

## OPTIMAL ATTITUDE CONTROL OF A RIGID BODY USING GEOMETRICALLY EXACT COMPUTATIONS ON $SO(3)$

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**ABSTRACT.** An efficient and accurate computational approach is proposed for a nonconvex optimal attitude control for a rigid body. The problem is formulated directly as a discrete time optimization problem using a Lie group variational integrator. Discrete time necessary conditions for optimality are derived, and an efficient computational approach is proposed to solve the resulting two-point boundary-value problem. This formulation wherein the optimal control problem is solved based on discretization of the attitude dynamics and derivation of discrete time necessary conditions, rather than development and discretization of continuous time necessary conditions, is shown to have significant advantages. In particular, the use of geometrically exact computations on  $SO(3)$  guarantees that this optimal control approach has excellent convergence properties even for highly nonlinear large angle attitude maneuvers.

### 1. INTRODUCTION

A discrete optimal control problem for attitude dynamics of a rigid body in the presence of an attitude dependent potential is considered. The objective is to minimize the square of the  $l_2$  norm of external control torques which transfer a given initial attitude and an initial angular momentum of the rigid body to a desired terminal attitude and a terminal angular momentum during a fixed maneuver time. The attitude of the rigid body is defined by the orientation of a body fixed frame with respect to a reference frame; the attitude is represented by a rotation matrix that is a  $3 \times 3$  orthogonal matrix with determinant of 1. The configuration manifold is the special orthogonal group,  $SO(3)$ .

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The dynamics of a rigid body has fundamental invariance properties. In the absence of nonconservative forces, total energy is preserved. A consequence of Noether's theorem is that symmetries in the Lagrangian result in conservation of the associated momentum map. Furthermore, the configuration manifold of the rigid body attitude dynamics is the Lie group  $SO(3)$ . General-purpose numerical integration methods, including the popular Runge–Kutta schemes, typically preserve neither first integrals nor the geometric characteristics of the configuration manifold. In particular, the orthogonal structure of rotation matrices is not reserved numerically with standard schemes.

A Lie group variational integrator that preserves those geometric features is presented in [19], and integrators on the configuration manifold  $SO(3)$  and  $SE(3)$  are developed in [10] and [15, 16], respectively. These integrators are obtained from a discrete variational principle, and they exhibit the symplectic and momentum preservation properties, and good energy behavior characteristic of variational integrators [21]. Since the rotation matrices are updated by a group operation, they automatically evolve on the rotation group without the need for reprojection techniques or constraints [6].

Optimal attitude control problems are studied in [23]. The angular velocity of a rigid body is treated as a control input; an optimal angular velocity that steers the rigid body is derived from the attitude kinematics. Continuous time optimal control problems on a Riemannian manifold are studied in [4], where necessary conditions for optimality are derived from a variational principle. An optimal control problem based on discrete mechanics is studied in [7]. The discrete equations of motion and the boundary conditions are imposed as constraints, and the optimal control problem is solved by a general-purpose parameter optimization tool. This approach requires large computation time, since the number of optimization parameters is proportional to the number of discrete time steps. Discrete necessary conditions for optimal control of the attitude dynamics of a rigid body are presented in [5].

This paper proposes an exact and efficient computational approach to solve an optimal control problem associated with the attitude dynamics of a rigid body that evolves on the configuration manifold  $SO(3)$ . We assume that the control inputs are parameterized by their value at each time step. A Lie group variational integrator on  $SO(3)$  that includes the effects of external control inputs is developed using the discrete Lagrange–d'Alembert principle. Discrete necessary conditions for optimality are obtained using a variational principle, while imposing the Lie group variational integrator as dynamic constraints.

The necessary conditions are expressed as a two-point boundary-value problem on  $T^*SO(3)$  and its dual. Sensitivity derivatives along an extremal solution are developed by following the procedures presented in [11], and

they are used to construct an algorithm that solves the boundary-value problem efficiently. Since the attitude of the rigid body is represented by a rotation matrix, and the orthogonal structure of rotation matrices is preserved by the Lie group variational integrator, the discretization of the optimal control problem does not exhibit singularities.

This paper is organized as follows. In Sec. 2, a Lie group variational integrator is developed using the discrete Lagrange–d’Alembert principle. Necessary conditions for optimality and a computational approach to solve the two-point boundary problem are presented in Sec. 3. Numerical results for the attitude control of an underactuated 3D pendulum, and for a fully actuated spacecraft in a circular orbit are given in Sec. 4.

## 2. EQUATIONS OF MOTION FOR THE ATTITUDE DYNAMICS OF A RIGID BODY

In this section, we define a rigid body model acting under a potential field and we develop discrete equations of motion for the attitude dynamics of the rigid body, referred to as a Lie group variational integrator. These discrete equations of motion are used as dynamic constraints for the optimal control problem presented in Sec. 3.

**2.1. Rigid body model.** We consider the attitude dynamics of a rigid body in the presence of an attitude dependent potential. The configuration manifold is the Lie group,  $SO(3)$ . We assume that the potential  $U(\cdot) : SO(3) \mapsto \mathbb{R}$  is determined by the attitude of the rigid body,  $R \in SO(3)$ . External control inputs generate moments about the mass center of the rigid body. A spacecraft on a circular orbit including gravity gradient effects [11], a 3D pendulum [10], or a free rigid body can be modeled in this way. The continuous equations of motion are given by

$$\dot{\Pi} + \Omega \times \Pi = M + Bu, \quad (1)$$

$$\dot{R} = RS(\Omega), \quad (2)$$

where  $\Omega \in \mathbb{R}^3$  is the angular velocity of the body expressed in the body fixed frame, and  $\Pi = J\Omega \in \mathbb{R}^3$  is the angular momentum of the body for a moment of inertia matrix  $J \in \mathbb{R}^{3 \times 3}$ . The vector  $M \in \mathbb{R}^3$  is the moment due to the potential,  $u \in \mathbb{R}^m$  is the external control input, and  $B \in \mathbb{R}^{3 \times m}$  is an input matrix. If the rank of the input matrix is less than 3, then the rigid body is underactuated. The matrix valued function,  $S(\cdot) : \mathbb{R}^3 \mapsto \mathfrak{so}(3)$  is an isomorphism defined such that  $S(x)y = x \times y$  for all  $x, y \in \mathbb{R}^3$ . The moment due to the potential is determined by the relationships,  $S(M) = \frac{\partial U}{\partial R}^T R - R^T \frac{\partial U}{\partial R}$ , or more explicitly,

$$M = r_1 \times u_{r_1} + r_2 \times u_{r_2} + r_3 \times u_{r_3}, \quad (3)$$

where  $r_i, u_{r_i} \in \mathbb{R}^{1 \times 3}$  are the  $i$ th row vectors of  $R$  and  $\frac{\partial U}{\partial R}$ , respectively. A detailed derivation of the above equations can be found in [10].

**2.2. Lie group variational integrator.** We describe a structure preserving numerical integrator for the attitude dynamics of the rigid body, referred to as a Lie group variational integrator. Since it is obtained from a discrete variational approach, and it exactly preserves the momentum and symplectic form, while exhibiting good energy behavior over exponentially long times. Since a Lie group numerical method [6] is explicitly adopted in the variational development, the rotation matrix automatically remains on  $\text{SO}(3)$ .

The Lie group variational integrator is obtained by following procedures commonly adopted in Lagrangian mechanics. The variational approach is based on the discretizing Hamilton principle rather than discretizing the continuous equations of motion. The velocity phase space of the continuous Lagrangian is replaced by discrete variables, and a discrete Lagrangian is chosen. Taking a variation of the action sum defined as the summation of the discrete Lagrangian, we obtain a Lagrangian form of the discrete equations of motion using the Lagrange–d’Alembert principle. A discrete version of the Legendre transformation yields discrete equations in Hamiltonian form.

The detailed derivation is presented in [10, 15]. In this paper, we extend these results to include the effects of external control inputs. Consider the fixed integration step size  $h \in \mathbb{R}$ . Let  $R_k \in \text{SO}(3)$  denote the attitude of the rigid body at time  $t = kh$ . We introduce a new variable  $F_k \in \text{SO}(3)$  defined by

$$F_k = R_k^T R_{k+1}, \quad (4)$$

which represents a relative attitude between integration steps. If we find  $F_k \in \text{SO}(3)$  at each integration step, the rotation matrix is updated by multiplication of two rotation matrices, i.e.,  $R_{k+1} = R_k F_k$ , which is the group operation on  $\text{SO}(3)$ . In particular, we ensure that  $F_k$  is a rotation matrix by expressing it as  $F_k = \exp(\xi_k)$ , where  $\xi_k \in \mathfrak{so}(3)$  is a skew-symmetric matrix and  $\exp(\cdot)$  denotes the matrix exponential. This guarantees that the rotation matrix evolves on  $\text{SO}(3)$  automatically. This is the approach of Lie group methods [6]. The following procedure provides an expression for  $F_k$  using the discrete Lagrange–d’Alembert principle.

Using the kinematic relationship (2),  $S(\Omega_k)$  can be approximated as

$$S(\Omega_k) = R_k^T \dot{R}_k \approx R_k^T \left( \frac{R_{k+1} - R_k}{h} \right) = \frac{1}{h} (F_k - I_{3 \times 3}).$$

Using the above equation, the kinetic energy of the rigid body is approximated as

$$\begin{aligned} T &= \frac{1}{2} \operatorname{tr}[S(\Omega_k) J_d S(\Omega_k)^T] \approx \frac{1}{2h^2} \operatorname{tr}[(F_k - I_{3 \times 3}) J_d (F_k - I_{3 \times 3})^T] \\ &= \frac{1}{h^2} \operatorname{tr}[(I_{3 \times 3} - F_k) J_d], \end{aligned}$$

where  $J_d \in \mathbb{R}^{3 \times 3}$  is a nonstandard moment of inertia matrix of the rigid body defined in terms of the standard moment of inertia matrix  $J \in \mathbb{R}^{3 \times 3}$  as  $J_d = \frac{1}{2} \operatorname{tr}[J] I_{3 \times 3} - J$ . Define a discrete Lagrangian  $L_d$  as

$$L_d(R_k, F_k) = \frac{1}{h} \operatorname{tr}[(I_{3 \times 3} - F_k) J_d] - hU(R_{k+1}). \quad (5)$$

This discrete Lagrangian is a first-order approximation of the integral of the continuous Lagrangian over one integration step. Therefore, the following action sum, defined as the summation of the discrete Lagrangian, approximates the action integral;  $\mathfrak{G}_d = \sum_{k=0}^{N-1} L_d(R_k, F_k)$ . Taking a variation of the action sum, we obtain the discrete equations of motion using the discrete Lagrange–d’Alembert principle. The variation of a rotation matrix can be expressed using the exponential of a Lie algebra element:

$$R_k^\epsilon = R_k \exp \epsilon \eta_k,$$

where  $\epsilon \in \mathbb{R}$  and  $\eta_k \in \mathfrak{so}(3)$  is the variation expressed as a skew-symmetric matrix. Thus, the infinitesimal variation is given by

$$\delta R_k = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} R_k^\epsilon = R_k \eta_k. \quad (6)$$

The Lagrange–d’Alembert principle states that the following equation is satisfied for all possible variations  $\eta_k \in \mathfrak{so}(3)$ :

$$\delta \sum_{k=0}^{N-1} \frac{1}{h} \operatorname{tr}[(I_{3 \times 3} - F_k) J_d] - hU(R_{k+1}) - \sum_{k=0}^{N-1} \frac{h}{2} \operatorname{tr}[\eta_{k+1} S(Bu_{k+1})] = 0. \quad (7)$$

Using the expression of the infinitesimal variation of a rotation matrix (6) and using the fact that the variations vanish at the end points, the above equation can be written as

$$\sum_{k=1}^{N-1} \operatorname{tr} \left[ \eta_k \left\{ \frac{1}{h} (F_k J_d - J_d F_{k-1}) + h R_k^T \frac{\partial U}{\partial R_k} - \frac{h}{2} S(Bu_k) \right\} \right] = 0.$$

Since the above expression should be zero for all possible variations  $\eta_k \in \mathfrak{so}(3)$ , the expression in the braces should be symmetric. Then, *the discrete equations of motion in Lagrangian form* are given by

$$\frac{1}{h} (F_{k+1} J_d - J_d F_k - J_d F_{k+1}^T + F_k^T J_d) = hS(M_{k+1}) + hS(Bu_{k+1}), \quad (8)$$

$$R_{k+1} = R_k F_k. \quad (9)$$

Using the discrete version of the Legendre transformation, *the discrete equations of motion in Hamiltonian form* are given by

$$hS(\Pi_k) = F_k J_d - J_d F_k^T, \quad (10)$$

$$R_{k+1} = R_k F_k, \quad (11)$$

$$\Pi_{k+1} = F_k^T \Pi_k + h(M_{k+1} + B u_{k+1}). \quad (12)$$

Given  $(R_k, \Pi_k)$ , we can obtain  $F_k$  by solving (10), and  $R_{k+1}$  is obtained by (11). The moment due to the potential  $M_{k+1}$  can be calculated by (3). Finally,  $\Pi_{k+1}$  is updated by (12). This yields a map  $(R_k, \Pi_k) \mapsto (R_{k+1}, \Pi_{k+1})$ , and this process can be repeated. The only implicit part is solving (10). We can express (10) in terms of a Lie algebra element  $S(f_k) = \text{logm}(F_k) \in \mathfrak{so}(3)$ , and find  $f_k \in \mathbb{R}^3$  numerically by a Newton iteration. The relative attitude  $F_k$  is obtained by the exponential map:  $F_k = \exp S(f_k)$ . Therefore, we are guaranteed that  $F_k$  is a rotation matrix.

The Lie group variational integrator has the desirable properties that they are symplectic and momentum preserving, and they exhibit good energy behavior for an exponentially long time period as they are derived from the discrete Hamilton principle. It also preserves the Lie group structure without the use of local charts, reprojection, or constraints, and it avoids singularities and ambiguities associated with other attitude parameterizations. This geometrically exact numerical integration method yields highly efficient and accurate computational algorithms for rigid body dynamics, and all of these properties are independent of the stepsize.

The order of the variational integrator is equal to the order of the corresponding discrete Lagrangian. Consequently, the above Lie group variational integrator is of first-order since (5) is a first-order approximation. While higher-order variational integrators can be obtained by modifying (5), we use the first-order integrator because it yields a compact form for the necessary conditions that preserves the geometry; these necessary conditions are developed in Section 3.

It is worthwhile noting that while the Lie group variational integrator adopted in this paper is formally first-order, it is the Lie group analogue of symplectic Euler, which is symplectically equivalent [20, 24] to the second-order accurate Störmer–Verlet method [3]. This means that one can obtain the Störmer–Verlet method by conjugating the symplectic Euler method with a symplectic transformation. In particular, numerical trajectories of symplectic Euler will shadow numerical trajectories obtained using Störmer–Verlet.

In practice, the shadowing result imparts the symplectic Euler method with the same desirable qualitative properties as Störmer–Verlet. Since on an appropriate choice of charts, our Lie symplectic Euler method reduces to symplectic Euler in coordinates, it follows that there is a corresponding

second-order Lie Störmer–Verlet method that our integrator is symplectically equivalent to, and whose discrete trajectories are shadowed by our method.

### 3. DISCRETE OPTIMAL CONTROL OF THE ATTITUDE DYNAMICS OF A RIGID BODY

We formulate a discrete optimal control problem for the attitude dynamics of a rigid body, and we derive necessary conditions for optimality using a variational principle. The necessary conditions are expressed as a two-point boundary-value problem, and a computational approach to solve the boundary-value problem is proposed using sensitivity derivatives.

**3.1. Problem formulation.** A discrete time optimal control problem in  $SO(3)$  is formulated as a maneuver of a rigid body from a given initial attitude  $R_0 \in SO(3)$  and an initial angular momentum  $\Pi_0 \in \mathbb{R}^3$  to a desired terminal attitude  $R_N^d \in SO(3)$  and a terminal angular momentum  $\Pi_N^d \in \mathbb{R}^3$  during a given maneuver time  $N$ . The performance index is the square of the  $l_2$  norm of the control input; the discrete equations of motion developed in the previous section are imposed as constraints:

$$\begin{aligned} &\text{given: } R_0, \Pi_0, R_N^d, \Pi_N^d, N, \\ &\min_{u_{k+1}} \mathcal{J} = \sum_{k=0}^{N-1} \frac{h}{2} \|u_{k+1}\|^2, \\ &\text{such that } R_N = R_N^d, \Pi_N = \Pi_N^d, \\ &\text{subject to (10), (11) and (12).} \end{aligned}$$

In [7], an optimal control problem based on discrete mechanics is considered. The control inputs at each discrete step are considered as optimization parameters, and the discrete equations of motion and the boundary conditions are imposed as constraints. The optimization problem is solved numerically by a general-purpose parameter optimization tool such as sequential quadratic programming (SQP). The same approach can be applied to the above optimization problem. However, it has a large computational burden since the number of optimization parameters,  $m \times N$ , is proportional to the number of integration steps. Usually, a large time step size is chosen to make the number of integration steps small, or the control inputs are approximated by collocation points. The resulting control inputs tends to be under-resolved and sub-optimal.

We derive necessary conditions for optimality using the standard calculus of variations. We assume that the control inputs are parameterized by their value at each time step. The necessary conditions are expressed as a two-point boundary-value problem.

**3.2. Necessary conditions of optimality.** Define an augmented performance index as

$$\mathcal{J}_a = \sum_{k=0}^{N-1} \frac{h}{2} \|u_{k+1}\|^2 + \lambda_k^{1,T} S^{-1}(\log m(F_k - R_k^T R_{k+1})) + \lambda_k^{2,T} \{-\Pi_{k+1} + F_k^T \Pi_k + h(M_{k+1} + B u_{k+1})\}, \quad (13)$$

where  $\lambda_k^1, \lambda_k^2 \in \mathbb{R}^3$  are Lagrange multipliers corresponding to the discrete equations of motion. The augmented performance index is chosen such that the dimension of the multipliers is equal to the dimension of the rotation matrix and the angular momentum vector. The discrete kinematics equation (11) is transformed into a matrix logarithm form. The constraints arising from the discrete kinematics equation (11) and the angular momentum equation (12) are explicitly applied. Equation (10) appears in the discrete equations of motion because we introduce the auxiliary variable  $F_k \in \text{SO}(3)$ . The constraint (10) is considered implicitly when taking a variation of the performance index.

Consider variations from a given trajectory denoted by  $\Pi_k, R_k, F_k$ , and  $u_k$ :

$$\Pi_k^\epsilon = \Pi_k + \epsilon \delta \Pi_k, \quad (14)$$

$$R_k^\epsilon = R_k \exp \epsilon S(\zeta_k) = R_k + \epsilon R_k S(\zeta_k) + \mathcal{O}(\epsilon^2), \quad (15)$$

$$F_k^\epsilon = F_k \exp \epsilon S(\xi_k) = F_k + \epsilon F_k S(\xi_k) + \mathcal{O}(\epsilon^2), \quad (16)$$

where  $\zeta_k, \xi_k \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$ . The real space  $\mathbb{R}^3$  is isomorphic to the Lie algebra  $\mathfrak{so}(3)$  according to the mapping  $S(\cdot) : \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ . The variations of the rotation matrices are expressed using the exponential of the Lie algebra elements. The corresponding infinitesimal variations of  $\Pi_k, R_k$ , and  $F_k$  are given by  $\delta \Pi_k, \delta R_k = R_k S(\zeta_k)$ , and  $\delta F_k = F_k S(\xi_k)$ , respectively.

The variation of the augmented performance index is obtained from the above expressions. Instead of taking a variation of the matrix logarithm in (13), we take a variation of the kinematics equation, (11) and we use it as a constrained variation. Since  $F_k = R_k^T R_{k+1}$  by (11), the variation  $\delta F_k$  is given by

$$\delta F_k = \delta R_k^T R_{k+1} + R_k^T \delta R_{k+1}.$$

Substituting the expression for the infinitesimal variation of  $R_k$ , we obtain

$$F_k S(\xi_k) = -S(\zeta_k) F_k + F_k S(\zeta_{k+1}).$$

Multiplying both sides of the above equation by  $F_k^T$  and using the property  $S(R^T x) = R^T S(x) R$  for all  $R \in \text{SO}(3)$  and  $x \in \mathbb{R}^3$ , we obtain

$$\xi_k = -F_k^T \zeta_k + \zeta_{k+1}. \quad (17)$$

We use (17) as a constrained variation equivalent to (11).



Now we develop another expression for a constrained variation using (10). Since we do not use (10) explicitly as a constraint in (13),  $\delta\Pi_k$  and  $\delta F_k$  are not independent. Taking a variation of (10), we obtain

$$hS(\delta\Pi_k) = F_k S(\xi_k) J_d + J_d S(\xi_k) F_k^T.$$

Using the properties,

$$S(Rx) = RS(x)R^T, \quad S(x)A + A^T S(x) = S(\{\text{tr}[A] I_{3 \times 3} - A\} x)$$

for all  $x \in \mathbb{R}^3$ ,  $A \in \mathbb{R}^{3 \times 3}$ , and  $R \in \text{SO}(3)$ , the above equation can be written as

$$\begin{aligned} hS(\delta\Pi_k) &= S(F_k \xi_k) F_k J_d + J_d F_k^T S(F_k \xi_k) \\ &= S(\{\text{tr}[F_k J_d] I_{3 \times 3} - F_k J_d\} F_k \xi_k). \end{aligned}$$

Thus,  $\xi_k$  is given by

$$\xi_k = \mathcal{B}_k \delta\Pi_k, \quad (18)$$

where  $\mathcal{B}_k = hF_k^T \{\text{tr}[F_k J_d] I_{3 \times 3} - F_k J_d\}^{-1} \in \mathbb{R}^{3 \times 3}$ . Equation (18) shows the relationship between  $\delta\Pi_k$  and  $\delta F_k$ .

Since the moment due to the potential  $M_k$  is dependent on the attitude of the rigid body, the variation of the moment  $\delta M_k$  can be written using a variation of the rotation matrix:

$$\delta M_k = \mathcal{M}_k \zeta_k, \quad (19)$$

where  $\mathcal{M}_k \in \mathbb{R}^{3 \times 3}$  is expressed in terms of the attitude of the rigid body, and the expression is determined by the potential field. We present detailed expressions of  $\mathcal{M}_k$  in Sec. 4 for a 3D pendulum and for a spacecraft in a circular orbit. Using (17) and (18),  $\delta M_{k+1}$  is given by

$$\delta M_{k+1} = \mathcal{M}_{k+1} \zeta_{k+1} = \mathcal{M}_{k+1} F_k^T \zeta_k + \mathcal{M}_{k+1} \mathcal{B}_k \delta\Pi_k. \quad (20)$$

Now, we take a variation of the augmented performance index (13) using the constrained variations (17), (18), and (20). Using (17), the variation of the performance index is given by

$$\begin{aligned} \delta \mathcal{J}_a &= \sum_{k=0}^{N-1} h \delta u_{k+1}^T u_{k+1} + \lambda_k^{1,T} \{\xi_k + F_k^T \zeta_k - \zeta_{k+1}\} \\ &\quad + \lambda_k^{2,T} \{-\delta\Pi_{k+1} + \delta F_k^T \Pi_k + F_k^T \delta\Pi_k + h\delta M_{k+1} + hB\delta u_{k+1}\}. \quad (21) \end{aligned}$$

Substituting (20) into (21) and rearranging, we obtain

$$\begin{aligned} \delta \mathcal{J}_a &= \sum_{k=0}^{N-1} h \delta u_{k+1}^T \{u_{k+1} + B^T \lambda_k^2\} - \zeta_{k+1}^T \lambda_k^1 + \zeta_k^T \{F_k \lambda_k^1 + hF_k \mathcal{M}_{k+1}^T \lambda_k^2\} \\ &\quad - \delta\Pi_{k+1}^T \lambda_k^2 + \delta\Pi_k^T \{F_k \lambda_k^2 + h\mathcal{B}_k^T \mathcal{M}_{k+1}^T \lambda_k^2\} + \xi_k^T \{-S(F_k^T \Pi_k) \lambda_k^2 + \lambda_k^1\}. \quad (22) \end{aligned}$$

Substituting (18) into (22) and using the fact that the variations  $\zeta_k, \delta\Pi_k$  vanish at  $k = 0, N$ , we obtain

$$\begin{aligned} \delta\mathcal{J}_a = & \sum_{k=1}^{N-1} h\delta u_k^T \{u_k + B^T \lambda_{k-1}^2\} + \zeta_k^T \{-\lambda_{k-1}^1 + F_k \lambda_k^1 + hF_k \mathcal{M}_{k+1}^T \lambda_k^2\} \\ & + \delta\Pi_k^T \left\{ -\lambda_{k-1}^2 + (F_k - \mathcal{B}_k^T S(F_k^T \Pi_k) + h\mathcal{B}_k^T \mathcal{M}_{k+1}^T) \lambda_k^2 + \mathcal{B}_k^T \lambda_k^1 \right\}. \end{aligned} \quad (23)$$

Since  $\delta\mathcal{J}_a = 0$  for all variations of  $\delta u_k, \zeta_k, \delta\Pi_k$  which are independent, the expression in the braces are zero. Thus we obtain necessary conditions for optimality as follows:

$$\Pi_{k+1} = F_k^T \Pi_k + h(M_{k+1} + B u_{k+1}), \quad (24)$$

$$hS(\Pi_k) = F_k J_d - J_d F_k^T, \quad (25)$$

$$R_{k+1} = R_k F_k, \quad (26)$$

$$u_{k+1} = -B^T \lambda_k^2, \quad (27)$$

$$\begin{bmatrix} \lambda_k^1 \\ \lambda_k^2 \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{k+1}^T & \mathcal{C}_{k+1}^T \\ \mathcal{B}_{k+1}^T & \mathcal{D}_{k+1}^T \end{bmatrix} \begin{bmatrix} \lambda_{k+1}^1 \\ \lambda_{k+1}^2 \end{bmatrix}, \quad (28)$$

where

$$\mathcal{A}_k = F_k^T, \quad (29)$$

$$\mathcal{B}_k = hF_k^T \{ \text{tr}[F_k J_d] I_{3 \times 3} - F_k J_d \}^{-1}, \quad (30)$$

$$\mathcal{C}_k = h\mathcal{M}_{k+1} F_k^T, \quad (31)$$

$$\mathcal{D}_k = F_k^T + S(F_k^T \Pi_k) \mathcal{B}_k + h\mathcal{M}_{k+1} \mathcal{B}_k. \quad (32)$$

In the above equations, the only implicit part is (25). For a given initial condition  $(R_0, \Pi_0, \lambda_0^1, \lambda_0^2)$ , we can find  $F_0$  by solving (25). Then,  $R_1$  is obtained from (26). Since  $u_1 = -B^T \lambda_0^2$  by (27), and  $M_1$  is a function of  $R_1, \Pi_1$  can be obtained using (24). We solve (25) to obtain  $F_1$  using  $\Pi_1$ . Finally,  $\lambda_1^1, \lambda_1^2$  are obtained from (28), since  $\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1$  are functions of  $R_1, \Pi_1, F_1$ . This yields a map  $(R_0, \Pi_0, \lambda_0^1, \lambda_0^2) \mapsto (R_1, \Pi_1, \lambda_1^1, \lambda_1^2)$ , and this process can be repeated.

**3.3. Two-point boundary-value problem.** The necessary conditions for optimality are given as a 12 dimensional two-point boundary-value problem on  $T^*\text{SO}(3)$  and its dual space. This problem is to find

1. attitude and angular momentum  $R_k$  and  $\Pi_k$ ,
2. multiplier variables  $\lambda_k^1$  and  $\lambda_k^2$ ,
3. control inputs  $u_k$

for  $k = \{0, 1, \dots, N\}$ , to satisfy simultaneously,

1. equations of motion (24), (25), (26),
2. multiplier equations (28),

3. optimality condition (27),
4. boundary conditions  $R_0, \Pi_0, R_N, \Pi_N$ .

An iterative numerical method for the two-point boundary-value problem is presented. A nominal solution that satisfies some of the above conditions is chosen, and this nominal solution is updated by successive linearization so that the remaining conditions are also satisfied as the process converges.

We use a neighboring extremal method [2]. A nominal solution satisfies all of the necessary conditions except the boundary conditions. The neighboring extremal method is characterized as an iterative algorithm for improving estimates of the unspecified multiplier initial conditions so as to satisfy the specified terminal boundary conditions in the limit. This is sometimes referred to as a shooting method. The optimality condition (27) is substituted into the equations of motion and the multiplier equations. The sensitivities of the specified terminal boundary conditions with respect to the unspecified initial multiplier conditions can be calculated by direct numerical differentiation, or they can be obtained by a linear analysis. The main advantage of the neighboring extremal method is that the number of iteration variables is small. It is equal to the dimension of the equations of motion. The difficulty is that the extremal solutions are sensitive to small changes in the unspecified initial multiplier values. Therefore, it is important to compute the sensitivities accurately.

We use linear analysis to compute the sensitivities. The sensitivity model is defined at the Lie algebra level as presented in [11]. It is natural to define the sensitivity model in the Lie algebra, since the Lie algebra is a linear vector space. The sensitivity derivatives in the Lie algebra are related to the original Lie group by the exponential map.

Using the variation models defined in (14), (15), the linearized equations of motion for the attitude dynamics can be written as

$$\begin{bmatrix} \zeta_{k+1} \\ \delta \Pi_{k+1} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{bmatrix} \begin{bmatrix} \zeta_k \\ \delta \Pi_k \end{bmatrix} - \begin{bmatrix} 0_{3 \times 3} \\ hBB^T \end{bmatrix} \delta \lambda_k^2. \quad (33)$$

Note that the homogeneous part of (33) is equivalent to equations that are the dual of (28). The variation of the equations of motion is equivalent to the dual of the multiplier equations, so the variation of the multiplier equations is equivalent to the second variation of the attitude dynamics equations. Together with (33), a tedious but straightforward application of (14), (15) to the multiplier equation gives the following expression for the sensitivity derivatives of the attitude, angular momentum, and the multipliers

$$\begin{bmatrix} x_{k+1} \\ \delta \lambda_{k+1} \end{bmatrix} = A_k \begin{bmatrix} x_k \\ \delta \lambda_k \end{bmatrix}, \quad (34)$$

where  $x_k = [\zeta_k^T, \delta\Pi_k^T]^T$ ,  $\delta\lambda_k = [\delta\lambda_k^{1,T}, \delta\lambda_k^{2,T}]^T \in \mathbb{R}^6$ , and  $A_k \in \mathbb{R}^{12 \times 12}$  can be suitably defined in terms of  $R_k, \Pi_k$ . The solution of (34) is given by

$$\begin{bmatrix} x_N \\ \delta\lambda_N \end{bmatrix} = \left( \prod_{k=0}^{N-1} A_k \right) \begin{bmatrix} x_0 \\ \delta\lambda_0 \end{bmatrix} \triangleq \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ \delta\lambda_0 \end{bmatrix}. \tag{35}$$

For the given two-point boundary-value problem  $x_0 = 0$  since the initial attitude and the initial angular momentum are given, and  $\lambda_N$  is free. Then we obtain

$$x_N = \Phi_{12}\delta\lambda_0. \tag{36}$$

The unspecified initial multipliers are  $\lambda_0$ , and the specified terminal boundary conditions are the terminal attitude  $R_N^d$  and the terminal angular momentum  $\Pi_N^d$ . Thus,  $\Phi_{12}$  represents the sensitivity of the specified terminal boundary conditions with respect to the unspecified initial multipliers. Using this sensitivity, an initial guess of the unspecified initial conditions is iterated to satisfy the specified terminal conditions in the limit.

TABLE 1. Newton–Armijo iteration procedures

- 1: Guess an initial multiplier  $\lambda_0$ .
- 2: Find  $\Pi_k, R_k, \lambda_k^1, \lambda_k^2$  for  $k = 1, 2, \dots, N$  using the initial conditions and (24)–(28).
- 3: Compute the error in satisfaction of the terminal boundary condition;  $\zeta_N = S^{-1}(\logm(R_N^T R_N^d))$ ,  $\delta\Pi_N = \Pi_N^d - \Pi_N$ .  
Error =  $\|[\zeta_N; \delta\Pi_N]\|$ .
- 4: Set  $\text{Error}^t = \text{Error}$ ,  $i = 1$ .
- 5: **while** Error  $> \epsilon_S$ .
- 6:     Find a line search direction;  $D = \Phi_{12}^{-1}$ .
- 7:     Set  $c = 1$ .
- 8:     **while**  $\text{Error}^t > (1 - 2\alpha c)\text{Error}$
- 9:         Choose a trial initial condition  $\lambda_0^t = \lambda_0 + cD[\zeta_N; \delta\Pi_N]$ .
- 10:         Find  $\Pi_k, R_k, \lambda_k^1, \lambda_k^2$  for  $k = 1, 2, \dots, N$  using the trial initial conditions and (24)–(28).
- 11:         Compute the error in satisfaction of the terminal boundary condition  
            $\zeta_N^t = S^{-1}(\logm(R_N^T R_N^d))$ ,  $\delta\Pi_N^t = \Pi_N^d - \Pi_N$ .  
           Error<sup>t</sup> =  $\|[\zeta_N^t; \delta\Pi_N^t]\|$ .
- 12:         Set  $c = c/2$ ,  $i = i + 1$ .
- 13:     **end while**
- 14:     Set  $\lambda_0 = \lambda_0^t$ , Error = Error<sup>t</sup>. (accept the trial)
- 15: **end while**

Any type of Newton iteration can be applied to this problem using the sensitivity derivative as a gradient. In this paper, we use a line search with backtracking algorithm, referred to as Newton-Armijo iteration in [9]. The procedure is summarized in Table 1, where  $i$  is the iteration index, and  $\epsilon_S, \alpha \in \mathbb{R}$  are a stopping criterion and a scaling factor, respectively. The outer loop finds a search direction by computing the sensitivity derivatives, and the inner loop performs a line search along the obtained direction; the error in satisfaction of the terminal boundary condition is determined on each iteration.

#### 4. NUMERICAL COMPUTATIONS

Numerical results are given for two optimal attitude control problems; optimal attitude control of an underactuated 3D pendulum and optimal attitude control of a fully actuated spacecraft on a circular orbit.

**4.1. 3D Pendulum.** A 3D pendulum is a rigid body supported at a frictionless pivot acting under the influence of uniform gravity [22]. The gravity potential, acting in the vertical or  $e_3$  direction, is given by

$$U(R) = -mge_3^T R\rho, \quad (37)$$

where  $m \in \mathbb{R}$  is the mass of the pendulum,  $g \in \mathbb{R}$  is the gravitational acceleration, and  $\rho \in \mathbb{R}^3$  represents a vector from the pivot point to the mass center of the pendulum in the body fixed frame. The pendulum model is shown in Fig. 1(a) with the pivot located at the origin, and we assume that the pendulum is axially symmetric. The gravity moment and its variations are given by

$$\begin{aligned} M &= mg\rho \times R^T e_3, \\ \delta M &= \mathcal{M}\zeta = mgS(\rho)S(R^T e_3)\zeta. \end{aligned}$$

There are two equilibrium manifolds; a hanging equilibrium manifold when  $R\rho = \|\rho\| e_3$ , an inverted equilibrium manifold when  $R\rho = -\|\rho\| e_3$ .

The properties of the axially symmetric pendulum are given by  $J = \text{diag}[0.156, 0.156, 0.3] \text{ kg m}^2$ ,  $m = 1 \text{ kg}$ , and  $\rho = [0, 0, \frac{3}{4}] \text{ m}$ . We assume that the component of the control input along the axis of symmetry is zero; this corresponds to an underactuated 3D pendulum. The corresponding input matrix is given by

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Two types of boundary conditions are considered. The first maneuver is to transfer the 3D pendulum from a hanging equilibrium to an inverted equilibrium. The second maneuver is a 180 degree rotation about the uncontrolled axis of symmetry starting in a hanging equilibrium. The terminal

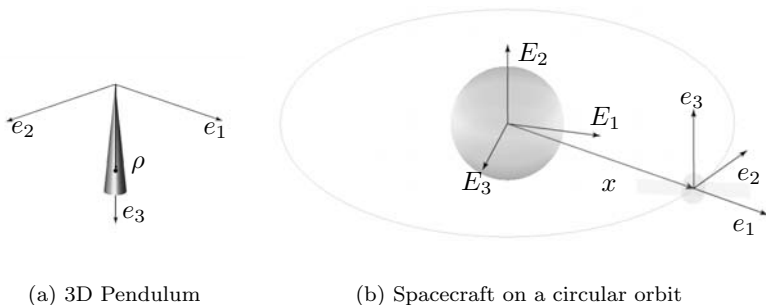


Fig. 1. Rigid body models

TABLE 2. Optimized performance index and violation of terminal boundary condition

Case	Model	$\mathcal{J}$	$\ \log m(R_N^{d,T} R_N)\ $	$\ \Pi_N^d - \Pi_N\ $
(i)	3D Pendulum	1.52	$1.77 \times 10^{-14}$	$7.08 \times 10^{-15}$
(ii)		40.22	$2.22 \times 10^{-16}$	$2.55 \times 10^{-14}$
(iii)	Spacecraft	23.35	$2.90 \times 10^{-15}$	$5.13 \times 10^{-15}$
(iv)		70.74	$7.31 \times 10^{-15}$	$1.48 \times 10^{-14}$

attitude also lies in the hanging equilibrium manifold. Each maneuver is completed in 1 sec. The time step size is  $h = 0.001$  sec and the number of integration steps is  $N = 1000$ . The corresponding boundary conditions are given by

- (i) Rotation from a hanging equilibrium to an inverted equilibrium.

$$R_0 = I_{3 \times 3}, \quad R_N^d = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\Pi_0 = 0_{3 \times 1}, \quad \Pi_N^d = 0_{3 \times 1}.$$

- (ii) Rotation from one hanging equilibrium to another hanging equilibrium.

$$R_0 = I_{3 \times 3}, \quad R_N^d = \text{diag}[-1, -1, 1],$$

$$\Pi_0 = 0_{3 \times 1}, \quad \Pi_N^d = 0_{3 \times 1}.$$

The optimized performance index and the violation of the terminal boundary condition are given in Table 2. The terminal boundary conditions are satisfied at the level of machine precision for both cases. Figures 2

and 3 show snapshots of the attitude maneuvers, the control input history, and the angular velocity response. (Simple animations which show these optimal attitude maneuvers of the 3D pendulum can be found at <http://www.umich.edu/~tylee>.) The third component of the angular velocity is constant; this is a conservation property of the controlled axially symmetric 3D pendulum.

The optimized attitude maneuver of the first case is an eigen-axis rotation about the fixed axis;  $[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0]$ . In the second case, the rotation about the axis of symmetry is induced from control moments about the first and second body fixed axes. The resulting attitude maneuver is more complicated, and it requires larger control inputs.

Figures 2(d) and 3(d) show the violation of the terminal boundary condition according to the number of iterations in a logarithm scale. The circles denote outer iterations to compute the sensitivity derivatives. The inner iterations correspond to backtracking to decrease the step length along the search direction. For all cases, the initial guesses of the unspecified initial multiplier are arbitrarily chosen such that the initial trial of control inputs is close to zero throughout the maneuver time. The error in satisfaction of the terminal boundary condition of the first case converges quickly to machine precision; only 7 iterations are required. A longer number of iterations is required in the second case, but the error converges exponentially to machine precision after the solution is close to the local minimum at the 55th iteration. These convergence rates are consistent with the quadratic rate of convergence of the Newton iteration, and demonstrate that the sensitivity derivatives are being computed accurately.

**4.2. Spacecraft on a Circular Orbit.** We consider a spacecraft on a circular orbit about a large central body, including gravity gradient effects [25]. The spacecraft model is shown at Fig. 1(b). The attitude of the spacecraft is represented with respect to the local vertical local horizontal (LVLH) axes. The gravity potential is given by

$$U(R) = -\frac{GM}{r_0} - \frac{1}{2}\omega_0^2 (\text{tr}[J] - 3e_3^T R J R^T e_3),$$

where  $G \in \mathbb{R}$  is the gravitational constant,  $M \in \mathbb{R}$  is the mass of the central body,  $r_0 \in \mathbb{R}$  is the orbital radius, and  $\omega_0 = \sqrt{\frac{GM}{r_0^3}} \in \mathbb{R}$  is the orbital angular velocity. The gravity moment and its variations are given by

$$M = 3\omega_0^2 R^T e_3 \times J R^T e_3,$$

$$\delta M = \mathcal{M}\zeta = 3\omega_0^2 \left[ -S(JR^T e_3)S(R^T e_3) + S(R^T e_3)JS(R^T e_3) \right] \zeta.$$

There are 24 distinct relative equilibria for which the principal axes are exactly aligned with the LVLH axes, and the spacecraft angular velocity is identical to the orbital angular velocity of the LVLH coordinate frame.

We assume that the spacecraft is fully actuated. The corresponding input matrix is  $B = I_{3 \times 3}$ . The mass, length and time dimensions are normalized by the mass of the spacecraft, a size scale factor of the spacecraft, and the orbital angular velocity  $\omega_0 \in \mathbb{R}$ , respectively. The mass property of the spacecraft is chosen as  $J = \text{diag}[1, 2.8, 2]$ .

Two boundary conditions are considered. Each maneuver is a large attitude change completed in a quarter of the orbit,  $T_f = \frac{\pi}{2}$ . The time step size is  $h = 0.001$  and the number of integration steps is  $N = 1571$ . The terminal angular momentum is chosen such that the terminal attitude is maintained after the maneuver.

(iii) Rotation maneuver about the LVLH axis  $e_1$ :

$$\begin{aligned} R_0 &= I_{3 \times 3}, & R_N^d &= \text{diag}[1, -1, -1], \\ \Pi_0 &= \omega_0 J R_0^T e_2, & \Pi_N^d &= \omega_0 J R_N^{d,T} e_2. \end{aligned}$$

(iv) Rotation maneuver about the LVLH axes  $e_1$  and  $e_2$ :

$$\begin{aligned} R_0 &= \text{diag}[1, -1, -1], & R_N^d &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \\ \Pi_0 &= \omega_0 J R_0^T e_2, & \Pi_N^d &= \omega_0 J R_N^{d,T} e_2. \end{aligned}$$

The optimized performance index and the violation of the constraints are given in Table 2. Figures 4 and 5 show the attitude maneuver of the spacecraft (clockwise direction), the control inputs, the angular velocity response, and the violation of the terminal boundary condition according to the number of iterations.

**4.3. Numerical properties.** One of the distinct features of the Lie group variational integrator is that it preserves the symplectic structure and the Lie group structure of the rigid body dynamics concurrently. The computational properties of the Lie group variational integrator have been compared with a symplectic integrator that does not preserve the Lie group structure, and a Lie group method that does not preserve the symplectic structure [16]. These numerical experiments demonstrate that it is critical for long-time accuracy to preserve both structures as the rigid body dynamics is a Lagrangian/Hamiltonian system evolving on a Lie group.

Therefore, the discrete dynamics are more faithful to the continuous equations of motion, and consequently more accurate solutions to the optimal control problem are obtained. The external control inputs break the Lagrangian and Hamiltonian system structure. But, the computational superiority of the discrete mechanics still holds for controlled systems. It has been shown that the discrete dynamics is more reliable even for controlled system as it computes the energy dissipation rate of controlled systems more accurately [21]. For example, this feature is extremely important in



computing accurate optimal trajectories for long term spacecraft attitude maneuvers using low energy control inputs.

The discrete dynamics does not only provide an accurate optimal control input, but also enables us to find it efficiently. The neighboring extremal method or the shooting method are numerically efficient in the sense that the number of optimization parameters is minimized. But, this approach may be prone to numerical ill-conditioning [1]. A small change in the initial multiplier can cause highly nonlinear behavior of the terminal attitude and angular momentum. It is difficult to compute the Jacobian matrix for Newton iterations accurately, and consequently, the numerical error may not converge to zero within machine precision.

However, the numerical examples presented in this section show excellent numerical convergence properties. They exhibit a quadratic rate of convergence. This is because the proposed computational algorithms on  $SO(3)$  are geometrically exact and numerically accurate. The attitude dynamics of a rigid body arises from Hamiltonian mechanics, which have neutral stability. The adjoint system is also neutrally stable. The proposed Lie group variational integrators and the discrete multiplier equations, obtained from variations expressed in the Lie algebra, can preserve the neutrally stability property. Therefore the sensitivity derivatives are computed accurately.

## 5. CONCLUSIONS

A discrete optimal control problem for the attitude dynamics of a rigid body in the presence of an attitude dependent potential is studied. The performance index is the  $l_2$  norm of external control inputs and boundary conditions on the attitude and the angular momentum are prescribed. The attitude is represented by a rotation matrix in the Lie group,  $SO(3)$ . This paper proposes three levels of geometrically exact computations on  $SO(3)$  to solve the optimal control problem; Lie group variational integrator, discrete-time necessary conditions for optimality, and discrete-time sensitivity derivatives.

The Lie group variational integrator obtained from a discrete variational principle preserves the geometric features of the attitude dynamics of the rigid body. It exhibits symplectic and momentum preservation properties, and good energy behavior characteristic of variational integrators. Since the rotation matrices are updated by a group operation, the Lie group structure is also preserved.

The necessary conditions of optimality are derived by a variational principle. The Lie group variational integrators are imposed as constraints, and the variation of the rotation matrices are expressed in terms of Lie algebra elements. The proposed discrete optimality conditions are the basis for a numerically efficient computational algorithms for the optimal attitude control problem, since the implicit part of the optimality conditions occurs

in a single equation of one variable. This implicit condition can be solved easily by Newton iteration. Other algorithms require iteration on the entire discrete time trajectory simultaneously.

The necessary conditions are expressed as a two-point boundary-value problem on  $T^*SO(3)$  and its dual space. The sensitivity derivatives are developed in the Lie algebra, and the two-point boundary-value problem is solved using a neighboring extremal method. The neighboring extremal method is efficient for this class of optimal control problems because the resulting problem of satisfying the terminal boundary conditions has a small number of variables. The main disadvantage is that a small change in the initial multipliers can produce a very large change in the terminal condition. This can result in numerical ill-conditioning. The nonlinearity also makes it hard to construct an accurate estimate of the Jacobian matrix that is needed for a Newton iteration. We address this issue in the paper by accurately computing the sensitivity derivatives in the Lie algebra of  $SO(3)$ . This yields an efficient method for solving the two-point boundary problem, and the error in the terminal boundary conditions converges exponentially to machine precision.

Numerical results for an optimal attitude control problem involving an underactuated axially symmetric 3D pendulum and for an optimal attitude control problem involving a fully actuated spacecraft on a circular orbit are given. The boundary conditions are chosen such that the resulting maneuvers are large angle attitude maneuver. It is shown that the proposed numerical computations on  $SO(3)$  are geometrically exact and highly efficient.

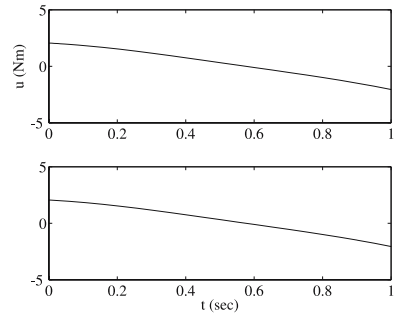
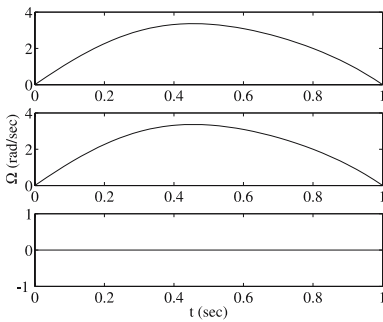
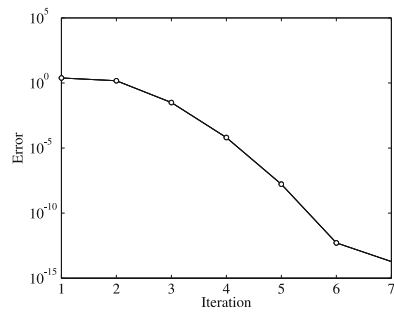
In this paper, the performance index is chosen as a smooth function of the control input. But, the proposed discrete-time optimal control approach using the Lie group variational integrator can be generalized in several ways, and actually, it has been applied to various optimal control problems for rigid bodies. For example, an optimal control problem on  $SE(3)$  is studied in [14], and this is extended to incorporate a combinatorial optimal control in [13]. Constraints on the control input are considered in [12, 17]. The future work includes an optimal control problem for a multibody system.

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(a) Attitude maneuver

(b) Control input  $u$ (c) Angular velocity  $\Omega$ 

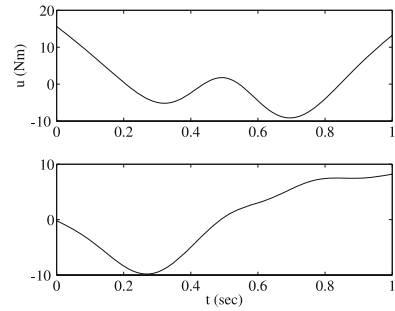
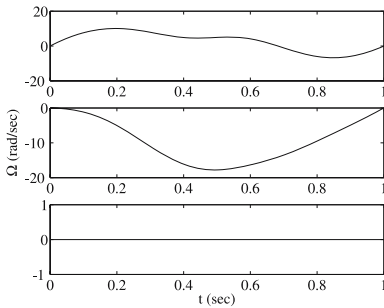
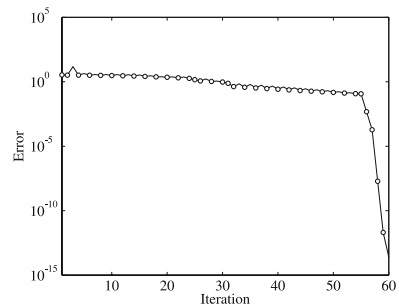
(d) Convergence rate

Fig. 2. Case (i): 3D pendulum rotation from a hanging equilibrium to an inverted equilibrium

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(a) Attitude maneuver

(b) Control input  $u$ (c) Angular velocity  $\Omega$ 

(d) Convergence rate

Fig. 3. Case (ii): 3D pendulum rotation from one hanging equilibrium to another hanging equilibrium

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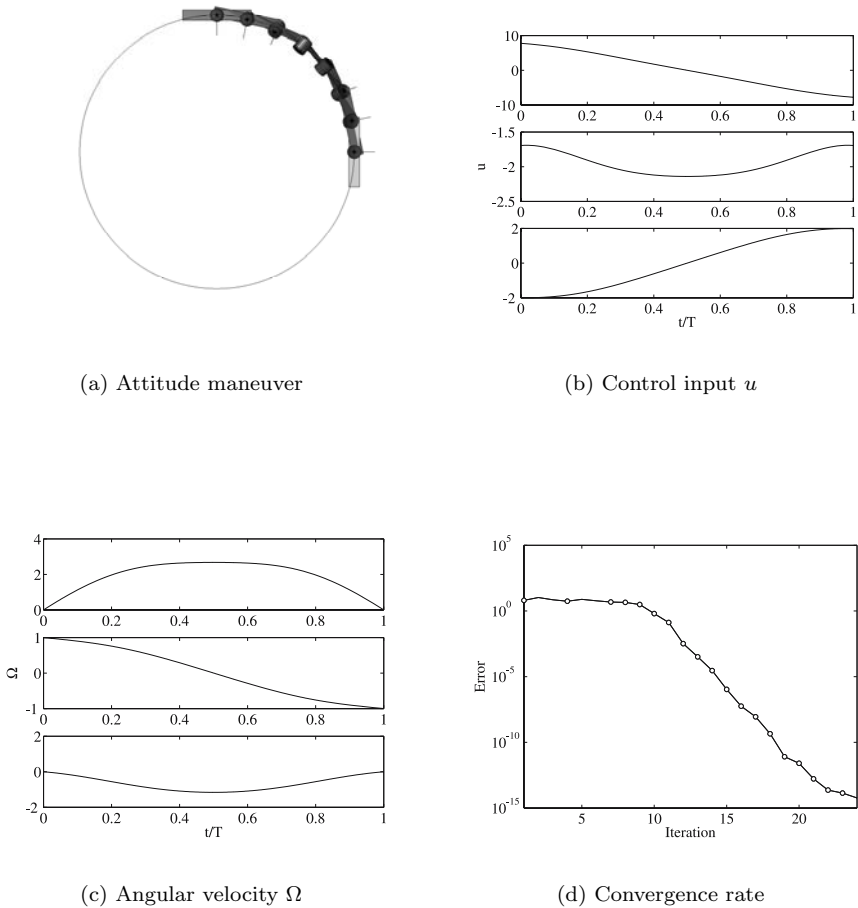


Fig. 4. Case (iii): Spacecraft rotation maneuver about the LVLH tangential axis

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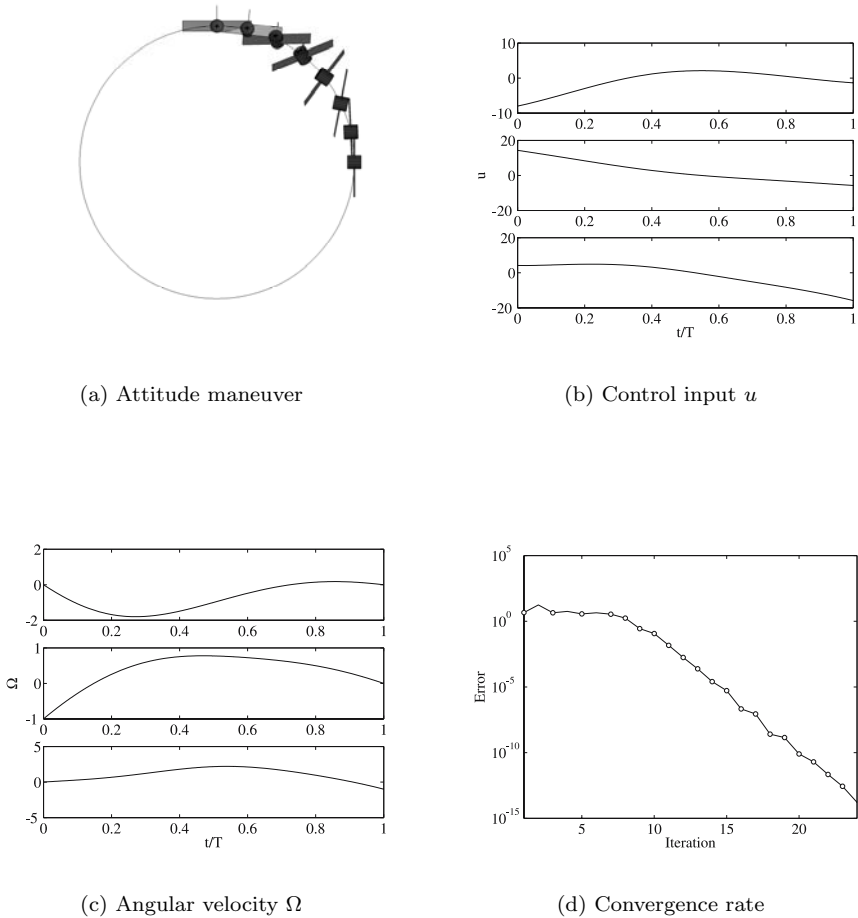


Fig. 5. Case (iv): Spacecraft rotation maneuver about the LVLH tangential and normal axes

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