M}_{0,1+k}(X,\beta)/S_k]}}{1 - q^{1/\mathbf{r}_1}\tilde{L}_1} \cdot \prod_{i=2}^{k+1} \mathrm{ev}_i^*(\mathbf{t}(L_i)) \right),$$

or, equivalently,

$$J(\mathbf{t}(q),Q) = \sum_a \Phi_a \left\langle\!\!\left\langle \frac{\Phi^a}{1 - q^{1/\mathbf{r}_1}\tilde{L}_1} \right\rangle\!\!\right\rangle^{S_\infty}_{\mathbf{t}(q)}.$$

The K-theoretic small J-function $J(0,Q)$ is defined by putting $\mathbf{t}(q) = 0$ in the K-theoretic big J-function.

For simplicity, we will refer to $J(\mathbf{t}(q),Q)$ as the K-theoretic big J-function. As explained in [61, §2.3], the K-theoretic big J-function is an element of the loop space $\mathcal{K}$. Let $\mathcal{L}_X \subseteq \mathcal{K}$ be the range of the K-theoretic big J-function $\mathbf{t}(q) \mapsto J(\mathbf{t}(q),Q), \mathbf{t}(q) \in \mathcal{K}_+$. In the case when X is a manifold, Givental showed in [27] that the range $\mathcal{L}_X$ is an overruled cone (but not Lagrangian) and has a finite-difference module structure. These properties still hold when X is a DM stack, and the details will be given below.

Let $p_1, \ldots, p_r$ be a nef integral basis of $H^2(X)$. Let P_i be a line bundle such that $p_i = -c_1(P_i)$ for $1 \leq i \leq r$. Let $Q_1, \ldots, Q_r$ be the Novikov variables dual to $P_1, \ldots, P_r$. We write $Q^\beta := \prod_{i=1}^r Q_i^{\beta_i}$, where $\beta_i = \langle \beta, p_i \rangle$. Because of the choices of P_i's, the expression Q^β does not contain negative powers of $Q_1, \ldots, Q_r$ if $\beta \in \mathrm{Eff}(X)$.

Definition 3.5. For $1 \leq i \leq r$, the $q^{Q_i \partial_{Q_i}}$ acts on functions in $Q_1, \ldots, Q_r$ as

$$f(Q_1, \ldots, Q_r) \mapsto f(Q_1, \ldots, Q_{i-1}, qQ_i, Q_{i+1}, \ldots, Q_r).$$

The following proposition is a generalization of [27, Theorem 2].

Proposition 3.6 ([60])**.**

$$\mathcal{L}_X = \bigcup_{\mathbf{t} \in \mathrm{K}_{\mathrm{orb}}(X)\widehat{\otimes}\Lambda_+} (1-q)\mathrm{S}_{\mathbf{t}}(q)\mathcal{K}_+.$$

Proposition 3.7 ([60])**.** *The range $\mathcal{L}_X$ is preserved by the operator $P_i q^{Q_i \partial_{Q_i}}$ for all $1 \leq i \leq r$.*

Let $\mathrm{L}_{\mathbf{t}} = \mathrm{L}_{\mathbf{t}}(q)$ and $\mathrm{S}_{\mathbf{t}} = \mathrm{S}_{\mathbf{t}}(q)$ be the endormphism-valued functions defined in the previous subsection. Following [39], we define an endomorphism A_i by

$$A_i = \mathrm{L}_{\mathbf{t}} \circ (P_i q^{Q_i\partial_{Q_i}} \mathrm{L}_{\mathbf{t}}^{-1}) = \mathrm{S}_{\mathbf{t}}^{-1} \circ (P_i q^{Q_i\partial_{Q_i}} \mathrm{S}_{\mathbf{t}}), \quad 1 \le i \le r.$$

It follows from Proposition 3.7 that A_i lies in $\oplus_r\big(\operatorname{End}(K(I_rX)) \otimes \mathbb{C}[q^{1/r}, q^{-1/r}]\big)\widehat{\otimes}\Lambda$. The operator $\mathcal{A}_i := A_i q^{Q_i\partial_{Q_i}}$ is referred to as the *q-shift operator* in [39]. We have

$$\mathrm{L}_{\mathbf{t}} \circ P_i q^{Q_i\partial_{Q_i}} = \mathcal{A}_i \circ \mathrm{L}_{\mathbf{t}}, \quad P_i q^{Q_i\partial_{Q_i}} \circ \mathrm{S}_{\mathbf{t}} = \mathrm{S}_{\mathbf{t}} \circ \mathcal{A}_i.$$

By using the same arguments as in the proof of [39, Proposition 2.10], one can show that the endomorphism A_i has an expansion of the form

$$A_i = P_i + \sum_{\substack{\beta \in \mathrm{Eff}(X) \\ \beta_i > 0}} A_{i,\beta}(q) Q^{\beta}, \tag{6}$$

where the first term is the tensor product with P_i and $A_{i,\beta}(q) \in \oplus_r\big(\operatorname{End}(K(I_rX)) \otimes \mathbb{C}[q^{1/r}]\big)$.

Lemma 3.8. *The operators $\nabla_a^q, \mathcal{A}_i = A_i q^{Q\partial_{Q_i}}$ satisfy the compatibility equations*

$$[\nabla_a^q, \nabla_b^q] = [\mathcal{A}_i, \mathcal{A}_j] = [\nabla_a^q, \mathcal{A}_i] = 0$$

for all $0 \le a, b < M$ and $1 \le i, j \le r$.

Proof. As explained in the proof of [39, Proposition 2.6], the compatibility equations follow from

$$\nabla_a^q \circ \mathrm{L}_{\mathbf{t}} = \mathrm{L}_{\mathbf{t}} \circ (1-q)\partial_a, \quad \mathcal{A}_i \circ \mathrm{L}_{\mathbf{t}} = \mathrm{L}_{\mathbf{t}} \circ P_i q^{Q_i\partial_{Q_i}}$$

and the fact that $(1-q)\partial_a$ and $P_i q^{Q_i\partial_{Q_i}}$ commute with each other. □

Corollary 3.9. *For $1 \le i \le r$, we define $A_{i,\mathrm{com}} = A_i|_{q=1} \in \operatorname{End}(K(IX))\widehat{\otimes}\Lambda$. Then $A_{i,\mathrm{com}}, i = 1, \dots, r$ commute with the quantum multiplication $\Phi_a \bullet_{\mathbf{t}}, 0 \le a \le M-1$.*

Proof. The compatibility equation $[\nabla_a^q, \mathcal{A}_i] = 0$ implies that

$$(1-q)\partial_a A_i = A_i\big(q^{Q_i\partial_{Q_i}} \circ (\Phi_a \bullet_{\mathbf{t}})\big) - (\Phi_a \bullet_{\mathbf{t}}) \circ A_i.$$

The corollary follows from the above equation by setting $q = 1$. □

Remark 3.10. We will apply the following strategy in Section 5 to compute quantum products. Let $\alpha \in \mathrm{K}_{\mathrm{orb}}(X)$. Suppose that we can find a polynomial $F_\alpha \in \mathbb{Q}[x_1, \dots, x_r]$ such that $F_\alpha(A_1, \dots, A_r)\mathbf{1}_X = \alpha$. Then for $\beta \in \mathrm{K}_{\mathrm{orb}}(X)$, we can obtain $\alpha \bullet_{\mathbf{t}} \beta$ by computing $F_\alpha(A_{1,\mathrm{com}}, \dots, A_{r,\mathrm{com}})\beta$. This is because

$$\begin{aligned} F_\alpha(A_{1,\mathrm{com}}, \dots, A_{r,\mathrm{com}})\beta &= F_\alpha(A_{1,\mathrm{com}}, \dots, A_{r,\mathrm{com}})(\beta \bullet_{\mathbf{t}} \mathbf{1}_X) \\ &= \beta \bullet_{\mathbf{t}} (F_\alpha\big(A_{1,\mathrm{com}}, \dots, A_{r,\mathrm{com}})\mathbf{1}_X\big) \\ &= \beta \bullet_{\mathbf{t}} \alpha. \end{aligned}$$

For later application in computing the small quantum K-ring, we introduce the following restriction of the K-theoretic big J-function:

$$J_{\mathbf{t}}(Q,q) := J(\mathrm{tr}_q(\mathbf{t}), Q), \quad \mathbf{t} \in \mathrm{K}_{\mathrm{orb}}(X).$$

We will simply refer to $J_{\mathbf{t}}(Q,q)$ as the J-function. The following proposition is a familiar result which relates the J-function to the S-operator.

Proposition 3.11. *Let* $\mathbf{t} \in \mathrm{K}_{\mathrm{orb}}(X)$. *Then* $J_{\mathbf{t}}(Q,q) = (1-q)\mathrm{S}_{\mathbf{t}}(q)(1)$.

Proof. The claim follows easily from the string equation stated in Proposition 2.10. □

3.3. **Quasimap theory and the K-theoretic small I-function.** In this subsection, we recall the definition of the K-theoretic small I-function, which is a generating series of K-theoretic (0+)-stable quasimap invariants. By the genus-zero mirror theorem in [61], the K-theoretic small I-function and J-function agree up to a change of variable.

We fix a GIT presentation $X = [W^{ss}(\theta)/G]$. Let $(C, x_1, \ldots, x_n)$ be a n-pointed, genus g twisted curve with balanced nodes and gerbe markings (see [2, §4]). Here we do not assume the gerbe markings are trivialized. A map $[u] : C \to X$ corresponds to a pair (P, u) with

$$P \to C$$

a principal G-bundle on C and

$$u : C \to P \times_G W$$

a section of the fiber bundle $P \times_G W \to C$. The map $[u]$ is called a *quasimap* to X if $[u]$ is representable and $[u]^{-1}([W^{us}/G])$ is zero-dimensional. Here W^{us} denotes the unstable locus. The locus $[u]^{-1}([W^{us}/G])$ is called the *base locus* of $[u]$ and points in the base locus are called *base points*.

The class β of a quasimap is defined to be the group homomorphism

$$\beta : \mathrm{Pic}([W/G]) \to \mathbb{Q}, \quad L \mapsto \deg([u]^*(L)).$$

We refer to the rational number $\deg(\beta) = \deg([u]^*(L_\theta))$ as the *degree* of the quasimap $[u]$. A group homomorphism $\beta : \mathrm{Pic}([W/G]) \to \mathbb{Q}$ is called an *effective* curve class if it is the class of some quasimap $[u]$. We denote by $\mathrm{Eff}(W, G, \theta)$ the semigroup of L_θ-effective curve classes on X. For convenience, we write $\beta \geq 0$ if $\beta \in \mathrm{Eff}(W, G, \theta)$ and $\beta > 0$ if the effective curve class is nonzero.

Fix a positive rational number ϵ. A quasimap is called *ϵ-stable* if the following three conditions hold:

(1) The base points are disjoint from the gerbe markings and nodes of $(C, x_1, \ldots, x_n)$.
(2) For every $y \in C$, we have $l(y) \leq 1/\epsilon$, where $l(y)$ is the length at y of the subscheme $[u]^{-1}([W^{us}(\theta)/G])$.
(3) The $\mathbb{Q}$-line bundle $(u^*L_\theta)^{\otimes \epsilon} \otimes \omega_{C,\log}$ is positive, where $\omega_{C,\log} := \omega_C(\sum_i x_i)$ is the log dualizing sheaf.

A quasimap is called (0+)-stable if it is ϵ-stable for every sufficiently small positive rational number ϵ, and ∞-stable if it is ϵ-stable for every sufficiently large ϵ.

Let $\underline{Q}^{\epsilon}_{g,n}(X,\beta)$ be the moduli stack of genus-g ϵ-stable quasimaps to X of curve class β with n gerbe markings. It is a Deligne–Mumford stack, proper over the affine quotient $W/\!\!/_0 G$, with a perfect obstruction theory (see [16]). The moduli stack of (0+)-stable quasimaps is denoted by $\underline{Q}^{0+}_{g,n}(X,\beta)$. According to the general constructions in [46, 49], the perfect obstruction theory induces a virtual structure sheaf

$$\mathcal{O}^{\mathrm{vir}}_{\underline{Q}^{\epsilon}_{g,n}(X,\beta)} \in K_\circ(\underline{Q}^{\epsilon}_{g,n}(X,\beta)).$$

To define the K-theoretic small I-function for the genus-0 theory, we will also need quasimap *graph spaces*. Given an effective curve class β, choose $\epsilon \in \mathbb{Q}_{>0}$ and $A \in \mathbb{Z}_{>0}$ such that $1/A < \epsilon < 1/\deg(\beta)$. We view $\mathbb{P}^1$ as the GIT quotient $\mathbb{C}^2/\!\!/\mathbb{C}^*$ with the polarization $\mathcal{O}_{\mathbb{P}^1}(A)$. Then the (0+)-stable quasimap graph space is defined by

$$\underline{QG}^{0+}_{0,1}(X,\beta) := \underline{Q}^{\epsilon}_{0,1}(X\times\mathbb{P}^1, \beta\times[\mathbb{P}^1]).$$

The definition is independent of the choice of A and ϵ. This moduli stack parametrizes quasimaps to $X\times\mathbb{P}^1$ with a unique rational component whose coarse moduli is mapped isomorphically onto $\mathbb{P}^1$.

Consider the $\mathbb{C}^*$-action on $\mathbb{P}^1$ given by

$$t[\zeta_0,\zeta_1] = [t\zeta_0,\zeta_1], \quad t\in\mathbb{C}^*.$$

Set $0 := [1:0]$ and $\infty := [0:1]$. The above $\mathbb{C}^*$-action on $\mathbb{P}^1$ induces an action on $\underline{QG}^{0+}_{0,1}(X,\beta)$. We denote the unique marking by $x_\star$. Let $\underline{F}^{0,\beta}_{\star,0}$ be the distinguished fixed-point component consisting of $\mathbb{C}^*$-fixed quasimaps such that the marking $x_\star$ is over ∞ and the entire class β is over $0\in\mathbb{P}^1$.

Let $QG^{0+}_{0,1}(X,\beta)$ be the moduli stack of (0+)-stable graph quasimaps with sections of the unique gerbe marking $x_\star$. Then $QG^{0+}_{0,1}(X,\beta)$ is the universal gerbe over $\underline{QG}^{0+}_{0,1}(X,\beta)$ corresponding to the unique marking $x_\star$. Let $F^{0,\beta}_{\star,0}$ be the distinguished fixed-point component in $QG^{0+}_{0,1}(X,\beta)$ consisting of $\mathbb{C}^*$-fixed quasimaps such that the marking $x_\star$ is over ∞ and the entire class β is over $0\in\mathbb{P}^1$. Then the forgetful map $\rho : F^{0,\beta}_{\star,0} \to \underline{F}^{0,\beta}_{\star,0}$ exhibits $F^{0,\beta}_{\star,0}$ as the trivial gerbe over $\underline{F}^{0,\beta}_{\star,0}$ banded by μ_a.

Let π_{IX} be the projection to IX from $IX\times\mathbb{P}^1$, ι be the involution on IX reversing the banding, and $\mathrm{ev}_\star : QG^{0+}_{0,1}(X,\beta) \to IX\times\mathbb{P}^1$ be the evaluation map at the marking $x_\star$. Recall that $\bar{I}X$ denotes the rigidified inertia stack of X. Let $\pi_{\bar{I}X} : \bar{I}X\times\mathbb{P}^1 \to \bar{I}X$, $\iota : \bar{I}X\to\bar{I}X$, and $\underline{\mathrm{ev}}_\star : \underline{QG}^{0+}_{0,1}(X,\beta) \to \bar{I}X\times\mathbb{P}^1$ be similarly defined maps. Write

$$\widetilde{\mathrm{ev}}_\star := \iota\circ\pi_{IX}\circ\mathrm{ev}_\star \quad\text{and}\quad \underline{\widetilde{\mathrm{ev}}}_\star := \iota\circ\pi_{\bar{I}X}\circ\underline{\mathrm{ev}}_\star.$$

We have the following diagram

$$\begin{tikzcd}
F^{0,\beta}_{\star,0} \arrow[r, "\widetilde{\mathrm{ev}}_\star"] \arrow[d, "\rho"'] & IX \arrow[d, "\varpi"] \\
\underline{F}^{0,\beta}_{\star,0} \arrow[r, "\underline{\widetilde{\mathrm{ev}}}_\star"] \arrow[u, bend right, "\sigma"'] \arrow[ur, "\widetilde{ev}_\star"] & \bar{I}X,
\end{tikzcd}$$

where the outer square commutes. Here σ is the section of the gerbe ρ associated with the trivial μ_a-torsor, and $\widetilde{ev}_\star$ is defined as the composition of $\underline{\widetilde{\mathrm{ev}}}_\star$ and σ.

The restriction of the absolute perfect obstruction theory of $\underline{QG}^{0+}_{0,1}(X,\beta)$ to $\underline{F}^{0,\beta}_{\star,0}$ decomposes into the moving and fixed parts. The fixed part of the obstruction theory defines a perfect obstruction theory on $\underline{F}^{0,\beta}_{\star,0}$ (c.f. [36]), which induces a virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}_{\underline{F}^{0,\beta}_{\star,0}} \in K_\circ(\underline{F}^{0,\beta}_{\star,0})$. The moving part of the obstruction theory defines the virtual normal bundle $N^{\mathrm{vir}}_{\underline{F}^{0,\beta}_{\star,0}/\underline{QG}^{0+}_{0,1}(X,\beta)} \in K^\circ_{\mathbb{C}^*}(\underline{F}^{0,\beta}_{\star,0})$, whose dual is denoted by $N^{\mathrm{vir},\vee}_{\underline{F}^{0,\beta}_{\star,0}/\underline{QG}^{0+}_{0,1}(X,\beta)}$.

We define the K-theoretic small I-function by

$$I(Q,q) := 1 + (1-q^{-1}) \cdot \sum_{\beta>0} Q^\beta (\widetilde{\mathrm{ev}}_\star)_* \left(\frac{\mathcal{O}^{\mathrm{vir}}_{\underline{F}^{0,\beta}_{\star,0}}}{\lambda^{\mathbb{C}^*}_{-1}\left(N^{\mathrm{vir},\vee}_{\underline{F}^{0,\beta}_{\star,0}/\underline{QG}^{0+}_{0,1}(X,\beta)}\right)} \right). \tag{7}$$

Remark 3.12. Let $\mathcal{O}^{\mathrm{vir}}_{F^{0,\beta}_{\star,0}}$ be the virtual structure sheaf of $F^{0,\beta}_{\star,0}$ and let $N^{\mathrm{vir}}_{F^{0,\beta}_{\star,0}/QG^{0+}_{0,1}(X,\beta)}$ be the virtual normal bundle of $F^{0,\beta}_{\star,0}$ in $QG^{0+}_{0,1}(X,\beta)$. Since the pullback of the perfect obstruction theory of $\underline{F}^{0,\beta}_{\star,0}$ coincides with that of $F^{0,\beta}_{\star,0}$, we have

$$\rho^*\mathcal{O}^{\mathrm{vir}}_{\underline{F}^{0,\beta}_{\star,0}} = \mathcal{O}^{\mathrm{vir}}_{F^{0,\beta}_{\star,0}} \quad \text{and} \quad \rho^* N^{\mathrm{vir}}_{\underline{F}^{0,\beta}_{\star,0}/\underline{QG}^{0+}_{0,1}(X,\beta)} = N^{\mathrm{vir}}_{F^{0,\beta}_{\star,0}/QG^{0+}_{0,1}(X,\beta)}.$$

Since σ is a section of the trivial gerbe ρ, we have

$$\sigma_*\left(\mathcal{O}_{\underline{F}^{0,\beta}_{\star,0}}\right) = \theta^{\mathbf{r}_\star}(\tilde{L}_\star),$$

where $\theta^{\mathbf{r}_\star}(\tilde{L}_\star) = \sum_{i=0}^{\mathbf{r}_\star-1} \tilde{L}_\star$ is the $\mathbf{r}_\star$-th Bott's cannibalistic class of the orbifold cotangent line bundle $\tilde{L}_\star$ at $x_\star$. An equivalent definition of the K-theoretic small I-function using the moduli stack of quasimap graph space with a trivialized gerbe marking is given by

$$I(Q,q) = 1 + (1-q^{-1}) \cdot \sum_{\beta>0} Q^\beta (\widetilde{\mathrm{ev}}_\star)_* \left(\frac{\theta^{\mathbf{r}_\star}(\tilde{L}_\star)\cdot \mathcal{O}^{\mathrm{vir}}_{F^{0,\beta}_{\star,0}}}{\lambda^{\mathbb{C}^*}_{-1}\left(N^{\mathrm{vir},\vee}_{F^{0,\beta}_{\star,0}/QG^{0+}_{0,1}(X,\beta)}\right)} \right).$$

As explained in [61, §2.3], the K-theoretic small I-function is an element of the loop space $\mathcal{K}$. We define

$$\mu^{>0}(Q,q) := [(1-q)I(Q,q) - (1-q)]_+ \in \mathcal{K}_+.$$

The following proposition is a special case of the mirror theorem proved in [61, Theorem 5.15]. It shows that $(1-q)I(Q,q)$ is a point in the range $\mathcal{L}_X$ of the permutation-equivariant K-theoretic big J-function.

Proposition 3.13 ([61])**.** *We have*

$$J(\mu^{>0}(Q,L),Q) = (1-q)I(Q,q).$$

Remark 3.14. If X is only quasiprojective but has a torus-action satisfying Assumption 2.15, we can define the torus-equivariant K-theoretic I-function and J-function. Proposition 3.13 still holds in the torus-equivariant setting.

4. K-theoretic small I-functions of Toric Deligne–Mumford stacks

4.1. **Set-up.** In this subsection, we recall some basic facts about toric DM stacks introduced in [10]. Let K be the algebraic torus $(\mathbb{C}^*)^r, r \geq 0$. We denote by $\mathbb{L} = \mathrm{Hom}(\mathbb{C}^*, K)$ the cocharacter lattice of K. We fix a finite collection $\rho_1, \ldots, \rho_m \in \mathbb{L}^\vee = \mathrm{Hom}(K, \mathbb{C}^*)$ of (not necessarily distinct) characters of K. For a subset $I \subseteq \{1, 2, \ldots, m\}$, we denote by $\bar{I}$ the complement of I. Define

$$\angle_I = \{\sum_{i\in I} a_i\rho_i \mid a_i \in \mathbb{R}_{>0}\} \subseteq \mathbb{L}^\vee \otimes \mathbb{R}$$

and

$$(\mathbb{C}^*)^I \times \mathbb{C}^{\bar{I}} = \{(z_1, \ldots, z_m) \mid z_i \neq 0 \text{ for } i \in I\} \subseteq \mathbb{C}^m.$$

Set $\angle_\emptyset := \{0\}$.

Let $\mathbb{C}_{\rho_i}$ denote the 1-dimensional representation of K corresponding to the character ρ_i. We consider the action of K on $W = \oplus_{i=1}^m \mathbb{C}_{\rho_i}$. For a stability condition $\theta \in \mathbb{L}^\vee \otimes \mathbb{R}$, we define the set of *anti-cones* to be

$$\mathcal{A}_\theta = \{I \subseteq \{1, 2, \ldots, m\} \mid \theta \in \angle_I\}.$$

Then the semistable locus $W^{ss} = W^{ss}(\theta)$ is given by

$$W^{ss} = \bigcup_{I\in\mathcal{A}_\theta} (\mathbb{C}^*)^I \times \mathbb{C}^{\bar{I}}.$$

We make the following assumption on the stability condition θ:

Assumption 4.1. We assume that

(1) $\{1, 2, \ldots, m\} \in \mathcal{A}_\theta$.
(2) For each $I \in \mathcal{A}_\theta$, the set $\{\rho_i \mid i \in I\}$ spans $\mathbb{L}^\vee \otimes \mathbb{R}$ over $\mathbb{R}$.

Under the above assumptions, the GIT quotient $X = [W^{ss}/K]$ is a non-empty toric Deligne–Mumford stack (in the sense of [10]), with quasi-projective coarse moduli space.

The action of $T = (\mathbb{C}^*)^m$ on W^{ss} descends to a $T' := T/K$-action on X. We also consider the ineffective T-action on X induced by the projection $T \to T'$. Let $\lambda_i \in R_T$ denote the equivariant parameter given by the projection $\pi_i : T \cong (\mathbb{C}^*)^m \to \mathbb{C}^*$ to the i-th factor. Then $R_T = \mathbb{Q}[\lambda_1^{\pm 1}, \ldots, \lambda_m^{\pm 1}]$.

For a character $\rho \in \mathbb{L}^\vee$, the associated line bundle

$$[(W^{ss} \times \mathbb{C}_\rho)/K]$$

on X will be denoted by L_ρ. For $1 \le i \le r$, we define $P_i := L^\vee_{\pi_i}$. These line bundles are equipped with the T-linearization $[z, v] \mapsto [t \cdot z, v], t \in T$. Note that the classes P_i generate the equivariant K-group $K_T(X)$.

Let D_j be the toric divisor

$$D_j := \{(z_1, \dots, z_m) \in W^{ss} \mid z_j = 0\}/K,$$

for $j \in \{1, \dots, m\}$. Let U_j be the T-equivariant line bundle $\mathcal{O}_X(-D_j)$. As the character of K is the free abelian group generated by π_i, there are unique integers m_{ij} such that

$$\rho_j = \sum_i m_{ij}\pi_i.$$

Then we have the multiplicative relations

$$U_j = \prod_{i=1}^{r} P_i^{m_{ij}} \lambda_j^{-1}.$$

According to [11], the ring $K_T(X)$ is described by Kirwan's relations

$$\prod_{j \notin I}(1 - U_j) = 0 \text{ whenever } I \notin \mathcal{A}_\theta.$$

Remark 4.2. According to [18, Remark 4.4], the first condition in Assumption 2.15 holds for the T-action on X. The second condition holds if we replace T by an n-fold cover $\tilde{T} \to T$, for some positive integer n.

There are canonical isomorphisms $K \cong \mathbb{L} \otimes \mathbb{C}^*$ and $\mathrm{Lie}(K) \cong \mathbb{L} \otimes \mathbb{C}$. The exponential map $\mathrm{Lie}(K) \to K$ is give by $\mathbb{L} \otimes \mathbb{C} \to \mathbb{L} \otimes \mathbb{C}^*, \ell \otimes c \to \ell \otimes \exp(2\pi\sqrt{-1}c)$. The kernel of the exponential map is $\mathbb{L} \subseteq \mathbb{L} \otimes \mathbb{C}$. Define $\mathbb{K} \subseteq \mathbb{L} \otimes \mathbb{Q}$ to be the set of $f \in \mathbb{L} \otimes \mathbb{Q}$ such that

$$I_f := \{i \in \{1, 2, \dots, m\} \mid \rho_i \cdot f \in \mathbb{Z}\} \in \mathcal{A}_\theta.$$

The lattice $\mathbb{L}$ acts on $\mathbb{K}$ via translation. According to [18, §4.8], the set $\{g \in K \mid (W^{ss})^g \neq \emptyset\}$ can be identified with $\mathbb{K}/\mathbb{L}$ via the exponential map.

The components of the inertia stack IX are indexed by elements of $\mathbb{K}/\mathbb{L}$. Let $f \in \mathbb{K}/\mathbb{L}$ and let $g = \exp(2\pi\sqrt{-1}f)$. Then $(W^{ss})^g = \mathbb{C}^{I_f} \cap W^{ss}$. Set

$$X^f := [(\mathbb{C}^{I_f} \cap W^{ss})/K].$$

Then

$$IX = \coprod_{f \in \mathbb{K}/\mathbb{L}} X^f.$$

Recall that the T-equivariant full orbifold K-group is given by

$$\mathrm{K}_{\mathrm{orb},T}(X) = \bigoplus_{f \in \mathbb{K}/\mathbb{L}} K_T(X^f).$$

We write $\mathbf{1}_f$ for the unit class in $K_T(X^f)$.

Let $S \subseteq \{1, 2, \ldots, m\}$ denote the set of indices i such that $\{1, \ldots, m\}\backslash\{i\} \notin \mathcal{A}_\theta$. Then every element of $\mathcal{A}_\theta$ contains S as a subset. We define

$$\mathcal{A}'_\theta = \{I\backslash S \mid I \in \mathcal{A}_\theta\}.$$

We have

$$H^2(X, \mathbb{R}) \cong (\mathbb{L}^\vee \otimes \mathbb{R}) / \sum_{i\in S} \mathbb{R}D_i.$$

Let D'_i denote the image of D_i in $H^2(X, \mathbb{R})$. Then the ample cone of X is given by

$$C'_\theta = \bigcap_{I\in\mathcal{A}'_\theta} \angle'_I,$$

where $\angle'_I := \sum_{i\in I} \mathbb{R}_{>0} D'_i$ is an open cone in $H^2(X, \mathbb{R})$, and Mori cone, the dual cone of C'_θ, is given by

$$\mathrm{NE}(X) = \{\beta \in H_2(X, \mathbb{R}) \mid \alpha \cdot \beta \geq 0 \text{ for all } \alpha \in C'_\theta\}$$

We have $\mathrm{Eff}(X) = \mathrm{NE}(X) \cap H_2(X, \mathbb{Z})$.

4.2. **Stacky loop spaces and K-theoretic toric I-functions.** In this subsection, we will compute the K-theoretic toric I-functions using the stacky loop spaces introduced in [16]. Let a be a positive integer. The weighted projective line $\mathbb{P}_{1,a}$ is the quotient stack $[\mathbb{C}^2\backslash\{0\}/\mathbb{C}^*]$, where $\mathbb{C}^*$ acts on $\mathbb{C}^2$ by weights 1 and a. Note that $0 := [1, 0]$ is a schematic point, while $\infty := [0, 1] \cong B\mu_a$ is a stacky point for $a > 1$.

We first consider a general stacky GIT quotient $X = W/\!\!/_\theta G$. Let $\mathfrak{X} := [W/G]$. We denote by $\mathrm{Hom}^{\mathrm{rep}}_\beta(\mathbb{P}_{1,a}, \mathfrak{X})$ the Hom-stack consiting of representable 1-morphisms $\mathbb{P}_{1,a} \to \mathfrak{X}$ with class $\beta \in \mathrm{Hom}_\mathbb{Z}(\mathrm{Pic}(X), \mathbb{Q})$. Let $T(G)$ denote a maximal torus of G. We recall the following lemma on the non-emptiness of $\mathrm{Hom}^{\mathrm{rep}}_\beta(\mathbb{P}_{1,a}, \mathfrak{X})$ from [16].

Lemma 4.3 ([16, Lemma 4.6])**.** *Every morphism $[u] \in \mathrm{Hom}^{\mathrm{rep}}_\beta(\mathbb{P}_{1,a}, \mathfrak{X})$ induces a canonical homomophism $\tilde{\beta} : \chi(T(G)) \to \mathbb{Q}$, well-defined up to the Weyl group action on the character group $\chi(T(G))$. Furthermore, the moduli stack $\mathrm{Hom}^{\mathrm{rep}}_\beta(\mathbb{P}_{1,a}, \mathfrak{X})$ is empty unless a is the minimal positive integer such that $a\tilde{\beta}(\eta) \in \mathbb{Z}$ for all $\eta \in \chi(T(G))$.*

Convention 4.4. In the rest of this subsection, we assume a is the unique positive integer associated to β as described in Lemma 4.3.

Given $\beta \in \mathrm{Eff}(W, G, \theta)$, we define

$$\underline{Q}_{\mathbb{P}_{1,a}}(X, \beta) \subseteq \mathrm{Hom}^{\mathrm{rep}}_\beta(\mathbb{P}_{1,a}, X)$$

to be the substack parametrizing representable morphisms $[u] : \mathbb{P}_{1,a} \to \mathfrak{X}$, mapping the generic point of $\mathbb{P}_{1,a}$ into X. This moduli stack is referred to as the *stacky loop space* in [16]. Let $\underline{F}_\beta$ be the distinguished $\mathbb{C}^*$-fixed closed substack of $\underline{Q}_{\mathbb{P}_{1,a}}(X, \beta)$ parametrizing elements such that the class β is exactly supported at 0.

For simplicity, we write $\underline{QG} := \underline{QG}^{0+}_{0,1}(X,\beta)$ and $\underline{Q}_{\mathbb{P}_{1,a}} := \underline{Q}_{\mathbb{P}_{1,a}}(X,\beta)$. Let $\pi_{\mathbb{P}^1}$ denote the projection to $\mathbb{P}^1$ from $\bar{I}X\times\mathbb{P}^1$ and let $\underline{\mathrm{ev}}_\star : \underline{QG} \to \bar{I}X\times\mathbb{P}^1$ be the evaluation map at the marking $x_\star$. The following lemma is a straightforward generalization of [16, Lemma 4.8 (2)] in K-theory.

Lemma 4.5. *(a) There is a natural isomorphism between an open neighborhood of $\underline{F}^{0,\beta}_{\star,0}$ in the closed substack $(\pi_{\mathbb{P}^1}\circ\underline{\mathrm{ev}}_\star)^{-1}(\infty)$ of $\underline{QG}$ and an open neighborhood of $\underline{F}_\beta$ in $\underline{Q}_{\mathbb{P}_{1,a}}$, under which $\underline{F}^{0,\beta}_{\star,0}\cong\underline{F}_\beta$. This isomorphism preserves the $\mathbb{C}^*\times T$-equivariant perfect obstruction theories.*

(b) Under the natural isomorphism between $\underline{F}^{0,\beta}_{\star,0}$ and $\underline{F}_\beta$, we have

$$\frac{(1-q^{-1})\cdot\mathcal{O}^{\mathrm{vir}}_{\underline{F}^{0,\beta}_{\star,0}}}{\lambda^{\mathbb{C}^*\times T}_{-1}\left(N^{\mathrm{vir},\vee}_{\underline{F}^{0,\beta}_{\star,0}/\underline{QG}}\right)} = \frac{\mathcal{O}^{\mathrm{vir}}_{\underline{F}_\beta}}{\lambda^{\mathbb{C}^*\times T}_{-1}\left(N^{\mathrm{vir},\vee}_{\underline{F}_\beta/\underline{Q}_{\mathbb{P}_{1,a}}}\right)}.$$

Proof. Statement (a) is Lemma 4.8 (2) in [16]. Statement (b) is the counterpart of [16, Lemma 4.8 (2)] in K-theory. The factor $(1-q^{-1})$ is introduced to cancel the K-theoretic Euler class of the 1-dimensional deformation corresponding to shifting the image of $x_\star$ away from $\infty\in\mathbb{P}^1$. We leave the easy details to the reader. $\square$

By choosing a section of the gerbe at $\infty\in\mathbb{P}_{1,a}$, we obtain an evaluation map $\mathrm{ev}_\infty : \underline{Q}_{\mathbb{P}_{1,a}}\to IX$ at ∞. Set $\check{\mathrm{ev}}_\infty := \iota\circ\mathrm{ev}_\infty$. By Lemma 4.5 and (7), we obtain the following.

Proposition 4.6.

$$I(Q,q) = 1+\sum_{\beta>0} Q^\beta(\check{\mathrm{ev}}_\infty)_*\left(\frac{\mathcal{O}^{\mathrm{vir}}_{\underline{F}_\beta}}{\lambda^{\mathbb{C}^*\times T}_{-1}\left(N^{\mathrm{vir},\vee}_{\underline{F}_\beta/\underline{Q}_{\mathbb{P}_{1,a}}}\right)}\right).$$

In the rest of this subsection, we focus on the case of toric DM stacks. Let $X = W/\!\!/_\theta K$ be the toric DM stack introduced in the previous subsection. Here K is the algebraic torus $(\mathbb{C}^*)^r$ which acts on $W = \oplus_{i=1}^m\mathbb{C}_{\rho_i}$ with weights $\rho_1,\dots\rho_m$. We recall the explicit descriptions of $\underline{Q}_{\mathbb{P}_{1,a}}(X,\beta)$ and $\underline{F}_\beta$ from [16].

Consider the graded ring $\mathbb{C}[x,y]$ with $\deg x = 1, \deg y = a$ and denote by $\mathbb{C}[x,y]_m$ its subspace of degree m. For $\beta\in\mathrm{Hom}_{\mathbb{Z}}(\chi(K),\mathbb{Q})$, we consider the finite-dimensional K-module

$$W_\beta := \bigoplus_{i=1}^m\left(\mathbb{C}[x,y]_{a\beta(L_{\rho_i})}\otimes\mathbb{C}_{\rho_i}\right).$$

According to [16, Corollary 5.2], we have

$$\underline{Q}_{\mathbb{P}_{1,a}}(X,\beta)\cong[W^{ss}_\beta/K] \tag{8}$$

Define

$$Z_\beta := \bigoplus_{i:\beta(L_{\rho_i})\in\mathbb{Z}_{\geq 0}}\mathbb{C}\cdot y^{\beta(L_{\rho_i})}\subseteq W_\beta$$

and $Z_\beta^{ss} := Z_\beta \cap W_\beta^{ss}$. Under the natural identification of Z_β with the K-submodule $\oplus_{i:\beta(L_{\rho_i})\in\mathbb{Z}_{\geq 0}}\mathbb{C}_{\rho_i}$, we have

$$Z_\beta = \bigcap_{i:\beta(L_{\rho_i})<0 \text{ or } \beta(L_{\rho_i})\notin\mathbb{Z}_{\geq 0}} D_{\rho_i}.$$

The distinguished $\mathbb{C}^*$-fixed point component $\underline{F}_\beta$ in $\underline{Q}_{\mathbb{P}_{1,a}}(X,\beta)$ satisfies

$$\underline{F}_\beta \cong [Z_\beta^{ss}/K]$$

under the isomorphism (8).[3]

Define $g_\beta = (e^{2\pi\sqrt{-1}\beta(L_{\pi_i})})_i \in K$ which acts on $W = \oplus_{i=1}^m \mathbb{C}_{\rho_i}$ as $(e^{2\pi\sqrt{-1}\beta(L_{\rho_i})})_i \in (\mathbb{C}^*)^m$. The component of IX corresponding to g_β is

$$X^{g_\beta} = [(W^{ss})^{g_\beta}/K] = \left[\left(W^{ss} \cap \bigcap_{i:\beta(\rho_i)\notin\mathbb{Z}} D_{\rho_i}\right)/K\right].$$

Note that the evaluation map $\mathrm{ev}_\infty : \underline{F}_\beta \to IX$ is a closed embedding and factors through X^{g_β}. The normal bundle of $\underline{F}_\beta = [Z_\beta^{ss}/K]$ in $X^{g_\beta} = [(W^{ss})^{g_\beta}/K]$ satisfies

$$\lambda_{-1}^{\mathbb{C}^*\times T}\left(N^{\mathrm{vir},\vee}_{[Z_\beta^{ss}/K]/[(W^{ss})^{g_\beta}/K]}\right) = \prod_{j:\rho(L_{\rho_j})\in\mathbb{Z}_{<0}} (1-U_j). \tag{9}$$

Let $\langle\nu\rangle$ denote the fractional part of a rational number ν.

Proposition 4.7.

$$\begin{aligned}\lambda_{-1}^{\mathbb{C}^*\times T}\left(N^{\mathrm{vir},\vee}_{\underline{F}_\beta/\underline{Q}_{\mathbb{P}_{1,a}}}\right) &= \lambda_{-1}^{\mathbb{C}^*\times T}\left(N^{\mathrm{vir},\vee}_{[Z_\beta^{ss}/K]/[W_\beta^{ss}/K]}\right)\\ &= \frac{\prod_{\nu:\langle\nu\rangle=\langle\beta(L_{\rho_j})\rangle, 0<\nu\leq\beta(L_{\rho_j})}(1-U_jq^\nu)}{\prod_{\nu:\langle\nu\rangle=\langle\beta(L_{\rho_j})\rangle, \beta(L_{\rho_j})<\nu<0}(1-U_jq^\nu)}\end{aligned}$$

and $\mathcal{O}^{\mathrm{vir}}_{\underline{F}_\beta} = \mathcal{O}_{\underline{F}_\beta} = \mathcal{O}_{[Z_\beta^{ss}/K]}$.

Proof. The proposition follows from the analysis of weights in the proof of [16, Proposition 5.3]. For the reader's convenience, we sketch the proof here. Let $[u] : \underline{Q}_{\mathbb{P}_{1,a}} \times \mathbb{P}_{1,a} \to X$ be the universal map and let $\pi : \underline{Q}_{\mathbb{P}_{1,a}} \times \mathbb{P}_{1,a} \to \underline{Q}_{\mathbb{P}_{1,a}}$ be the projection. As explained in [16, §5.2], the virtual normal bundle $N^{\mathrm{vir}}_{\underline{F}_\beta/\underline{Q}_{\mathbb{P}_{1,a}}}$ is the moving part of $\oplus_{i=1}^m R^\bullet\pi_*[u]^*L_{\rho_i}$,

[3] Our description of Z_β is different from that in [16]. This is because we use $\mathbb{P}_{1,a}$ while the authors of [16] use $\mathbb{P}_{a,1}$. To obtain the statements recalled in this subsection, the reader should just switch x and y in the corresponding statements in [16, §5].

and the virtual structure sheaf of $\underline{F}_\beta$ is defined by the fixed part of the obstruction bundle $\oplus_{i=1}^m R^1\pi_*[u]^*L_{\rho_i}$. According to [16], we have

$$\left(\bigoplus_{i=1}^m R^\bullet\pi_*[u]^*L_{\rho_i}\right)\Big|_{\underline{F}_\beta}$$
$$= \bigoplus_{i:\beta(L_{\rho_i})\geq 0}\left(U_j^{-1}\otimes H^0(\mathbb{P}_{1,a},\mathcal{O}(a\beta(L_{\rho_i})))\right)\oplus\bigoplus_{i:\beta(L_{\rho_i})<0}\left(U_j^{-1}\otimes H^1(\mathbb{P}_{1,a},\mathcal{O}(a\beta(L_{\rho_i})))\right).$$

In the case $\beta(L_\rho)\geq 0$, the $\mathbb{C}^*$-weights of the $\mathbb{C}^*$-module $H^0(\mathbb{P}_{1,a},\mathcal{O}(a\beta(L_\rho)))$ are

$$-\beta(L_\rho), -\beta(L_\rho)+1,\ldots,-\beta(L_\rho)+\lfloor\beta(L_\rho)\rfloor.$$

In the case $\beta(L_\rho)<0$, the $\mathbb{C}^*$-weights of the $\mathbb{C}^*$-module $H^1(\mathbb{P}_{1,a},\mathcal{O}(a\beta(L_\rho)))$ are

$$-\beta(L_\rho)-1, -\beta(L_\rho)-2,\ldots,-\beta(L_\rho)+\lfloor\beta(L_\rho)+1\rfloor.$$

This implies the formula of the K-theoretic Euler class of the virtual normal bundle.

Since the restriction of the obstruction bundle has no $\mathbb{C}^*$-fixed part, we conclude that $\mathcal{O}^{\mathrm{vir}}_{\underline{F}_\beta}=\mathcal{O}_{\underline{F}_\beta}$.

□

Theorem 4.8.

$$I(Q,q)=1+\sum_{\beta>0}Q^\beta\prod_{j=1}^m\frac{\prod_{\nu:\langle\nu\rangle=\langle\beta(L_{\rho_j})\rangle,\nu\leq 0}(1-U_jq^\nu)}{\prod_{\nu:\langle\nu\rangle=\langle\beta(L_{\rho_j})\rangle,\nu\leq\beta(L_{\rho_j})}(1-U_jq^\nu)}\mathbf{1}_{g_\beta^{-1}},$$

where $\mathbf{1}_{g_\beta^{-1}}$ denotes the unit class in $K_T(X^{g_\beta^{-1}})$.

Proof. Recall that the map $\check{\mathrm{ev}}_\infty$ is the composition of the closed embedding $[Z_\beta^{ss}/K]\hookrightarrow X^{g_\beta}$ and the involution on IX. Then the theorem follows from Proposition 4.6, Proposition 4.7, and (9). □

Let $J_0(Q,q):=J(0,Q)$ be the K-theoretic small J-function.

Corollary 4.9. *If $\beta(\det\mathbb{T}_{[W/K]})=\sum_{j=1}^m\beta(L_{\rho_j})>1$ for all effective $\beta\neq 0$ and $\beta(L_{\rho_j})\geq 0$ for all j and effective β, then $\mu^{>0}(Q,q)=0$ and $J_0(Q,q)=(1-q)I(Q,q)$.*

Proof. By the assumption that $\beta(L_{\rho_j})\geq 0$ for all j and effective β, we can write the K-theoretic I-function as

$$I(Q,q)=1+\sum_{\beta>0}\frac{Q^\beta}{p_\beta(q)}\mathbf{1}_{g_\beta^{-1}},$$

where $p_\beta(q)$ is a polynomial in $q^{1/a}$ whose constant term is 1. Here a is the order of g_β. Note that $(1-q)/p_\beta(q)\in\mathcal{K}_-$. This is because $p_\beta(q)$ has a nonzero constant term and the highest power of q in $p_\beta(q)$ is greater than or equal to $\sum_{j=1}^m\beta(L_{\rho_j})$, which is greater than 1 by assumption. Hence

$$\mu^{>0}(Q,q)=[(1-q)I(Q,q)-(1-q)]_+=0.$$

By the K-theoretic mirror theorem stated in Proposition 3.13, we have $J_0(Q,q) = (1-q)I(Q,q)$. $\square$

Remark 4.10. It is well-known that the cohomological small I and J-functions coincide if the first assumption in Corollary 4.9 holds, i.e., $\beta(\det \mathbb{T}_{[W/K]}) > 1$ for all effective $\beta \neq 0$. In K-theory, we also need the second assumption in Corollary 4.9 because of the following. Suppose there exists j such that $\beta(L)_{\rho_j} < 0$. The coefficient of Q^β in $(1-q)I(Q,q)$ is of the form

$$\frac{r(q)}{q^w p(q)},$$

where $r(q)$ and $p(q)$ are polynomials in $q^{1/a}$ with nonzero constant terms and w is a positive multiple of $1/a$. The projection of the above rational function onto $\mathcal{K}_+$ is in general nonzero because its partial fraction decomposition contains terms which are negative powers of $q^{1/a}$. Hence the function $\mu^{>0}(Q,q)$ is nontrivial.

5. The small orbifold quantum K-ring of weighted projective spaces

In this section, we focus on weighted projective spaces. Let $n \in \mathbb{Z}_{>0}$ and let $w_0, \ldots, w_n$ be a sequence of positive integers. Set $V = \mathbb{C}^{n+1}$. The weighted projective space $\mathbb{P}^{\mathbf{w}} = \mathbb{P}(w_0, \ldots, w_n)$ is defined as the quotient

$$[(V - \{0\})/\mathbb{C}^*],$$

where $\mathbb{C}^*$ acts with weights $-w_0, \ldots, -w_n$. Here we follow the convention of [19]; see [19, Remark 2.1] for an explanation of the choice of negative weights.

The components of the inertia stack $I\mathbb{P}^{\mathbf{w}}$ are indexed by elements of the set

$$F = \left\{ \frac{j}{w_i} \,\middle|\, 0 \leq j < w_i, 0 \leq i \leq n \right\}.$$

For $f \in F$, we denote by V^f the linear subspace of V fixed by $\exp(2\pi\sqrt{-1}f) \in \mathbb{C}^*$, i.e. $V^f = \mathbb{C}^{I_f}$ where $I_f := \{i \mid w_i \cdot f \in \mathbb{Z}\}$. Define $\mathbb{P}(V^f) := [(V^f - \{0\})/\mathbb{C}^*]$. Then we have

$$I\mathbb{P}^{\mathbf{w}} = \coprod_{f \in F} \mathbb{P}(V^f).$$

Note that the connected component $\mathbb{P}(V^f)$ itself is a weighted projective space. The involution ι on $I\mathbb{P}^{\mathbf{w}}$ exchanges $\mathbb{P}(V^f)$ with $\mathbb{P}(V^{\langle -f \rangle})$, $f \neq 0$, and is the identity on $\mathbb{P}(V^0)$.

Recall that the full orbifold K-group of $\mathbb{P}^{\mathbf{w}}$ is

$$\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}}) = \bigoplus_{f \in F} K(\mathbb{P}(V^f)).$$

Its dimension is $M := \sum_{j=0}^n w_j^{w_j}$. With respect to the tensor product on $K(\mathbb{P}^{\mathbf{w}})$, we have an algebra isomorphism

$$K(\mathbb{P}^{\mathbf{w}}) \cong \frac{\mathbb{C}[P, P^{-1}]}{\langle (1-P^{w_0}) \cdots (1-P^{w_n}) \rangle}.$$

The relation corresponds to the K-theory class defined by the Koszul complex associated to the morphism $\oplus_{j=0}^{n}\mathcal{O}_{\mathbb{P}^{\mathbf{w}}}(-w_j) \to \mathcal{O}_{\mathbb{P}^{\mathbf{w}}}$. Similarly, for each $f \in F$, we have

$$K(\mathbb{P}(V^f)) \cong \frac{\mathbb{C}[P, P^{-1}]}{\left\langle \prod_{i \in I_f}(1 - P^{w_i}) \right\rangle}.$$

Set

$$N_f = \sum_{w_i : w_i \in I_f} w_i.$$

Then $N_f = \dim K(\mathbb{P}(V^f))$.

Let Q be the Novikov variable dual to $\mathcal{O}_{\mathbb{P}^{\mathbf{w}}}(1)$. We choose the ground λ-algebra to be

$$\Lambda = \mathbb{C}[[Q^{1/\mathrm{lcm}(w_0, \ldots, w_n)}]].$$

Let $f_1, \ldots f_k$ be the elements of F arranged in increasing order. Set $f_{k+1} = 1$. We define an ordered basis of $\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda$ as follows. Let $\mathbf{1}_f$ denote the unit class in $K(\mathbb{P}(V^f))$. In particular, $\mathbf{1}_0$ is the K-theory class of the structure sheaf of $\mathbb{P}^{\mathbf{w}}$, which is also denoted by 1. Let $P \in \mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})$ be the image of $\mathcal{O}_{\mathbb{P}^{\mathbf{w}}}(-1) \in K(\mathbb{P}^{\mathbf{w}})$ under the inclusion $K(\mathbb{P}^{\mathbf{w}}) \subseteq \mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})$. For each f_i, let $a_1^{(f_i)}, a_2^{(f_i)}, \ldots, a_{N_{f_i}}^{(f_i)}$ be the sequence obtained by arranging the terms

$$\frac{m}{w_j}, \quad \text{where } w_j \in I_{f_i} \text{ and } m \in \{0, \ldots, w_j - 1\},$$

in increasing order. We define a basis for $K(\mathbb{P}(V^{f_i}))$ by

$$v_1^{(f_i)} = \mathbf{1}_{f_i}, \quad v_j^{(f_i)} = \prod_{m=1}^{j-1} \left(1 - \xi_{a_m^{(f_i)}} P\right)\mathbf{1}_{f_i} \quad \text{for } 1 < j \leq N_{f_i},$$

where $\xi_{a_m^{(f_i)}} := e^{2\pi\sqrt{-1}a_m^{(f_i)}}$ is a root of unity. We thus obtain the following Λ-basis for $\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda$:

$$\begin{aligned}
&v_1^{(f_1)}, v_2^{(f_1)} \ldots v_{N_{f_1}}^{(f_1)},\\
&v_1^{(f_2)}, v_2^{(f_2)} \ldots v_{N_{f_2}}^{(f_2)},\\
&\ldots,\\
&v_1^{(f_k)}, v_2^{(f_k)} \ldots v_{N_{f_k}}^{(f_k)},
\end{aligned}$$

where the terms in the i-th row form a basis for $K(\mathbb{P}(V^{f_i}))$.

Let $J_{\mathbb{P}^{\mathbf{w}}}(Q, q)$ be the K-theoretic small J-function of $\mathbb{P}^{\mathbf{w}}$. Since $\mathbb{P}^{\mathbf{w}}$ satisfies the conditions in Corollary 4.9, we have $J_{\mathbb{P}^{\mathbf{w}}}(Q, q) = (1 - q)I(Q, q)$. It follows from Theorem 4.8 and Corollary 1.2 that the K-theoretic small J-function of $\mathbb{P}^{\mathbf{w}}$ is given by

$$J_{\mathbb{P}^{\mathbf{w}}}(Q, q) = (1 - q) \sum_{d: d \geq 0, \langle d \rangle \in F} \frac{Q^d}{\prod_{j=0}^{n} \prod_{\nu : \langle \nu \rangle = \langle d w_j \rangle, 0 < \nu \leq d w_j} (1 - P^{w_j} q^\nu)} \mathbf{1}_{\langle d \rangle},$$

satisfying finite-difference equation

$$\prod_{j=0}^{n}\prod_{\ell=0}^{w_j-1}(1-q^{-\ell}(Pq^{Q\partial_Q})^{w_j})J_{\mathbb{P}^{\mathbf{w}}}(Q,q)=QJ_{\mathbb{P}^{\mathbf{w}}}(Q,q). \tag{10}$$

We denote the operator on the left hand side by

$$D^{(1)}:=\prod_{j=0}^{n}\prod_{\ell=0}^{w_j-1}(1-q^{-\ell}(Pq^{Q\partial_Q})^{w_j}).$$

Note that

$$D^{(1)}=\prod_{j=0}^{n}\prod_{\ell=0}^{w_j-1}\prod_{m=0}^{w_j-1}(1-\zeta_{w_j}^{m}q^{-\ell/w_j}Pq^{Q\partial_Q}),$$

where $\zeta_r=\exp(2\pi\sqrt{-1}/r)$ is a primitive r-th root of unity. For each $i\in\{1,\dots,k\}$ we define the operators

$$\widetilde{D}_j^{(f_i)}=\begin{cases}\mathrm{id} & j=1,\\ \prod_{m=1}^{j-1}\left(1-\xi_{a_m^{(f_i)}}q^{-f_i}Pq^{Q\partial_Q}\right) & 2\le j\le N_{f_i}+1,\end{cases}$$

and

$$D_j^{(f_i)}=\widetilde{D}_j^{(f_i)}\cdot\prod_{\ell:\ell<i}\widetilde{D}^{(f_\ell)}_{N_{f_\ell}+1}.$$

By definition, we have $D_1^{(f_{i+1})}=D^{(f_i)}_{N_{f_i}+1}, i<k$ and $D^{(1)}=D^{(f_k)}_{N_{f_k}+1}$.

Recall that $\Lambda_+\subseteq\Lambda$ is the maximal ideal generated by Q of positive degrees. For $d\in\mathbb{Q}_{>0}$, we denote by $\Lambda_{\ge d}$ (resp. $\Lambda_{>d}$) the ideal of Λ generated by Q of degree $\ge d$ (resp. $>d$).

Lemma 5.1. *We have*

$$\frac{1}{1-q}D_j^{(f_i)}J_{\mathbb{P}^{\mathbf{w}}}(Q,q)=Q^{f_i}v_j^{(f_i)}\mod \Lambda_{>f_i}\mathcal{K}_- \tag{11}$$

for $1\le i\le k$ *and* $1\le j\le N_{f_i}$ *and*

$$\frac{1}{1-q}D^{(1)}J_{\mathbb{P}^{\mathbf{w}}}(Q,q)=Q\mathbf{1}_0\mod\Lambda_{>1}\mathcal{K}_-. \tag{12}$$

Proof. Equation (12) follows from the finite-difference equation (10) and the asymptotic expansion

$$QJ_{\mathbb{P}^{\mathbf{w}}}(Q,q)/(1-q)=Q\mathbf{1}_0\mod\Lambda_{>1}\mathcal{K}_-.$$

We write

$$J_{\mathbb{P}^{\mathbf{w}}}(Q,q)/(1-q)=\sum_{m=1}^{k}\frac{Q^{f_m}}{\prod_{j=0}^{n}\prod_{\nu:\langle\nu\rangle=\langle f_m w_j\rangle,0<\nu\le f_m w_j}(1-P^{w_j}q^{\nu})}\mathbf{1}_{f_m}+r(Q,q),$$

where $r(Q,q) \in \Lambda_{\geq 1}\mathcal{K}_-$. For $1 \leq m \leq k$, consider the coefficient of $\mathbf{1}_{f_m}$ in the above expression:

$$g_m(Q,q) := \frac{Q^{f_m}}{\prod_{j=0}^{n} \prod_{\nu:\langle \nu \rangle = \langle f_m w_j \rangle, 0<\nu \leq f_m w_j} (1 - P^{w_j} q^{\nu})}. \tag{13}$$

It follows from (12) that $D^{(1)} g_m(Q,q) = 0$ for $1 \leq m \leq k$ and $D^{(1)} r(Q,q) = Q\mathbf{1}_0$ modulo $\Lambda_{>1}\mathcal{K}_-$. Note that for $1 \leq i \leq k$, $1 \leq j \leq N_{f_i}$, and $f \geq 1$, $D_j^{(f_i)} Q^f$ is a polynomial in q, and its degree in q is strictly less than that of $D^{(1)} Q^f$. Hence

$$D_j^{(f_i)} r(Q,q) = 0 \mod \Lambda_{\geq 1}\mathcal{K}_-.$$

To prove (11), we only need to analyze the Laurent polynomial part of $D_j^{(f_i)} g_m(Q,q)$ for $1 \leq m \leq k$. We have

$$D_j^{(f_i)} Q^{f_m} = \left(\prod_{\ell:\ell<i} \prod_{c=1}^{N_{f_\ell}} \left(1 - \xi_{a_c^{(f_\ell)}} q^{f_m - f_\ell} P\right) \right) \cdot \left(\prod_{c=1}^{j-1} (1 - \xi_{a_c^{(f_i)}} q^{f_m - f_i} P) \right) Q^{f_m}. \tag{14}$$

If $i < m$, then (14) is a polynomial in q, and by Lemma 5.2, its degree in q is less than that of the denominator of (13). Hence $D_j^{(f_i)} g_m(Q,q) \in \Lambda_{>f_i}\mathcal{K}_-$ if $m > i$.

If $i > m$, then the RHS of (14) contains a factor

$$\prod_{c=1}^{N_{f_m}} \left(1 - \xi_{a_c^{(f_m)}} P\right) = \prod_{i \in I_{f_m}} (1 - P^{w_i})$$

which vanishes as an element in $K(\mathbb{P}(V^{f_m}))$. Hence (14) and, therefore, $D_j^{(f_i)} g_m(Q,q)$ are zero if $i > m$.

If $i = m$, by Lemma 5.2, the degree of (14) in q is equal to that of the denominator of (13). We have

$$\begin{aligned} D_j^{(f_i)} Q^{f_i} \mathbf{1}_{f_i} = & \left(\prod_{\ell:\ell<i} \prod_{c=1}^{N_{f_\ell}} \left(1 - \xi_{a_c^{(f_\ell)}} q^{f_i - f_\ell} P\right) \right) \cdot \left(\prod_{c=1}^{j-1} \left(1 - \xi_{a_c^{(f_i)}} P\right) \right) Q^{f_i} \mathbf{1}_{f_i} \\ = & Q^{f_i} v_j^{(f_i)} \cdot \prod_{j=0}^{n} \prod_{\substack{\nu:\langle \nu \rangle = \langle f_i w_j \rangle \\ 0<\nu \leq f_i w_j}} (1 - P^{w_j} q^{\nu}). \end{aligned}$$

In the second line, we used the definition of $v_j^{(f_i)}$ and the second identity in Lemma 5.2. This concludes the proof of the lemma.

$\square$

Lemma 5.2. *We have the following identities*

$$\sum_{j=0}^{n} \sum_{\substack{\nu:\langle\nu\rangle=\langle f_m w_j\rangle \\ 0<\nu\le f_m w_j}} \nu = \sum_{\ell=1}^{m-1} (f_m - f_\ell) N_{f_\ell}$$

and

$$\prod_{j=0}^{n} \prod_{\substack{\nu:\langle\nu\rangle=\langle f_m w_j\rangle \\ 0<\nu\le f_m w_j}} (1 - P^{w_j} q^{\nu}) = \prod_{\ell=1}^{m-1} \prod_{c=1}^{N_{f_\ell}} \left(1 - \xi_{a_c^{(f_\ell)}} q^{f_m - f_\ell} P\right)$$

for $1 \le m \le k$.

Proof. We obtain the first identity by taking the degrees of both sides of the second identity. It suffices to prove the second identity. We have

$$\begin{aligned}
\prod_{j=0}^{n} \prod_{\substack{\nu:\langle\nu\rangle=\langle f_m w_j\rangle \\ 0<\nu\le f_m w_j}} (1 - P^{w_j} q^{\nu}) &= \prod_{j=0}^{n} \prod_{\substack{a:0\le a<f_m w_j \\ a\in\mathbb{Z}}} (1 - P^{w_j} q^{f_m w_j - a}) \\
&= \prod_{j=0}^{n} \prod_{\substack{f:0\le f<f_m \\ w_j f\in\mathbb{Z}}} (1 - P^{w_j} q^{w_j(f_m - f)}) \\
&= \prod_{f:0\le f<f_m} \Big(\prod_{w_j\in I_f} (1 - P^{w_j} q^{w_j(f_m - f)}) \Big) \\
&= \prod_{\ell=1}^{m-1} \prod_{c=1}^{N_{f_\ell}} \left(1 - \xi_{a_c^{(f_\ell)}} q^{f_i - f_\ell} P\right).
\end{aligned}$$

□

Let $S := \mathrm{S}_0(q)$ be the restriction of the S-operator of $\mathbb{P}^{\mathbf{w}}$ at $\mathbf{t} = 0$. By definition, we have

$$S = \mathrm{id} + S', \tag{15}$$

where id denotes the identity map and $S' \in \oplus_r \left(\mathrm{End}(K(I_r X)) \otimes \mathbb{C}(q^{1/r})\right) \widehat{\otimes} \Lambda_+$ satisfying $S'(\gamma) \in \Lambda_+ \mathcal{K}_-$ for any $\gamma \in \mathrm{K}_{\mathrm{orb}}(X)$. Let $\mathcal{A} = S^{-1}(P q^{Q\partial_Q} S) q^{Q\partial_Q}$ be the q-shift operator defined in Section 3.2.

Lemma 5.3. *We have*

$$\frac{1}{1-q} D_j^{(f_i)} J_{\mathbb{P}^{\mathbf{w}}}(Q, q) = Q^{f_i} S(v_j^{(f_i)}) \tag{16}$$

for $1 \le i \le k, 1 \le j \le N_{f_i}$ *and*

$$\frac{1}{1-q} D^{(1)} J_{\mathbb{P}^{\mathbf{w}}}(Q, q) = Q S(\mathbf{1}_0). \tag{17}$$

Proof. Equation (17) follows from the finite-difference equation (10) and Proposition 3.11. We prove (16) by induction. For $i = j = 1$, the formula (16) follows from Proposition 3.11. Assume that (16) holds for some (i, j) with $1 \le i \le k, 1 \le j \le N_i$. There are two cases. In the first case, we assume $j < N_i$. By definition, we have

$$D_{j+1}^{(f_i)} = \big(1 - \xi_{a_j^{(f_i)}} q^{-f_i} P q^{Q\partial_Q}\big) D_j^{(f_i)}.$$

Then it follows that

$$\begin{aligned}
\frac{1}{1-q} D_{j+1}^{(f_i)} J_{\mathbb{P}^{\mathsf{w}}}(Q,q) &= \frac{1}{1-q}\big(1 - \xi_{a_j^{(f_i)}} q^{-f_i} P q^{Q\partial_Q}\big) D_j^{(f_i)} J_{\mathbb{P}^{\mathsf{w}}}(Q,q) \\
&= \big(1 - \xi_{a_j^{(f_i)}} q^{-f_i} P q^{Q\partial_Q}\big) Q^{f_i} S(v_j^{(f_i)}) \\
&= Q^{f_i}\big(1 - \xi_{a_j^{(f_i)}} P q^{Q\partial_Q}\big) S(v_j^{(f_i)}) \\
&= Q^{f_i} S\big((1 - \xi_{a_j^{(f_i)}} \mathcal{A}) v_j^{(f_i)}\big) \\
&= Q^{f_i} S\big((1 - \xi_{a_j^{(f_i)}} P) v_j^{(f_i)} + r(Q,q)\big) \\
&= Q^{f_i} S\big(v_{j+1}^{(f_i)}\big) + Q^{f_i} S\big(r(Q,q)\big),
\end{aligned}$$

where $r(Q,q) \in \Lambda_+\mathcal{K}_+$. In the fifth line, we used the asymptotic expansion of the q-shift operator in (6). To prove the induction step in this case, we show that $r(Q,q) = 0$. By the asymptotic expansion (15), we have

$$S(v_{j+1}^{(f_i)}) = v_{j+1}^{(f_i)} + u(Q,q),$$

where $u(Q,q) \in \Lambda_+\mathcal{K}_-$, and

$$S\big(r(Q,q)\big) = r(Q,q) + w(Q,q),$$

where $w(Q,q) \in \Lambda_+\mathcal{K}$, and the lowest degree of Q in $w(Q,q)$ is larger than that in $r(Q,q)$. Suppose $r(Q,q) \neq 0$. Let $Q^{d_0} f(q)$ be the (nonzero) term in $r(Q,q)$ with the lowest degree of Q. Then $Q^{d_0} f(q)$ is also the term with the lowest degree of Q in $S(r(Q,q))$. Note that $f(q)$ is a Laurent polynomial in q. Then it follows from Lemma 5.1, (11) that $Q^{d_0} f(q) = 0$, which is a contradiction. Hence $r(Q,q) = 0$.

In the second case, we assume $j = N_{f_i}, i < k$. We have

$$\begin{aligned}
\frac{1}{1-q} D_1^{(f_{i+1})} J_{\mathbb{P}^{\mathsf{w}}}(Q,q) &= \frac{1}{1-q}\big(1 - \xi_{a_{N_{f_i}}^{(f_i)}} q^{-f_i} P q^{Q\partial_Q}\big) D_{N_{f_i}}^{(f_i)} J_{\mathbb{P}^{\mathsf{w}}}(Q,q) \\
&= \big(1 - \xi_{a_{N_{f_i}}^{(f_i)}} q^{-f_i} P q^{Q\partial_Q}\big) Q^{f_i} S(v_{N_{f_i}}^{(f_i)}) \\
&= Q^{f_i}\big(1 - \xi_{a_{N_{f_i}}^{(f_i)}} P q^{Q\partial_Q}\big) S(v_{N_{f_i}}^{(f_i)}) \\
&= Q^{f_i} S\big((1 - \xi_{a_{N_{f_i}}^{(f_i)}} \mathcal{A}) v_{N_{f_i}}^{(f_i)}\big).
\end{aligned}$$

Note that $\big(1-\xi_{a^{(f_i)}_{N_{f_i}}}P\big)v^{(f_i)}_{N_{f_i}}=0$ in $K(\mathbb{P}(V^{f_i}))$. By the asymptotic expansion of the q-shift operator in (6), we write

$$\big(1-\xi_{a^{(f_i)}_{N_{f_i}}}\mathcal{A}\big)v^{(f_i)}_{N_{f_i}}=Q^{d_0}f(q)+r(Q,q)\in\Lambda_+\mathcal{K}_+,$$

where $Q^{d_0}f(q), d_0>0$ is the term with the lowest degree of Q. We claim that

$$r(Q,q)=0 \quad \text{and} \quad Q^{f_i+d_0}f(q)=Q^{f_{i+1}}v^{(f_{i+1})}_1.$$

The above equalities imply the induction step. Suppose $r(Q,q)$ is nonzero. Let $Q^{d_1}g(q)$ be the term with the lowest degree of Q in $r(Q,q)$. We have $d_1>d_0$. Note that $Q^{f_i+d_0}f(q)$ and $Q^{f_i+d_1}g(q)$ are the terms with the two lowest degrees in $Q^{f_i}S\big(\big(1-\xi_{a^{(f_i)}_{N_{f_i}}}\mathcal{A}\big)v^{(f_i)}_{N_{f_i}}\big)$. Since $f(q)$ and $g(q)$ are Laurent polynomial in q, it follows from Lemma 5.1 that

$$Q^{f_i+d_0}f(q)+Q^{f_i+d_1}g(q)=Q^{f_{i+1}}v^{(f_{i+1})}_1.$$

This implies that $Q^{f_i+d_0}f(q)=Q^{f_{i+1}}v^{(f_{i+1})}_1$ and $g(q)=0$. The latter is a contradiction and therefore $r(Q,q)=0$. This concludes the proof of the claim.

Finally, we note that the above argument also implies that

$$\big(1-\xi_{a^{(f_k)}_{N_{f_k}}}\mathcal{A}\big)v^{(f_k)}_{N_{f_k}}=Q^{1-f_k}\mathbf{1}_0.$$

This gives another proof of (17). □

Set $f_{k+1}:=1$ and $v^{(f_{k+1})}_1:=\mathbf{1}_0$. The following corollary follows from the proof of the above lemma.

Corollary 5.4. *(a)* $(1-\xi_{a^{(f_i)}_j}\mathcal{A})v^{(f_i)}_j=v^{(f_i)}_{j+1}, \quad 1\le i\le k, 1\le j<N_{f_i}.$

(b) $(1-\xi_{a^{(f_i)}_{N_{f_i}}}\mathcal{A})v^{(f_i)}_{N_{f_i}}=Q^{f_{i+1}-f_i}v^{(f_{i+1})}_1, \quad 1\le i\le k.$

Theorem 5.5. *We have*

(a) $v^{(f_1)}_1=\mathbf{1}_0$.

(b) $(1-\xi_{a^{(f_i)}_j}P)\bullet_0 v^{(f_i)}_j=v^{(f_i)}_{j+1}, \quad 1\le i\le k, 1\le j<N_{f_i}.$

(c) $(1-\xi_{a^{(f_i)}_{N_{f_i}}}P)\bullet_0 v^{(f_i)}_{N_{f_i}}=Q^{f_{i+1}-f_i}v^{(f_{i+1})}_1, \quad 1\le i\le k.$

Proof. Statement (a) follows from the definition. By the proof of Lemma 5.3, we have

$$(1-\xi_{a^{(f_i)}_j}\mathcal{A})\mathbf{1}_0=1-\xi_{a^{(f_i)}_j}P.$$

Then statements (b) and (c) follow from Corollary 5.4 and Remark 3.10. □

Let $Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda := (\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda, \bullet_0)$ denote the small orbifold quantum K-ring of $\mathbb{P}^{\mathbf{w}}$. In the rest of the section, we will only consider $Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda$. For simplicity, we denote the small quantum product by $\cdot$ and write

$$\prod_{i=1}^{n} \alpha_i^{m_i} = \underbrace{\alpha_1 \cdot \ldots \cdot \alpha_1}_{m_1} \cdot \ldots \cdot \underbrace{\alpha_n \cdot \ldots \cdot \alpha_n}_{m_n}.$$

Corollary 5.6. *We have the following relation in $Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda$*

$$\prod_{j=0}^{n}(1-P^{w_j})^{w_j} = Q\mathbf{1}_0 \tag{18}$$

and a ring isomorphism

$$Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda \cong \frac{\Lambda[P,P^{-1}]}{\left\langle \prod_{j=0}^{n}(1-P^{w_j})^{w_j} - Q\mathbf{1}_0 \right\rangle}. \tag{19}$$

Proof. By Theorem 5.5, we have

$$\prod_{i=1}^{k}\prod_{j=1}^{N_{f_i}}(1-\xi_{a_j^{(f_i)}}P)\cdot \mathbf{1}_0 = Q\mathbf{1}_0$$

in $Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda$. We obtain the quantum ring relation by observing that

$$\prod_{i=1}^{k}\prod_{j=1}^{N_{f_i}}(1-\xi_{a_j^{(f_i)}}P) = \prod_{j=0}^{n}(1-P^{w_j})^{w_j}.$$

The ring isomorphism (19) follows from the fact that both sides have the same dimension $M = \sum_{j=0}^{n} w_j^{w_j}$. ∎

Remark 5.7. The relation (18) was also obtained by González–Woodward in [35, Example 1.3], where they denote P^{w_j} by X_j^{-1} for $0 \le j \le n$. However, their small quantum K-ring is generated by $P^{\pm w_0}, \ldots, P^{\pm w_n}$ and, therefore, can be viewed as a subring of $Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda$. We expect that for a general orbifold target space, González–Woodward's quantum K-ring is a subring of the full orbifold quantum K-ring introduced in this paper.

Let $\lceil f \rceil$ denote the least integer greater than or equal to f. Recall that we set $f_{k+1} = 1$ and $\mathbf{1}_1 := \mathbf{1}_0$.

Corollary 5.8. *For $1 \le i, j \le k$, we have the following relations in $Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}^{\mathbf{w}})_\Lambda$:*

(a) $Q^{f_{i+1}-f_i}\mathbf{1}_{f_{i+1}} = \prod_{b\in I_{f_i}}(1-P^{w_b})\mathbf{1}_{f_i}$.

(b) $Q^{f_i}\mathbf{1}_{f_i} = \prod_{\ell<i}\prod_{b\in I_{f_\ell}}(1-P^{w_b}) = \prod_{a=0}^{n}(1-P^{w_a})^{\lceil f_i w_a\rceil}$.

(c) *Furthermore,*

$$\mathbf{1}_{f_i}\cdot\mathbf{1}_{f_j} = Q^{f_{i,j}-\langle f_i+f_j\rangle}\prod_{a=0}^{n}(1-P^{w_a})^{\langle -f_i w_a\rangle + \langle -f_j w_a\rangle - \langle -(f_i+f_j)w_a\rangle}\mathbf{1}_{f_{i,j}}, \tag{20}$$

where $f_{i,j}$ is the smallest element in $F \cup \{f_{k+1}\}$ such that $f_{i,j} \geq \langle f_i + f_j \rangle$.

Proof. (a) and the first identity in (b) follow easily from Theorem 5.5 and the identity

$$\prod_{m=1}^{N_{f_\ell}} (1 - \xi_{a_m^{(f_\ell)}} P) = \prod_{b \in I_{f_\ell}} (1 - P^{w_b}).$$

The second identity in (b) follows from the fact that for any $a \in \{0, \dots, n\}$, we have a bijection

$$\{\ell \mid 1 \leq \ell < i, a \in I_{f_\ell}\} \to \left\{ \frac{c}{w_a} \,\middle|\, c \in \mathbb{Z}_{\geq 0}, \frac{c}{w_a} < f_i \right\}, \quad \ell \mapsto f_\ell.$$

The cardinality of the latter set is $\lceil f_i w_a \rceil$.

To prove (20), we assume $Q \neq 0$. Then by using (b), we rewrite the left hand side of (20) as

$$\mathbf{1}_{f_i} \cdot \mathbf{1}_{f_j} = Q^{-f_i - f_j} \prod_{a=0}^{n} (1 - P^{w_a})^{\lceil f_i w_a \rceil + \lceil f_j w_a \rceil} \tag{21}$$

and the right hand side as

$$\begin{aligned} & Q^{f_{i,j} - \langle f_i + f_j \rangle} \prod_{a=0}^{n} (1 - P^{w_a})^{\langle -f_i w_a \rangle + \langle -f_j w_a \rangle - \langle -(f_i + f_j) w_a \rangle} \mathbf{1}_{f_{i,j}} \\ & \qquad = Q^{-\langle f_i + f_j \rangle} \prod_{a=0}^{n} (1 - P^{w_a})^{\langle -f_i w_a \rangle + \langle -f_j w_a \rangle - \langle -(f_i + f_j) w_a \rangle} \cdot \prod_{a=0}^{n} (1 - P^{w_a})^{\lceil f_{i,j} w_a \rceil} \\ & \qquad = Q^{-\langle f_i + f_j \rangle} \prod_{a=0}^{n} (1 - P^{w_a})^{\langle -f_i w_a \rangle + \langle -f_j w_a \rangle - \langle -(f_i + f_j) w_a \rangle + \lceil f_{i,j} w_a \rceil}. \end{aligned} \tag{22}$$

By comparing (21) and (22) and using the relation (18), we see that (20) follows from the following claim.

Claim 5.9. For any a, the following identities hold:

$$\lceil f_i w_a \rceil + \lceil f_j w_a \rceil = \langle -f_i w_a \rangle + \langle -f_j w_a \rangle - \langle -(f_i + f_j) w_a \rangle + \lceil f_{i,j} w_a \rceil,$$

if $f_i + f_j < 1$, and

$$\lceil f_i w_a \rceil + \lceil f_j w_a \rceil = \langle -f_i w_a \rangle + \langle -f_j w_a \rangle - \langle -(f_i + f_j) w_a \rangle + \lceil f_{i,j} w_a \rceil + w_a,$$

if $f_i + f_j \geq 1$.

Set $m = \lceil (f_i + f_j) w_a \rceil$. By definition, we have $m - 1 < (f_i + f_j) w_a \leq m$, which is equivalent to

$$\frac{m-1}{w_a} < f_i + f_j \leq \frac{m}{w_a}.$$

This implies that

$$(m-1)/w_a < f_{i,j} \leq m/w_a,$$

if $f_i + f_j < 1$, and
$$(m-1)/w_a < f_{i,j} + 1 \leq m/w_a,$$
if $f_i + f_j \geq 1$. Hence we have
$$\begin{aligned} \lceil (f_i + f_j) w_a \rceil &= \lceil f_{i,j} w_a \rceil, && \text{if } f_i + f_j < 1, \text{ and} \\ \lceil (f_i + f_j) w_a \rceil &= \lceil f_{i,j} w_a \rceil + w_a, && \text{if } f_i + f_j \geq 1. \end{aligned}$$
We conclude the proof of the claim by using the identity $\lceil x \rceil - \langle -x \rangle = x$, for any $x \in \mathbb{Q}$. $\square$

Remark 5.10. The relations of the full orbifold K-ring of $\mathbb{P}^{\mathbf{w}}$ have been computed by Goldin–Harada–Holm–Kimura [34]. Set $\ell := \mathrm{lcm}(w_0, \dots, w_n)$. Then the class $\mathbf{1}_f, f \in F$ corresponds to $\alpha_{\ell\langle -f \rangle}$ in [34, §4]. By setting Q to zero in the identities in (a) and (c) of Corollary 5.8, we recover the relations (4.4) and (4.1) in [34].

Example 5.11. We illustrate the structure of the small orbifold quantum K-ring of $\mathbb{P}(2,1)$, the stacky GIT quotient of $\mathbb{C}^2$ by the $\mathbb{C}^*$-action
$$t \cdot (z_0, z_1) := (t^{-2} z_0, t^{-1} z_1)$$
for $t \in \mathbb{C}^*$ and $(z_0, z_1) \in \mathbb{C}^2$. Note that $0 := [1 : 0] \in \mathbb{P}(2,1)$ is a stacky point and isomorphic to $B\mu_2$. Let $\Lambda = \mathbb{C}[[Q^{1/2}]]$. We have
$$I\mathbb{P}(2,1) = \mathbb{P}(2,1) \coprod B\mu_2$$
and
$$\mathrm{K}_{\mathrm{orb}}(\mathbb{P}(2,1))_\Lambda = K(\mathbb{P}(2,1))_\Lambda \oplus K(B\mu_2)_\Lambda.$$
Let $\mathbf{1}_0$ and $\mathbf{1}_{\frac{1}{2}}$ be the K-theory classes of the structure sheaves of $\mathbb{P}(2,1)$ and $B\mu_2$, respectively. One can generate a basis of $\mathrm{K}_{\mathrm{orb}}(\mathbb{P}(2,1))_\Lambda$ from $\mathbf{1}_0$ by multiplying $1 \pm P$ using the small quantum product:
$$\begin{tikzcd} \mathbf{1}_0 \arrow[r, "(1-P)\cdot"] & (1-P)\mathbf{1}_0 \arrow[r, "(1-P)\cdot"] & (1-P)^2\mathbf{1}_0 \arrow[dll, "(1+P)\cdot"] \\ Q^{\frac{1}{2}}\mathbf{1}_{\frac{1}{2}} \arrow[r, "(1-P)\cdot"] & Q^{\frac{1}{2}}(1-P)\mathbf{1}_{\frac{1}{2}} \arrow[dl, "(1+P)\cdot"] & \\ Q\mathbf{1}_0 & & \end{tikzcd}$$
The three elements in the first line form a basis for $K(\mathbb{P}(2,1))_\Lambda$ and the two elements in the second line form a basis for $K(B\mu_2)_\Lambda$. If we invert Q, then we have the following ring isomorphism
$$Q\mathrm{K}_{\mathrm{orb}}(\mathbb{P}(2,1))_\Lambda \cong \frac{\mathbb{C}[P, P^{-1}]}{\langle (1-P)(1-P^2)^2 - Q\mathbf{1}_0 \rangle}.$$
Using Corollary 5.8, we obtain the relations
$$Q^{\frac{1}{2}}\mathbf{1}_{\frac{1}{2}} = (1-P^2)(1-P)\mathbf{1}_0 \quad \text{and} \quad \mathbf{1}_{\frac{1}{2}} \cdot \mathbf{1}_{\frac{1}{2}} = (1-P)\mathbf{1}_0.$$

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