VARIATIONAL STRUCTURES IN COCHAIN PROJECTION BASED VARIATIONAL DISCRETIZATIONS OF LAGRANGIAN PDES

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Abstract. Compatible discretizations, such as finite element exterior calculus, provide a discretization framework that respect the cohomological structure of the de Rham complex, which can be used to systematically construct stable mixed finite element methods. Multisymplectic variational integrators are a class of geometric numerical integrators for Lagrangian and Hamiltonian field theories, and they yield methods that preserve the multisymplectic structure and momentum-conservation properties of the continuous system. In this paper, we investigate the synthesis of these two approaches, by constructing discretization of the variational principle for Lagrangian field theories utilizing structure-preserving finite element projections. In our investigation, compatible discretization by cochain projections plays a pivotal role in the preservation of the variational structure at the discrete level, allowing the discrete variational structure to essentially be the restriction of the continuum variational structure to a finite-dimensional subspace. The preservation of the variational structure at the discrete level will allow us to construct a discrete Cartan form, which encodes the variational structure of the discrete theory, and subsequently, we utilize the discrete Cartan form to naturally state discrete analogues of Noether's theorem and multisymplecticity, which generalize those introduced in the discrete Lagrangian variational framework by Marsden et al. [29]. We will study both covariant spacetime discretization and canonical spatial semi-discretization, and subsequently relate the two in the case of spacetime tensor product finite element spaces.

Contents

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1. Introduction

The problem of structure-preservation in numerical discretizations of partial differential equations has primarily been studied in two disjoint stages, the first involving the semi-discretization of the spatial degrees of freedom, and the second having to do with the time-integration of the resulting coupled system of ordinary differential equations. Implicit in such an approach is the use of tensor product meshes in spacetime. In the context of spatial semi-discretization, the notion of structure-preservation is focused on compatible discretizations (see Arnold [5], and references therein), that preserve in some manner the functional and geometric relationships between the different function spaces that arise in the partial differential equation, and in the context of time-integration, geometric numerical integrators (see Hairer et al. [20], and references therein) aim to preserve geometric invariants like the symplectic or Poisson structure, energy, momentum, and the nonlinear manifold structure of the configuration spaces, like its Lie group, homogeneous space, or Riemannian structure.

Lagrangian partial differential equations are an important class of partial differential equations that exhibit geometric structure that can benefit from numerical discretizations that preserve such structure. This can either be viewed as an infinite-dimensional Lagrangian system with time as the independent variable, or a finite-dimensional Lagrangian multisymplectic field theory [30] with space and time as independent variables. Lagrangian variational integrators [28; 29] are a popular method for systematically constructing symplectic integrators of arbitrarily high-order, and satisfy a discrete Noether's theorem that relates group-invariance with momentum conservation. A group-invariant (and hence momentum-preserving) variational integrator can be constructed from group-equivariant interpolation spaces [14].

In this paper, we will demonstrate how compatible discretization, multisymplectic variational integrators, and group-equivariant interpolation spaces can be combined to yield a natural geometric structure-preserving discretization framework for Lagrangian field theories.

Multisymplectic Formulation of Classical Field Theories. The variational principle for Lagrangian PDEs involve a multisymplectic formulation [29; 30]. The base space X consists of independent variables, denoted by $(x^0, \dots, x^n) \equiv (t, x)$, where $x^0 \equiv t$ is time, and $(x^1, \dots, x^n) \equiv x$ are space variables. The dependent field variables, $(y^1, \dots, y^m) \equiv y$, form a fiber over each spacetime basepoint. The independent and field variables form the configuration bundle, $\pi : Y \to X$. The configuration of the system is specified by a section of Y over X, which is a continuous map $\phi : X \to Y$, such that $\pi \circ \phi = 1_X$. This means that for every $(t, x) \in X$, $\phi((t, x))$ is in the fiber over (t, x), which is $\pi^{-1}((t, x))$.

For ODEs, the Lagrangian depends on position and its time derivative, which is an element of the tangent bundle TQ, and the action is obtained by integrating the Lagrangian in time. In the multisymplectic case, the Lagrangian density is dependent on the field variables and the partial derivatives of the field variables with respect to the spacetime variables, and the action integral is obtained by integrating the Lagrangian density over a region of spacetime. The multisymplectic analogue of the tangent bundle is the first jet bundle J^1Y, consisting of the configuration bundle Y, and the first partial derivatives of the field variables with respect to the independent variables. In coordinates, we have $\phi(x^0, \dots, x^n) = (x^0, \dots x^n, y^1, \dots y^m)$, which allows us to denote the partial

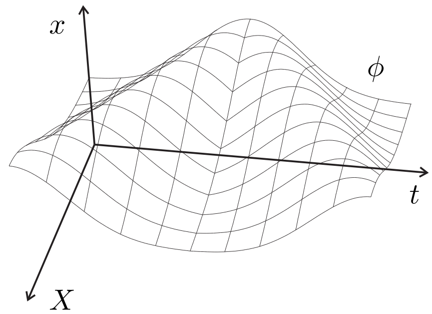


FIGURE 1. A section of the configuration bundle: the horizontal axes represent spacetime, and the vertical axis represent dependent field variables. The section ϕ gives the value of the field variables at every point of spacetime.

derivatives by $v^a_\mu = y^a{}_{,\mu} = \partial y^a/\partial x^\mu$. We can think of J^1Y as a fiber bundle over X. Given a section $\phi : X \to Y$, we obtain its first jet extension, $j^1\phi : X \to J^1Y$, that is given by

$$j^1\phi(x^0, \dots, x^n) = \left(x^0, \dots, x^n, y^1, \dots, y^m, y^1{}_{,0}, \dots, y^m{}_{,n}\right),$$

which is a section of the fiber bundle J^1Y over X. We refer to sections of J^1Y of the form $j^1\phi$, where ϕ is a section of Y, as holonomic. The Lagrangian density is a bundle map $\mathcal{L} : J^1Y \to \Omega^{n+1}(X)$. Given the action functional, $S[\phi] = \int_X \mathcal{L}(j^1\phi)$, Hamilton's principle states that $\delta S = 0$ (subject to compactly supported variations). As we will see, this is the basis of Lagrangian multisymplectic variational integrators [29].

The variational structure of a Lagrangian field theory is given by the Cartan form, which in coordinates has the expression

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial v^a_\mu} dy^a \wedge d^n x_\mu + \left(L - \frac{\partial L}{\partial v^a_\mu} v^a_\mu\right) d^{n+1}x. \tag{1.1}$$

This can be defined intrinsically as the pullback of the canonical $(n+1)$-form on the dual jet bundle by the covariant Legendre transform $\mathbb{F}\mathcal{L} : J^1Y \to J^1Y^*$. Then, the action can be expressed as $S[\phi] = \int_X \mathcal{L}(j^1\phi) = \int_X (j^1\phi)^*\Theta_{\mathcal{L}}$. The variation of the action is then expressed as

$$dS[\phi] \cdot V = -\int_X (j^1\phi)^*(j^1V \lrcorner\, \Omega_{\mathcal{L}}) + \int_{\partial X} (j^1\phi)^*(j^1V \lrcorner\, \Theta_{\mathcal{L}}),$$

where $\Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}}$ defines the multisymplectic form and j^1V denotes the jet prolongation of the vector field V (for details, see Gotay et al. [15]). Hence, the variation of the action is completely specified by the Cartan form; we will show that a finite element discretization of the variational principle gives rise to a discrete form and subsequently we will express variational properties of the discrete system in terms of the discrete Cartan form.

In this paper, we will take the configuration bundle $Y = H\Lambda^k(X)$, the space of square integrable $k-$forms on X with square integrable exterior derivative. In this setting, the appropriate analogue of the jet bundle is $J^1_{H\Lambda^k} := H\Lambda^k \times dH\Lambda^k$, where the jet extension of a field $\phi \in H\Lambda^k$ is $j^1\phi = (x, \phi, d\phi)$ (i.e., we take the Lagrangian theory to depend on the exterior derivative of the field and not depending more generally on all first-order derivatives; for scalar fields, $k = 0$, these are equivalent).

Finite Element Exterior Calculus. The notion of compatible discretization is a research area that has garnered significant interest and activity in the finite element community, motivated by the seminal work of Arnold et al. [6] on finite element exterior calculus that provides a broad generalization of Hiptmair's work on mixed finite elements for electromagnetism [21]. This arises from the fundamental role that the de Rham complex of exterior differential forms plays in mixed formulations of elliptic partial differential equations, and the realization that many of the most successful mixed finite element spaces, such as Raviart–Thomas and Nédélec elements, can be viewed as finite element subspaces of the de Rham complex that satisfy a bounded cochain projection property, so that the set of mixed finite elements form a subcomplex that provides stable approximations of the original problem.

Group-equivariant interpolation. The study of group-equivariant approximation spaces [14] for functions taking values on manifolds is motivated by the applications to geometric structure-preserving discretization of Lagrangian and Hamiltonian PDEs with symmetries. In particular, when the Lagrangian density for a Lagrangian PDE with symmetry is discretized using a Lagrangian multisymplectic variational integrator constructed from an approximation space that is equivariant with respect to the symmetry group, the resulting numerical method automatically preserves the momentum map associated with the symmetry of the PDE. In essence, such variational discretizations exhibit a discrete analogue of Noether's theorem, which connects symmetries of the Lagrangian with momentum conservation laws.

Many intrinsic geometric flows such as the Ricci flow and the Einstein equations involves computing the evolution of a Riemannian or pseudo-Riemannian metric on spacetime. Additionally, these intrinsic geometric flows can often be formulated variationally, so it is natural to consider group-equivariant approximation spaces taking values on Riemannian or pseudo-Riemannian metrics with a view towards constructing variational discretizations that preserve the associated momentum maps.

A now standard approach to constructing an approximation space for functions taking values on a Riemannian manifold that is equivariant with respect to Riemannian isometries is the method of geodesic finite elements introduced independently by Sander [33] and Grohs [18]. Given a Riemannian manifold (M, g), the geodesic finite element $\varphi : \Delta^n \to M$ associated with a set of linear space finite elements $\{v_i : \Delta^n \to \mathbb{R}\}_{i=0}^n$ is given by the Fréchet (or Karcher) mean,

$$\varphi(x) = \arg\min_{p\in M} \sum\nolimits_{i=0}^{n} v_i(x)(\text{dist}(p, m_i))^2,$$

where the optimization problem involved can be solved using optimization algorithms developed for matrix manifolds (see Absil et al. [2], and references therein). The spatial derivatives of the geodesic finite element can be computed in terms of an associated optimization problem. The advantage of the geodesic finite element approach is that it inherits the approximation properties of the underlying linear space finite element, but it can be expensive to compute, since it entails solving an optimization problem on a manifold.

An alternative approach to group-equivariant interpolation for functions taking values on symmetric spaces was introduced in Gawlik and Leok [14], which, in particular, is applicable to the interpolation of Riemannian and pseudo-Riemannian metrics. It uses the generalized polar decomposition [32] to construct a local diffeomorphism between a symmetric space and a Lie triple system, which lifts a scalar-valued interpolant to a symmetric space-valued interpolant.

Lagrangian Variational Integrators. Variational integrators (see [28], and references therein) are a class of geometric structure-preserving numerical integrators that are based on a discretization of Hamilton's principle. They are particularly appropriate for the simulation of Lagrangian and Hamiltonian ODEs and PDEs, as they automatically preserve many geometric invariants, including the symplectic structure, momentum maps associated with symmetries of the system, and exhibit bounded energy errors for exponentially long times.

In the case of Lagrangian ODEs, variational integrators are based on constructing computable approximations $L_d : Q \times Q \to \mathbb{R}$ of the exact discrete Lagrangian,

$$L_d^E(q_0, q_1, h) = \operatorname{ext}_{\substack{q \in C^2([0,h],Q) \\ q(0)=q_0, q(h)=q_1}} \int_0^h L(q(t), \dot{q}(t))dt,$$

which can be viewed as Jacobi's solution of the Hamilton–Jacobi equation. Given a discrete Lagrangian L_d, one introduces the discrete action sum $\mathbb{S}_d = \sum_{k=0}^{n-1} L_d(q_k, q_{k+1})$, and then the discrete Hamilton's principle states that $\delta \mathbb{S}_d = 0$, for fixed boundary conditions q_0 and q_n. This leads to the discrete Euler–Lagrange equations,

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) = 0,$$

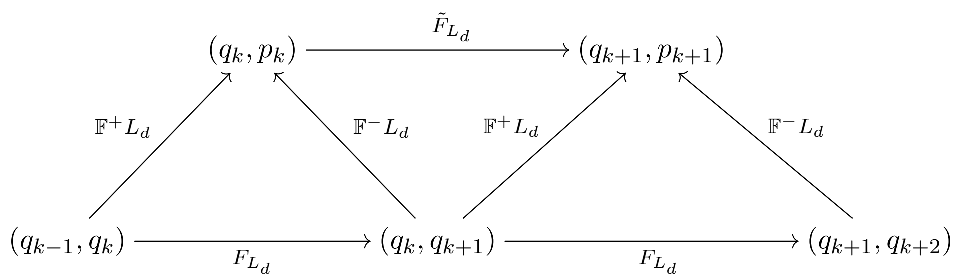
where D_i denotes the partial derivative with respect to the i-th argument. This implicitly defines the discrete Lagrangian map $F_{L_d} : (q_{k-1}, q_k) \mapsto (q_k, q_{k+1})$ for initial conditions (q_{k-1}, q_k) that are sufficiently close to the diagonal of $Q \times Q$. It is also equivalent to the implicit discrete Euler–Lagrange equations,

$$p_k = -D_1 L_d(q_k, q_{k+1}), \qquad p_{k+1} = D_2 L_d(q_k, q_{k+1}),$$

which implicitly defines the discrete Hamiltonian map $\tilde{F}_{L_d} : (q_k, p_k) \mapsto (q_{k+1}, p_{k+1})$, which is automatically symplectic. This clearly follows from the fact that these equations are precisely the characterization of a symplectic map in terms of a Type I generating function. The two equations in the implicit discrete Euler–Lagrange equations can be used to define the discrete Legendre transforms, $\mathbb{F}^{\pm} L_d : Q \times Q \to T^*Q$:

$$\begin{aligned} \mathbb{F}^+ L_d &: (q_0, q_1) \to (q_1, p_1) = (q_1, D_2 L_d(q_0, q_1)), \\ \mathbb{F}^- L_d &: (q_0, q_1) \to (q_0, p_0) = (q_0, -D_1 L_d(q_0, q_1)). \end{aligned}$$

The following commutative diagram illustrates the relationship between the discrete Hamiltonian flow map, discrete Lagrangian flow map, and the discrete Legendre transforms,



If the discrete Lagrangian is invariant under the diagonal action of a Lie group G, i.e., $L_d(q_0, q_1) = L_d(gq_0, gq_1)$, for all $g \in G$, then the discrete Noether's theorem states that there is a discrete momentum map that is automatically preserved by the variational integrator. The bounded energy error of variational integrators can be understood by performing backward error analysis [8; 19], which then shows that the discrete flow map is approximated with exponential accuracy by the exact flow map of the Hamiltonian vector field of a modified Hamiltonian.

Multisymplectic Hamiltonian Variational Integrators. For Hamiltonian PDEs (see, for example, Marsden and Shkoller [27]) the action is a functional on the field and multimomenta values (more precisely, sections of the restricted dual jet bundle),

$$S[\phi, p] = \int [p^\mu \partial_\mu \phi - H(\phi, p)] d^{n+1}x,$$

where the integration is over some $(n+1)$-dimensional region. The variational principle gives the De Donder–Weyl equations $\partial_\mu p^\mu = -\partial H / \partial \phi$, $\partial_\mu \phi = \partial H / \partial p^\mu$. Defining $z = (\phi, p^0, \ldots, p^n)$ and K^μ

as the $(n+2)\times(n+2)$ skew-symmetric matrix with value -1 in the $(0,\mu+1)$, 1 in the $(\mu+1,0)$ entry, and 0 in every other entry (with indexing from 0 to $n+1$), the De Donder–Weyl equations can be written in the form

$$K^0\partial_0 z + \cdots + K^n\partial_n z = \nabla_z H.$$

This formulation of Hamiltonian PDEs was studied in Bridges [10]; in particular, it was shown that such a system admits a multisymplectic conservation law of the form $\partial_\mu\omega^\mu(V,W)=0$, where the ω^μ are two-forms corresponding to K^μ and the conservation law holds when evaluated on first variations V,W. For discretizing such equations, multisymplectic integrators have been developed which admit a discrete analogue of this multisymplectic conservation law (see, for example, Bridges and Reich [12]). Such multisymplectic integrators have traditionally not been approached from a variational perspective.

However, in Tran and Leok [34], we developed a systematic method for constructing variational integrators for multisymplectic Hamiltonian PDEs which automatically admit a discrete multisymplectic conservation law and a discrete Noether's theorem by virtue of the discrete variational principle. The construction is based on a discrete approximation of the boundary Hamiltonian that was introduced in Vankerschaver et al. [35],

$$H_{\partial U}(\varphi_A,\pi_B) = \text{ext}\Big[\int_B p^\mu\phi d^n x_\mu - \int_U (p^\mu\partial_\mu\phi - H(\phi,p))d^{n+1}x\Big],$$

where $\partial U = A\sqcup B$, boundary conditions are placed on the field value ϕ on A and normal momenta value on B, and one extremizes over the sections (ϕ,p) over U satisfying the specified boundary conditions. The boundary Hamiltonian is a generating functional in the sense that the Type II variational principle generates the normal momenta value along A and the field value along B,

$$\frac{\delta H_{\partial U}}{\delta\varphi_A} = -p^n|_A, \quad \frac{\delta H_{\partial U}}{\delta\pi_B} = \phi|_B.$$

A variational integrator is then constructed by first approximating the boundary Hamiltonian using a finite-dimensional function space and quadrature, and subsequently enforcing the Type II variational principle. For example, with particular choices of function spaces and quadrature, Tran and Leok [34] recover the class of multisymplectic partitioned Runge–Kutta methods.

In this paper, we take a different approach in several regards. First, we focus on Lagrangian field theories as opposed to Hamiltonian field theories. For Hamiltonian field theories, the momenta are related to the field and its derivative by the Legendre transform; this falls out from the variational principle so one does not need to enforce it beforehand. Thus, in this sense, the momenta and field values can be considered as independent before enforcing the variational principle. On the other hand, for Lagrangian field theories, the Lagrangian depends on both the field value and its first derivative, so one cannot naïvely treat the two as independent; that is, the Lagrangian depends on holonomic sections of the jet bundle. As we will see, this will mean that we need to pay particular attention to the holonomic condition when discretizing via a finite element projection. Furthermore, as opposed to constructing variational integrators from a generating functional (the analogue in the Lagrangian framework would be the boundary Lagrangian, see Vankerschaver et al. [35]), in this paper, we instead investigate directly discretizing the variational principle $\delta S = 0$ utilizing projections into finite-dimensional subspaces. Finally, for simplicity, we do not utilize any quadrature approximations of the various integrals which we encounter; for strong nonlinearities in the Lagrangian, one generally has to utilize quadrature to construct an efficient discretization. However, the theory that we outline is also applicable to the case of applying a quadrature rule, given that one applies the quadrature rule to the action before enforcing the variational principle, so that the resulting discretization is still variational; we will elaborate on this in Remark 2.2. For this reason, we will assume exact integration in order to keep the exposition simple.

Main Contributions. This paper studies the variational finite element discretization of Lagrangian field theories from two perspectives; we begin by investigating directly discretizing the full variational principle over the full spacetime domain, which we refer to as the "covariant" approach, and subsequently study semi-discretization of the instantaneous variational principle on a globally hyperbolic spacetime, which we refer to as the "canonical" approach. This paper can be considered a discrete analogue to the program initiated in Gotay et al. [15, 16], which lays the foundation for relating the covariant and canonical formulations of Lagrangian field theories through their (multi)symplectic structures and momentum maps. One of the goals of understanding the relation between these two different formulations is to systematically relate the covariant gauge symmetries of a gauge field theory to its initial value constraints. This is seen, for example, in general relativity, where the diffeomorphism gauge invariance gives rise to the Einstein constraint equations over the initial data hypersurface (see, for example, Gourgoulhon [17]). When one semi-discretizes such gauge field theories, the discrete initial data must satisfy an associated discrete constraint. We aim to make sense of the discrete geometric structures in the covariant and canonical discretization approaches as a foundation for understanding the discretization of gauge field theories.

In Section 2, we begin by formulating a discrete variational principle in the covariant approach, utilizing the finite element construction to appropriately restrict the variational principle. We show that a cochain projection from the underlying de Rham complex onto the finite element spaces yields a natural discrete variational principle that is compatible with the holonomic jet structure of a Lagrangian field theory. In Section 2.1, we then show that discretizing by cochain projections leads to a naturality relation between the continuous variational problem and the discrete variational problem; this naturality then implies that discretization and the variational principle commute and also, that discretizing at the level of the configuration bundle or at the level of the jet bundle are equivalent. Subsequently, by decomposing the finite element spaces into boundary and interior components, we define a discrete Cartan form in analogy with the continuum Cartan form which will, in a sense, encode the discrete variational structure. With particular choices of finite element spaces, this discrete Cartan form recovers the notion of the discrete Cartan form introduced by Marsden et al. [29]; however, we note that our notion of a discrete Cartan form is more general and furthermore, since our discrete variational problem is naturally related to the continuum variational problem, we are able to explicitly discuss in what sense the discrete Cartan form converges to the continuum Cartan form. Using this discrete Cartan form, in Sections 2.2 and 2.3, we state and prove discrete analogues of the multisymplectic form formula and Noether's theorem. Finally, in Section 2.4, we reinterpret and concisely summarize the preceding sections by interpreting the discrete variational structures as elements of a discrete variational complex.

In Section 3, we study the semi-discretization of the canonical formulation of a Lagrangian field theory on a globally hyperbolic spacetime. In Section 3.1, we discretize the instantaneous variational principle utilizing cochain projections onto finite element spaces over a Cauchy surface, which gives rise to a semi-discrete Euler–Lagrange equation. In Section 3.2, we relate this semi-discrete Euler–Lagrange equation to a Hamiltonian flow on a symplectic semi-discrete phase space. We will discuss in what sense the symplectic structure on the semi-discrete phase space arises from a symplectic structure on the continuum phase space. Subsequently, we will investigate the energy-momentum map structure associated to the semi-discrete phase space in Section 3.3, and discuss how, under appropriate equivariance conditions on the projection, the energy-momentum map structure on the semi-discrete phase space arises as the pullback of the energy-momentum map structure on the continuum phase space. This lays a foundation for understanding initial value constraints when discretizing field theories with gauge symmetries. Finally, in Section 3.4, we relate the covariant and canonical discretization approaches in the case of tensor product finite element spaces.

The underlying theme of this paper is that, when one discretizes the variational principle utilizing compatible discretization techniques, the associated (covariant or canonical) discretization inherits discrete variational structures which can be viewed as pullbacks or projections of the associated continuum variational structures. These discrete variational structures allow one to investigate structure-preservation under discretization of important physical properties, such as momentum conservation, symplecticity, and (gauge) symmetries.

2. Covariant Discretization of Lagrangian Field Theories

In this section, we discretize the covariant Euler–Lagrange equations which arise from the variational principle $\delta S[\phi] = 0$ for the action $S : \phi \mapsto \int_X \mathcal{L}(j^1\phi)$ where $\phi \in H\Lambda^k$ is a section of the configuration bundle and $j^1\phi = (x, \phi, d\phi)$. To utilize the finite element method, we take our base space X to be a bounded $(n+1)-$dimensional polyhedral domain with boundary ∂X, equipped with a finite element triangulation $\mathcal{T}_h$. We will assume X has a Riemannian or Lorentzian metric. For this discretization, we perform the variation over a finite element space, and subsequently study how the multisymplectic and covariant momentum map structures are affected by discretization. In particular, we show how these structures are preserved for particular choices of finite element spaces, namely spaces whose projections are cochain maps or group-equivariant interpolation spaces.

First, we derive the Euler–Lagrange equations in the Hilbert space setting, where we take the first jet bundle to be $J^1_{H\Lambda^k} = H\Lambda^k \times dH\Lambda^k$. Fixing the trace of ϕ on ∂X, the variational principle is to find $\phi \in H\Lambda^k$ such that $\delta S[\phi] \cdot v = 0$ for all $v \in \mathring{H}\Lambda^k$. This yields

$$
\begin{aligned}
0 = \delta S[\phi] \cdot v &= \int_X \Big(\delta_2 \mathcal{L}(j^1\phi) \cdot v + \delta_3 \mathcal{L}(j^1\phi) \cdot dv \Big) \\
&= (\partial_2 \mathcal{L}(j^1\phi), v)_{L^2 H\Lambda^k} + (\partial_3 \mathcal{L}(j^1\phi), dv)_{L^2 H\Lambda^{k+1}} \\
&= (\partial_2 \mathcal{L}(j^1\phi), v)_{L^2 H\Lambda^k} + (d^* \partial_3 \mathcal{L}(j^1\phi), dv)_{L^2 H\Lambda^k}
\end{aligned}
$$

where $j^1\phi = (x, \phi, d\phi)$, δ_i denotes the variation with respect to the i^{th} argument, and in the second line we apply the Riesz representation theorem to express $\delta_i \mathcal{L}(j^1\phi)$ as an element $\partial_i \mathcal{L}(j^1\phi)$ of $H\Lambda^k$ (noting that for a general class of Lagrangians $\mathcal{L}(j^1\phi)$, e.g. polynomial in ϕ and $d\phi$, $\delta_i \mathcal{L}(j^1\phi)$ is a bounded linear functional, since the domain is compact). Note that this can equivalently be stated as $dS[\phi] \cdot V = 0$ for all $V \in \mathfrak{X}(\mathring{H}\Lambda^k)$.

There is a slight subtlety in the Lagrangian formulation of the variation principle, in that the variation with respect to the derivative of the field is regarded as independent from the variation of the field. To be more precise, one should regard the third argument of $\mathcal{L}(x, \phi, \psi)$, $\psi \in dH\Lambda^k$, as independent of ϕ and subsequently enforce that the argument is holonomic utilizing the variational principle, i.e., that $(x, \phi, \psi) = j^1\phi = (x, \phi, d\phi)$. To do this, we use the Hamilton–Pontryagin principle, introducing a Lagrange multiplier to enforce the holonomic condition. The action reads

$$
S[\phi, \psi, p] = \int_X \mathcal{L}(x, \phi, \psi) - (p, d\phi - \psi)_{L^2 \Lambda^{k+1}}
$$

where $\phi \in H\Lambda^k, \psi \in dH\Lambda^k, p \in L^2\Lambda^{k+1}$. Since $\psi \in dH\Lambda^k$, we can express it as $\psi \equiv d\chi$. We enforce that S is stationary with respect to variations $\tilde{\phi} \in \mathring{H}\Lambda^k$, $d\tilde{\chi} \in d\mathring{H}\Lambda^k, \tilde{p} \in L^2\Lambda^{k+1}$ (note there is no boundary condition assumed on the variation associated with p, since this is an auxilliary variable). This yields

$$
\begin{aligned}
0 &= \delta_1 S[\phi, d\chi, p] \cdot \tilde{\phi} = (\partial_2 \mathcal{L}(x, \phi, d\chi), \tilde{\phi})_{L^2\Lambda^k} - (p, d\tilde{\phi})_{L^2\Lambda^{k+1}}, \\
0 &= \delta_2 S[\phi, d\chi, p] \cdot d\tilde{\chi} = (\partial_3 \mathcal{L}(x, \phi, d\chi), d\tilde{\chi})_{L^2\Lambda^{k+1}} + (p, d\tilde{\chi})_{L^2\Lambda^{k+1}}, \\
0 &= \delta_3 S[\phi, d\chi, p] \cdot \tilde{p} = (\tilde{p}, d\phi - d\chi)_{L^2\Lambda^{k+1}}.
\end{aligned}
$$

The third equation holds for all $\tilde{p} \in L^2\Lambda^{k+1}$ and hence $d\phi = d\chi$ in $L^2\Lambda^{k+1}$. Substituting this into the first two equations, setting $\tilde{\phi} = \tilde{\chi}$, and adding the resulting equations together gives the weak Euler–Lagrange equations that we derived formally from directly working with the action $S[\phi] = \int_X \mathcal{L}(j^1\phi)$. Thus, it is in this sense that the variational principle treats the derivative of the field as independent of the field. Although the resulting equations were equivalent in this case, we will see shortly that when we discretize the variational principle, one has to be careful about the holonomic condition.

To formulate a discrete variational principle, let $\{\Lambda_h^m\}_{m=0}^{n+1}$ be a subcomplex of finite element spaces approximating $\{H\Lambda\}$ with projections $\pi_h^m : H\Lambda^m \to \Lambda_h^m$. This provides an approximation of $J^1_{H\Lambda^k} = H\Lambda^k \times dH\Lambda^k$ by $\pi_h^k H\Lambda^k \times \pi_h^{k+1}(dH\Lambda^k)$. Consider the associated degenerate action $S_h : H\Lambda^k \to \mathbb{R}$ defined by

$$S_h[\phi] = \int_X \mathcal{L}(x, \pi_h^k\phi, \pi_h^{k+1}d\phi); \tag{2.1}$$

we refer to this as a degenerate action since the projections have nontrivial kernels, as projections from infinite-dimensional spaces to finite-dimensional subspaces. We immediately see that the formal variation of S_h where one treats the derivative of ϕ as independent from ϕ will not work in this setting, since the argument of $\mathcal{L}$ is not holonomic (i.e., of the form $(x, \psi, d\psi)$ for some ψ). To resolve this issue, we utilize the Hamilton–Pontryagin approach, where we weakly enforce the condition that the argument is holonomic. The degenerate Hamilton–Pontryagin action is

$$S_h[\phi, d\chi, p] = \int_X \mathcal{L}(x, \pi_h^k\phi, \pi_h^{k+1}d\chi) - (\pi_h^{k+1}p, d\pi_h^k\phi - \pi_h^{k+1}d\chi)_{L^2\Lambda^{k+1}}.$$

Enforcing stationarity with respect to variations $\tilde{\phi} \in \mathring{H}\Lambda^k, d\tilde{\chi} \in d\mathring{H}\Lambda^k, \tilde{p} \in L^2\Lambda^{k+1}$ gives

$$\begin{aligned}
0 &= \delta_1 S_h[\phi, d\chi, p] = (\partial_2\mathcal{L}(x, \pi_h^k\phi, \pi_h^{k+1}d\chi), \pi_h^k\tilde{\phi})_{L^2\Lambda^k} - (\pi_h^{k+1}p, d\pi_h^k\tilde{\phi})_{L^2\Lambda^{k+1}},\\
0 &= \delta_2 S_h[\phi, d\chi, p] = (\partial_3\mathcal{L}(x, \pi_h^k\phi, \pi_h^{k+1}d\chi), \pi_h^{k+1}d\tilde{\chi})_{L^2\Lambda^{k+1}} + (\pi_h^{k+1}p, \pi_h^{k+1}d\tilde{\chi})_{L^2\Lambda^{k+1}},\\
0 &= \delta_3 S_h[\phi, d\chi, p] = (\pi_h^{k+1}\tilde{p}, d\pi_h^k\phi - \pi_h^{k+1}d\chi)_{L^2\Lambda^{k+1}}.
\end{aligned}$$

Even ignoring issues of degeneracy of the Lagrangian itself (e.g., due to gauge freedom), these equations do not uniquely determine $(\phi, d\chi, p)$ due to the nontrivial kernel of the projections; however, they do determine $(\phi, d\chi, p)$ if we restrict to the images of the projections, i.e., that the fields are in the associated finite-dimensional subspaces. In this case, performing an analogous substitution to the case of the continuum Hamilton–Pontryagin principle gives

$$(\partial_2\mathcal{L}(x, \pi_h^k\phi, d\pi_h^k\phi), \pi_h^k\tilde{\phi})_{L^2\Lambda^k} + (\partial_3\mathcal{L}(x, \pi_h^k\phi, d\pi_h^k\phi), \pi_h^k\tilde{\phi})_{L^2\Lambda^{k+1}} = 0. \tag{2.2}$$

One could ask whether this equation arises from an action $S_h[\phi]$ without utilizing the Hamilton–Pontryagin principle, so that the associated discrete variational structures are manifest and directly comparable to the continuum variational structures. The issue with the action (2.1) is that the argument of $\mathcal{L}$ is not holonomic; however, if we assume that the projections are cochain projections, i.e., that $\pi_h^{k+1}d = d\pi_h^k$, then the argument of $\mathcal{L}$ is holonomic. Assuming cochain projections, we can formally apply the variational principle to

$$S_h[\phi] = \int_X \mathcal{L}(x, \pi_h^k\phi, \pi_h^{k+1}d\phi) = \int_X \mathcal{L}(x, \pi_h^k\phi, d\pi_h^k\phi)$$

and recover equation (2.2). Hence, for the rest of the paper, we will assume that the projections are cochain projections (with the caveat that in Section 3 regarding semi-discretization, we will assume that the projections are cochain projections with respect to the spatial exterior derivative).

Assumption 2.1 (Cochain Projections)**.** *The projections* $\pi_h^m : H\Lambda^m \to \Lambda_h^m$ *are cochain projections, i.e., that* $\pi_h^{k+1}d = d\pi_h^k$.

Furthermore, we will generally denote the projections as π_h, where the degree of the differential forms that they act on are implicitly understood.

Thus, with the assumption of cochain projections, we can interpret the variational principle associated to the degenerate action as the variational principle applied to the original action S restricted to Λ^k_h; we will make this statement more precise in Section 2.1. Instead of enforcing the variational principle $\delta S[\phi]\cdot v = 0$ for all compactly supported $v \in H\Lambda^k$, the finite-dimensional reduction to the problem is given by enforcing the variational principle $\delta S[\phi]\cdot v = 0$ for all $v \in \Lambda^k_h$ with vanishing trace on the boundary, we denote the space of such v by $\mathring{\Lambda}^k_h$. With the assumption of cochain projections, we can formally treat the variation in the derivative of $\phi \in \Lambda^k_h$ as independent of ϕ, and hence we can compute the (discrete) Euler–Lagrange equations formally, as one would do in the continuum case. The variational principle thus yields a discrete weak form of the Euler–Lagrange equation: find $\phi \in \Lambda^k_h$ such that

$$0 = \delta S[\phi]\cdot v = \big(\partial_2\mathcal{L}(j^1\phi), v\big)_{L^2\Lambda^k} + \big(\partial_3\mathcal{L}(j^1\phi), dv\big)_{L^2\Lambda^{k+1}}, \text{ for all } v \in \mathring{\Lambda}^k_h. \tag{2.3}$$

Integrating by parts, this gives

$$0 = \big(\partial_2\mathcal{L}(j^1\phi), v\big)_{L^2\Lambda^k} + \big(d^*\partial_3\mathcal{L}(j^1\phi), v\big)_{L^2\Lambda^k} + \int_{\partial X} v\wedge *\partial_3\mathcal{L}(j^1\phi), \text{ for all } v \in \mathring{\Lambda}^k_h, \tag{2.4}$$

where the codifferential d^* is interpreted in the weak sense. Note the boundary term vanishes since $v \in \mathring{\Lambda}^k_h$, but we include it explicitly since it will be necessary in the formulation of the multisymplectic form formula and Noether's theorem (where one generally has nonzero variations on the boundary).

We refer to these equivalent equations, (2.3) and (2.4), as the discrete Euler–Lagrange equations (DEL). Fixing a basis of shape functions $\{v_i\}$ for $\mathring{\Lambda}^k_h$, expressing $\phi = \phi^j v_j$, and choosing $v = v_i$, (2.3) is equivalent to a (generally nonlinear) system of equations for the unknown components ϕ^i. Letting $[i]$ denote the set of indices j such that $\text{supp}(v_j)\cap\text{supp}(v_i)$ has positive measure, the system of equations can be written

$$\big(\partial_2\mathcal{L}(j^1(\sum_{j\in[i]}\phi^j v_j)), v_i\big)_{L^2\Lambda^k} + \big(\partial_3\mathcal{L}(j^1(\sum_{j\in[i]}\phi^j v_j)), dv_i\big)_{L^2\Lambda^{k+1}} = 0,\ i = 1,\dots,\dim\mathring{\Lambda}^k_h.$$

Example 2.1 (Nonlinear Poisson/Wave Equation)**.** *We consider the nonlinear (scalar) Poisson/wave equation in $1+1$ spacetime dimensions on a rectangular domain, $X = [a,b]\times[c,d]$,*

$$\partial_t^2\phi + \epsilon\partial_x^2\phi + N'(\phi) = 0,$$

where for the Poisson equation $\epsilon = +1$ and for the wave equation $\epsilon = -1$. The Lagrangian is given by $L(\phi,\phi_t,\phi_x) = \frac{1}{2}(\partial_t\phi)^2 + \epsilon\frac{1}{2}(\partial_x\phi)^2 - N(\phi)$, or equivalently,

$$\mathcal{L}(\phi, d\phi) = \frac{1}{2}d\phi\wedge\star d\phi - N(\phi)dt\wedge dx.$$

where the metric corresponding to the Poisson equation is $g = diag(1,1)$ and corresponding to the wave equation is $g = diag(1,-1)$. Compute $\partial_3\mathcal{L} = d\phi$, $\partial_2\mathcal{L} = -N'(\phi)$, so the discrete Euler–Lagrange equation reads: find $\phi \in \Lambda^0_h$ such that

$$(d\phi, dv) - (N'(\phi), v) = 0, \text{ for all } v \in \mathring{\Lambda}^0_h.$$

(where the product $(\cdot,\cdot)$ is computed with the appropriate metric signature). We subdivide X into a regular rectangular mesh and use a tensor-product basis of hat functions $\psi_{ij}(t,x) = \chi_i(t)\xi_j(x)$ subordinate to this mesh.

Expressing $\phi = \phi^{ij}\psi_{ij}$ and taking $v = \psi_{mn}$, the above equation reads

$$\sum_{ij\in[mn]} \Big(\phi^{ij}(\chi_i'(t),\chi_m'(t))(\xi_j(x),\xi_n(x)) + \epsilon\phi^{ij}(\chi_i(t),\chi_m(t))(\xi_j'(x),\xi_n'(x))\Big) - (N'(\phi),\psi_{mn}) = 0.$$

Since $[m] = \{m-1, m, m+1\}$, this gives a nine-point integrator on the interior elements of the mesh. Explicitly, we compute the stiffness and mass matrix elements

$$\{(\chi_i'(t),\chi_m'(t))\}_{i\in[m]} = \frac{1}{\Delta t}\{-1,2,-1\},$$
$$\{(\chi_i(t),\chi_m(t))\}_{i\in[m]} = \Delta t\left\{\frac{1}{6},\frac{2}{3},\frac{1}{6}\right\}$$

(and similarly for the x direction). This gives

$$\frac{\phi^{m+1\tilde{n}} - 2\phi^{m\tilde{n}} + \phi^{m-1\tilde{n}}}{\Delta t^2} + \epsilon\frac{\phi^{\tilde{m}n+1} - 2\phi^{\tilde{m}n} + \phi^{\tilde{m}n-1}}{\Delta x^2} + \frac{1}{\Delta t\Delta x}(N'(\phi),\psi_{mn}) = 0,$$

where $\phi^{m\tilde{n}} = \frac{1}{6}(\phi^{mn+1} + 4\phi^{mn} + \phi^{mn-1})$ and $\phi^{\tilde{m}n} = \frac{1}{6}(\phi^{m+1n} + 4\phi^{mn} + \phi^{m-1n})$. Noting that $(N'(\phi),\psi_{mn}) = \delta\mathcal{N}/\delta\phi^{mn}$, where $\mathcal{N} = \int N(\phi)dt\wedge dx$, this reproduces the nine-point variational integrator derived by Chen [13]. As was shown in Chen [13], using mid-point quadrature, this method reduces to the multisymplectic integrator derived by Marsden et al. [29]. We will continue this example in the subsequent section to provide an example of the discrete Cartan form, see Example 2.2.

In order to provide local statements of the multisymplectic form formula and Noether's theorem, we now localize the DEL. For a region $U \subset X$, we say that a node i is an interior point of U if U contains all simplices touching i. Denote $\bar{U}$ as the union of all simplices touching interior nodes i of U; we say that U is regular if $U = \bar{U}$. We define the admissible variations with respect to a regular region U as the space of all $v \in \mathring{\Lambda}^k_h$ such that $v|_U \in \mathring{\Lambda}^k_h(U)$. We define the localized action $S_U[\phi] = \int_U \mathcal{L}(j^1\phi)$ and the associated localized DEL,

$$0 = \delta S_U[\phi]\cdot v = \big(\partial_2\mathcal{L}(j^1\phi), v\big)_{L^2\Lambda^k(U)} + \big(\partial_3\mathcal{L}(j^1\phi), dv\big)_{L^2\Lambda^{k+1}(U)} \tag{2.5}$$
$$= \big(\partial_2\mathcal{L}(j^1\phi), v\big)_{L^2\Lambda^k(U)} + \big(d^*\partial_3\mathcal{L}(j^1\phi), v\big)_{L^2\Lambda^k(U)} + \int_{\partial U} v\wedge *\partial_3\mathcal{L}(j^1\phi)$$

which is enforced for all regular U and admissible v (as before, the boundary term vanishes for admissible v, but we write it explicitly as it will arise later).

Proposition 2.1. *The localized DEL (2.5), ranging over all regular U and admissible v, are equivalent to the DEL (2.4)*

Proof. To see that the localized DEL imply the DEL, choose $U = X$ which is trivially regular; the space of admissible variations with respect to X is then just $\mathring{\Lambda}^k_h$. To see that the DEL imply the localized DEL, let U be regular and v be admissible. Since $\text{supp}(v) \subset U$, the integrals over X in the DEL can be replaced by integrals over U. □

2.1. **Variational Structure of Discretization.** In this section, we aim to elucidate the variational structure that arises from discretizing the variational principle utilizing cochain projections. Recalling that the Cartan form (1.1) encodes the variational structure of a Lagrangian field theory, we will construct a discrete analogue of the Cartan form, which will naturally encode the variational structure of the discretized theory.

We first show that the restricted variational principle over the finite-dimensional subspace $\mathring{\Lambda}^k_h$ can be interpreted as a Galerkin variational integrator. Restricting the configuration space to $\mathring{\Lambda}^k_h$, we can view the action as a function of the components ϕ^i in the expansion $\phi = \phi^i v_i$.

$$S[\phi^i] = \int \mathcal{L}(x, \phi^i v_i, \phi^i dv_i).$$

Taking the variation of S with respect to ϕ^j,

$$\begin{aligned}\frac{\delta S[\phi^i]}{\delta \phi^j} &= \int \Big(\frac{\delta \mathcal{L}}{\delta \phi} \cdot \frac{\delta(\phi^i v_i)}{\delta \phi^j} + \frac{\delta \mathcal{L}}{\delta(d\phi)} \cdot \frac{\delta(\phi^i dv_i)}{\delta \phi^j} \Big) = \int \Big(\frac{\delta \mathcal{L}}{\delta \phi} \cdot v_j + \frac{\delta \mathcal{L}}{\delta(d\phi)} \cdot dv_j \Big) \\ &= (\partial_2 \mathcal{L}, v_j) + (\partial_3 \mathcal{L}, dv_j),\end{aligned}$$

which shows that the conditions $\delta S/\delta \phi^j = 0$ is equivalent to the DEL (2.4). Similarly, the localized DEL (2.5) is equivalent to the conditions $\delta S_U/\delta \phi^j = 0$ for all interior nodes j. That is, the DEL can be interpreted as a Galerkin variational integrator. From this viewpoint of the DEL, we see that given appropriate choices of function spaces (and possibly a choice of quadrature rule), our discrete Euler–Lagrange equation reproduces multisymplectic variational integrators based on finite differences or nodal value finite element spaces (e.g., as discussed in Marsden et al. [29] and Chen [13]). However, the discrete variational principle in the form $\delta S[\phi] \cdot v = 0$, for $\phi \in \Lambda^k_h$ and $v \in \mathring{\Lambda}^k_h$, is expressed explicitly at the level of function spaces and hence, will allow us to examine the discrete variational structure more directly. Along with allowing more general approximating finite element spaces, this also has the advantage of stating properties of the discrete variational principle at the level of function spaces. Consequently, as we will see, properties such as multisymplecticity and Noether's theorem can be stated in a geometric way, which makes no explicit reference to finite differencing or quadrature.

When the variational principle to the Lagrangian system defined by $\mathcal{L}$ is enforced only over the subspace $\mathring{\Lambda}^k_h$ of the configuration bundle, the discrete dynamics (2.4) are recovered. To recast (2.4) in terms of the Cartan form, we note that variations $\delta S[\phi] \cdot v$ is equivalently given by the differential of the action paired with a vertical vector field $dS_\phi \cdot V$. This follows since $T(H\Lambda^k) \cong H\Lambda^k \times H\Lambda^k$, so $V(\phi)$ can be identified with some $v \in H\Lambda^k$. We define a discrete vertical vector field as an element of $\mathfrak{X}(\Lambda^k_h)$. Then, in particular, a constant vertical vector field $V \in \mathfrak{X}(\Lambda^k_h)$ can be identified with $v \in \Lambda^k_h$; i.e., viewing $\mathfrak{X}(\Lambda^k_h)$ as the space of (sufficient regularity) maps $\Lambda^k_h \to \Lambda^k_h$, we have $V(\phi) = v$ for all ϕ; in which case, we denote $V = V_v$ and the space of such vector fields as V_h (note the time-ϵ flow of V_v on any section $\phi \in \Lambda^k_h$ is given by $\phi + \epsilon v$). We denote the spaces $\mathring{\mathfrak{X}}(\Lambda^k_h)$ and $\mathring{V}_h$ as the subspaces of the above spaces which vanish on ∂X. Then (2.4) is equivalent to finding $\phi \in \mathring{\Lambda}^k_h$ such that

$$0 = dS[\phi] \cdot V = -\int_X (j^1\phi)^*(j^1 V \lrcorner\, \Omega_{\mathcal{L}}), \text{ for all } V \in \mathring{\mathfrak{X}}(\Lambda^k_h), \tag{2.6}$$

or equivalently using constant vector fields,

$$0 = \delta S[\phi] \cdot v = dS[\phi] \cdot V_v = -\int_X (j^1\phi)^*(j^1 V_v \lrcorner\, \Omega_{\mathcal{L}}), \text{ for all } V_v \in \mathring{V}_h, \tag{2.7}$$

where $j^1 V$ is the vector field jet prolongation of V.

By the above, we can view the Lagrangian structure associated to the equations (2.7) as the restriction of the full Lagrangian structure to the discrete space. The next natural question to ask would be: is there some sense in which the discrete equations, which arises as a restriction of the variational principle, can instead be viewed as a variational principle on the full configuration bundle? Since we assume that the projection maps $\pi_h : H\Lambda^m \to \Lambda^m_h$ are cochain projections on the Hilbert de Rham complex, there is a natural relation between the dynamics of the restricted Lagrangian structure and variations on the full space of the degenerate Lagrangian. To see this, recall that the Lagrangian density $\mathcal{L} : J^1_k \to \wedge^{n+1}(T^*X)$ is a bundle map, so it induces a map on the space of sections, $\mathcal{L} : J^1_{H\Lambda^k} \to \Lambda^{n+1}(X)$. In equation (2.1), we defined a degenerate Lagrangian density, $\mathcal{L}_h : J^1_{H\Lambda^k} \to \Lambda^{n+1}(X)$ given by $\mathcal{L}_h(x, \phi, d\psi) = \mathcal{L}(x, \pi_h\phi, \pi_h d\psi)$ with associated degenerate action $S_h[\phi] = \int_X \mathcal{L}_h(j^1\phi)$. In the case of a cochain projection, we can then view the variations of S restricted to Λ^k_h as variations of S_h on the full configuration bundle.

Proposition 2.2. ***(Naturality of Discrete Variational Structure)***
The restricted variational structures are related to the degenerate variational structures by

$$\mathcal{L}(j^1\pi_h\phi) = \mathcal{L}_h(j^1\phi), \tag{2.8}$$

$$dS[\pi_h\phi] \cdot (T\pi_h{\cdot}V_v) = dS_h[\phi] \cdot V_v, \tag{2.9}$$

for $\phi \in H\Lambda^k$ and $v \in H\Lambda^k$.

Proof. For (2.8), since π_h is a cochain projection,

$$\mathcal{L}(j^1\pi_h\phi) = \mathcal{L}(x, \pi_h\phi, d\pi_h\phi) = \mathcal{L}(x, \pi_h\phi, \pi_h d\phi) = \mathcal{L}_h(x, \phi, d\phi) = \mathcal{L}_h(j^1\phi).$$

Then, (2.9) follows similarly, noting that since V_v generates the flow $\phi + \epsilon v$ on ϕ, $T\pi_h \cdot V_v$ generates the flow $\pi_h\phi + \epsilon\pi_h v$ on $\pi_h\phi$, which gives

$$\begin{aligned} dS[\pi_h\phi] \cdot (T\pi_h \cdot V_v) &= \delta S[\pi_h\phi] \cdot \pi_h v \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} S[\pi_h\phi + \epsilon\pi_h v)] = \frac{d}{d\epsilon}\Big|_{\epsilon=0} S[\pi_h(\phi + \epsilon v)] \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} S_h[\phi + \epsilon v] = \delta S_h[\phi] \cdot v = dS_h[\phi] \cdot V_v, \end{aligned}$$

where $S_h = S \circ \pi_h$ follows from the cochain map property. □

The naturality equations (2.8) and (2.9) reveal that the process of discretization of the full field dynamics, in the case of using cochain projections for discretization, is itself associated to an action on the full field space; i.e., the discretization is compatible with the structure of a Lagrangian theory. A corollary is that the equation (2.7) can be seen as either arising from the variation of the full action S at a discrete field $\pi_h\phi$, or as from the variation of the discrete action S_h at the full field ϕ. This shows that the variations associated to S_h on the full field space are degenerate, since they are equivalently given by the variations of S on the projected space. Thus, the finite-dimensionality of the restricted variational principle on S can be interpreted as the degeneracy of the variational principle of S_h on the full space, where two fields are equivalent if their difference is in $\ker(\pi_h)$ (and similarly for two vertical variations if their difference is in $\ker(T\pi_h)$). In other words, our finite-dimensional variational problem on the discrete space arises as a degenerate variational problem over the infinite-dimensional space, where the set of equivalence classes forms a finite-dimensional space, with the canonical representative $i_h\pi_h\phi$ for the equivalence class of ϕ, where $i_h : \Lambda^k_h \hookrightarrow H\Lambda^k$ is the inclusion map.

Furthermore, the above naturality relation shows that projecting the equations obtained from the variational principle applied to the continuum action is equivalent to first discretizing the action through the projection and subsequently applying the variational principle. Thus, when discretizing via cochain projections, the variational principle and discretization commute:

$$\begin{array}{ccc} S : H\Lambda^k \to \mathbb{R} & \xrightarrow{\text{Discretize}} & S_h : \Lambda_h^k \to \mathbb{R} \\ \Big\downarrow {\scriptstyle \text{Variational Principle}} & & \Big\downarrow {\scriptstyle \text{Variational Principle}} \\ \text{Weak EL} & \xrightarrow{\text{Discretize}} & \text{Discrete EL} . \end{array}$$

This generalizes the result of Leok [24] where it was shown that discretization via discrete exterior calculus and the variational principle commute in the case of electromagnetism. In particular, the result of Leok [24] follows from the above, since one can view discrete exterior calculus in the framework of finite element exterior calculus as a particular low-order example; namely, through the use of Whitney forms.

As a final remark on the above naturality relation, a more fundamental issue for discretization is whether one should discretize at the level of the configuration bundle or the jet bundle. One can discretize sections of the configuration bundle, via $\phi \mapsto \pi_h^k \phi$ and subsequently work with the restricted Lagrangian $\mathcal{L}(j^1(\pi_h \phi))$, or one can discretize sections of the jet bundle, via $j^1\phi \mapsto (\pi_h^k \times \pi_h^{k+1}) j^1\phi$; in general, these methods are not equivalent. However, in the case of cochain projections, these two discretization processes are equivalent; i.e., the following diagram commutes

$$\begin{array}{ccc} \phi & \xmapsto{\pi_h^k} & \pi_h \phi \\ \Big\downarrow {\scriptstyle j^1} & & \Big\downarrow {\scriptstyle j^1} \\ j^1\phi & \xmapsto{\pi_h^k \times \pi_h^{k+1}} & j^1(\pi_h \phi), \end{array}$$

so there is no ambiguity in which discretization procedure to use. Furthermore, regarding Assumption 2.1, the above diagram shows that we only need the existence of the space Λ_h^{k+1} and the projection π_h^{k+1} such that the above diagram commutes and thus, one can perform the discretization solely using Λ_h^k and π_h^k, without reference or implementation of Λ_h^{k+1} and π_h^{k+1}. In particular, as discussed in, for example, Arnold et al. [6, 7] and Arnold [5], there is a large class of classical finite element spaces for which such cochain projections exist, so our theory is broadly applicable.

In order to state discrete analogues of the multisymplectic form formula and Noether's theorem, we will have to consider variations which are nonzero on the boundary of a regular region U. To do this, consider the following decomposition. Let U be a regular region and let $v \in \Lambda_h^k$. Consider v restricted to U. In general, since we are not assuming v be an admissible variation relative to U, v may have nonzero trace along ∂U. Decompose $v = v_\partial + v_{in}$ where v_∂ denotes the boundary component of v consisting of the expansion of v with respect to all shape functions which have nonzero trace on ∂U; while $v_{in} = v - v_\partial$ corresponds to the expansion of v into shape functions with vanishing trace on the boundary. Let $\mathcal{T}[\partial U]$ denote the set of all top-dimensional elements in $\mathcal{T}_h$ on which shape functions with nonvanishing trace on ∂U are supported. We extend this decomposition to vector fields analogously $V = V_\partial + V_{in}$, noting again by our previous discussion that for $V \in \mathfrak{X}(\Lambda_h^k)$, $V(\phi)$ can be identified with some $v \in \Lambda^k$ (and analogously for vector fields on the jet bundle).

Remark 2.1. *If one considers Lagrange polynomial nodal shape functions (corresponding to point value degrees of freedom), then the shape functions which are nonzero on the boundary are those associated to the nodes on ∂U. In this case, $\mathcal{T}[\partial U]$ consists of those top-dimensional elements touching the boundary, i.e., the one-ring of the boundary ∂U. For general (local) shape functions,* *internal nodes may give rise to shape functions which are nonzero on the boundary, so $\mathcal{T}[\partial U]$ will generally consist of the elements touching ∂U and the elements touching those elements, i.e., the two-ring of the boundary ∂U. In any case, we consider discretization by the finite element method due to the local support property of the shape functions, which will allow the discrete Cartan form defined below to be localized on $\mathcal{T}[\partial U]$.*

We can now consider variations which do not vanish on ∂U. In particular, we compute for a solution ϕ_h of the discrete Euler–Lagrange equation and $V \in \mathfrak{X}(\Lambda^k_h)$,

$$\begin{aligned} dS_U[\phi_h]\cdot V &= -\int_U (j^1\phi_h)^*(j^1V \lrcorner \Omega_{\mathcal{L}}) + \int_{\partial U} (j^1\phi_h)^*(j^1 V \lrcorner \Theta_{\mathcal{L}}) \\ &= -\int_U (j^1\phi_h)^*(j^1V_{in} \lrcorner \Omega_{\mathcal{L}}) - \int_U (j^1\phi_h)^*(j^1V_\partial \lrcorner \Omega_{\mathcal{L}}) + \int_{\partial U} (j^1\phi_h)^*(j^1V \lrcorner \Theta_{\mathcal{L}}) \\ &= -\sum_{T\in\mathcal{T}[\partial U]} \int_T (j^1\phi_h)^*(j^1V_\partial \lrcorner \Omega_{\mathcal{L}}) + \int_{\partial U} (j^1\phi_h)^*(j^1V \lrcorner \Theta_{\mathcal{L}}), \end{aligned}$$

where the term involving V_{in} vanishes by the DEL. This boundary variation formula will be our candidate for a discrete Cartan form, as it encodes the contribution to the action from V nonvanishing on and near the boundary, and will allow us to state discrete analogues of the multisymplectic form formula and Noether's theorem.

Definition 2.1 (Discrete Cartan Form)**.** *The discrete Cartan one-form, evaluated at a solution ϕ_h of the discrete Euler–Lagrange equation (2.4), is defined by*

$$\Theta^h_U(\phi_h)\cdot V \equiv dS_U[\phi_h]\cdot V_\partial = \int_{\partial U} (j^1\phi_h)^*(j^1V \lrcorner \Theta_{\mathcal{L}}) - \sum_{T\in\mathcal{T}[\partial U]} \int_T (j^1\phi_h)^*(j^1V_\partial \lrcorner \Omega_{\mathcal{L}}), \tag{2.10a}$$

or in terms of the Lagrangian density,

$$\Theta^h_U(\phi_h)\cdot V = \int_{\partial U} \Big(*\partial_3\mathcal{L}(j^1\phi_h)\Big)\wedge V(\phi_h) + \sum_{T\in\mathcal{T}[\partial U]} \int_T (\partial_2\mathcal{L}(j^1\phi_h) + d^*\partial_3\mathcal{L}(j^1\phi_h))\wedge\star V(\phi_h)_\partial. \tag{2.10b}$$

Even though the continuum Cartan form only involves integration on ∂U, we will see why this is the appropriate definition in the discrete setting. In stating the discrete analogues of the multisymplectic form formula and Noether's theorem, we will contrast this to the (integrated) Cartan form of the continuum theory,

$$\Theta_U(\phi)\cdot V \equiv \int_{\partial U} (j^1\phi)^*(j^1V \lrcorner \Theta_{\mathcal{L}}) = \int_{\partial U} \Big(*\,\partial_3\mathcal{L}(j^1\phi)\Big) \wedge V(\phi). \tag{2.11}$$

Remark 2.2 (Quadrature)**.** *Although, in our exposition, we have assumed that with the given Lagrangian and choice of finite element space one can evaluate the integrals involved exactly, one can more generally utilize quadrature to approximate the action before enforcing the variational principle. For a regular region U, let us consider quadrature nodes $\{c_a \in U\}$ and associated quadrature weights $\{b_a\}$. With finite element shape functions $\{v_j\}$ and expressing the density as $\mathcal{L} = L d^{n+1}x$, the associated discrete action is given by applying quadrature,*

$$\mathbb{S}_U[\{\phi^j\}] = \sum_a b_a L(j^1(\phi^i v_i))|_{c_a}. \tag{2.12}$$

The variation in the direction $w = w^k v_k$ is given by

$$\delta\mathbb{S}_U[\{\phi^j\}]\cdot\{w^k\} = \sum_a b_a \frac{\partial}{\partial\phi^k}\Big[L(j^1(\phi^i v_i))\Big|_{c_a}\Big] w^k. \tag{2.13}$$

The associated discrete Euler–Lagrange equation is given by enforcing the variational principle for variations w with vanishing trace on ∂U. Then, the discrete Cartan form with quadrature (at a solution of the discrete Euler–Lagrange equation), is defined by taking an arbitrary variation and removing the term on the interior which vanishes by the discrete Euler–Lagrange equation. In particular, it is given by summing over all a such that c_a is contained in the support of some shape function with nonvanishing trace on the boundary; we denote the set of all such a by $\mathcal{I}[\partial U]$. Hence, the discrete Cartan form with quadrature is given by

$$\Theta_U^h(\phi)\cdot W = \sum_{a\in\mathcal{I}[\partial U]} b_a \frac{\partial}{\partial\phi^k}\Big[L(j^1(\phi^i v_i))\big|_{c_a}\Big] w^k,$$

where $W(\phi) = w^k v_k$.

Using this discrete Cartan form, an analogous statement of discrete multisymplecticity that we state below holds in this setting, with the caveat that the first variations are defined relative to the discrete Euler–Lagrange equations with quadrature. Similarly, an analogous statement to the discrete Noether's theorem below also holds in this setting, with the caveat that the group action leaves the discrete action with quadrature, equation (2.12), invariant. This is a direct consequence of the fact that the formulation with quadrature is still variational, since we applied the quadrature rule to the action, before enforcing the variational principle (see Section 2.4). In general, if one applies quadrature after enforcing the variational principle, i.e., to the equations of motion (2.4), the system is not variational. To see this, we compute the variation of the action first,

$$\delta S_U[\phi^j v_j]\cdot(w^k v_k) = \int_X [\partial_2\mathcal{L}(j^1(\phi^i v_i)) + d^*\partial_3\mathcal{L}(j^1(\phi^i v_i))]\wedge\star w^k v_k,$$

(for w with vanishing trace on ∂U) and subsequently apply quadrature, so that the above becomes

$$\sum_a b_a\Big[*\Big([\partial_2\mathcal{L}(j^1(\phi^i v_i)) + d^*\partial_3\mathcal{L}(j^1(\phi^i v_i))]\wedge\star w^k v_k\Big)\Big]\Big|_{c_a}.$$

In general, this is not equal to (2.13), except when ϕ a scalar field, using nodal interpolating shape functions and quadrature points at those nodes, in which case they are the same. Thus, for a variational formulation, one should generally apply quadrature before enforcing the variational principle. For the rest of the paper, we will return to the assumption that one can evaluate the various integrals exactly, but keeping in mind that similar results hold in the case of quadrature.

We make several additional remarks regarding this candidate (2.10a) for a discrete Cartan form. We defined the discrete Cartan form as the variation of the action (which may be nonvanishing on the boundary) at a solution of the discrete Euler–Lagrange equations. Even though this functional involves integration over top-dimensional regions $T\in\mathcal{T}[\partial U]$, it only depends on the degrees of freedom which contribute to the nonzero value of V on ∂U and so makes sense as a candidate for a discrete Cartan form. In the continuum variational problem, boundary variations can be supported arbitrarily close to ∂U, whereas in the finite element variational problem, this is not the case, so the discrete Cartan form (which encodes the contribution of the variation of the action by boundary variations) should indeed contain the additional terms involving integration over the elements of $\mathcal{T}[\partial U]$. These terms shrink relative to the integral over ∂U in the following heuristic sense. The terms involving $\mathcal{T}[\partial U]$ are $O(h)$ smaller than the term over ∂U: the cardinality of $\mathcal{T}[\partial U]$ scales like the number of boundary faces in ∂U, which is $O(h^{-n})$; on the other hand, the size of T is $O(h^{n+1})$, so the terms in the discrete Cartan form involving the sum over $\mathcal{T}[\partial U]$ is $O(h)$, whereas the first term is $O(1)$ for a fixed region U. Thus, as $h\to 0$, for a fixed region U, the Cartan form formally only involves the first contribution, as expected. In other words, as we refine the mesh, ∂U stays (roughly) the same, while the region containing only elements touching ∂U shrinks (and a similar remark applies to the discrete multisymplectic form formula and the additional terms involving the sum over $\mathcal{T}[\partial U]$, so that the multisymplectic form formula is formally recovered in the limit). This can be combined with bounds on the integrands to show convergence more rigorously, which we will sketch when discussing Noether's theorem.

We now show that Defintion 2.1 recovers the notion of the discrete Cartan form introduced in Marsden et al. [29] and further examined in Chen [13], in the case that the degrees of freedom are the nodal values of the field with nodal interpolating shape functions. As previously remarked, in this case, the shape functions which are nonzero on ∂U are those associated to nodes on ∂U. Consider a single node i on ∂U and let v_i be the shape function associated to the degree of freedom on the node; note that v_i (restricted to U) is supported in some $T_i \in \mathcal{T}[\partial U]$ and denote $F_i = \partial T_i \cap \partial U$. Consider a variation by an amount Vv_i ($V \in \mathbb{R}$). Marsden et al. [29] and Chen [13] define the discrete Cartan form associated to this node as $\frac{\delta S_U[\phi^j v_j]}{\delta \phi^i} V$, viewing the action as a function of the components in the expansion of $\phi = \phi^j v_j$. Then, compute

$$\begin{aligned}
\frac{\delta S_U[\phi^j v_j]}{\delta \phi^i} V &= \int_U \Big(\frac{\delta \mathcal{L}}{\delta \phi} \cdot \frac{\delta(\phi^j v_j)}{\delta \phi^i} V + \frac{\delta \mathcal{L}}{\delta (d\phi)} \cdot \frac{\delta(\phi^j dv_j)}{\delta \phi^i} V \Big) \\
&= \int_U \Big(\frac{\delta \mathcal{L}}{\delta \phi} \cdot V v_i + \frac{\delta \mathcal{L}}{\delta (d\phi)} \cdot V dv_i \Big) = \int_U \Big(\partial_2 \mathcal{L} \wedge \star V v_i + \partial_3 \mathcal{L} \wedge \star V dv_i \Big) \\
&= \int_U (\partial_2 \mathcal{L} + d^* \partial_3 \mathcal{L}) \wedge \star V v_i + \int_{\partial U} (\ast \partial_3 \mathcal{L}) \wedge V v_i \\
&= \int_{T_i} (\partial_2 \mathcal{L} + d^* \partial_3 \mathcal{L}) \wedge \star V v_i + \int_{F_i} (\ast \partial_3 \mathcal{L}) \wedge V v_i
\end{aligned}$$

Then, summing over all such variations on each node on ∂U, one recovers our discrete Cartan form, equation (2.10b). There are several generalizations which our discrete Cartan form makes relative to the discrete Cartan form of Marsden et al. [29] and Chen [13]. First, note that their Cartan form is defined in terms of the nodal values of the field, which implicitly suppresses the fact that the Cartan form involves integration over both ∂U and elements of $\mathcal{T}[\partial U]$. Our explicit formula for the discrete Cartan form lends itself more easily to showing convergence to the continuum Cartan form, as we sketched heuristically above and will discuss further when discussing Noether's theorem. That the discrete Cartan form involves integration over elements neighboring the boundary is inevitable, since a variation of the field value on the boundary induces changes to the field values on elements of $\mathcal{T}[U]$. Furthermore, since we allow for general finite element spaces, we immediately get several generalizations. First, note that the dimension of the spacetime is arbitrary in our formulation, so this discrete Cartan form holds beyond the $1+1$ spacetime dimensions that they utilize explicitly in their framework (although this is not a fundamental restriction in their theory). Furthermore, our framework allows for arbitrary degree of differential forms, as opposed to just scalar fields. In particular, the degrees of freedom associated to the boundary variations need not be nodal values, but can be determined by more general degrees of freedom, such as moments or flux type degrees of freedom (e.g., when considering a theory involving vector fields, which we identify with $1-$forms via the metric). Furthermore, these degrees of freedom determining the boundary variations may be close to (i.e., in $\mathcal{T}[\partial U]$) but not necessarily on ∂U.

Example 2.2 (Discrete Cartan Form for the Nonlinear Poisson/Wave Equation)**.** *Recall the discretization of the nonlinear Poisson/wave equation given in Example 2.1. Consider a regular region $U \subset X$; for simplicity, we take U to be a rectangular region $U = [t_0, t_M] \times [x_0, x_N]$ (with no loss of generality, since any regular region is a union of such rectangular regular regions), where the vertices of U are given by $\{(t_i, x_j)\}_{i,j=0}^{M,N}$ where $t_i = t_0 + i\Delta t, x_j = x_0 + j\Delta x$. We index the piecewise linear nodal interpolating shape functions $\psi_{ij}(t,x) = \chi_i(t)\xi_j(x)$ by the node (t_i, x_j) which it* *interpolates; i.e.,* $\chi_i(t_k)\xi_j(x_l) = \delta_{ik}\delta_{jl}$. *Let*

$$\phi_h = \sum_{i,j=0}^{M,N} \phi_h^{ij}\chi_i\xi_j$$

be a solution of the associated discrete Euler–Lagrange equation, restricted to U.

Recall the definition of the discrete Cartan form as the variation of the action by $w \in \Lambda_h^0(U)$ *(with generally nonvanishing trace on* ∂U*). Letting* $w = w_{in} + w_\partial \in \Lambda_h^0(U)$ *and* $W \in \mathfrak{X}(\Lambda_h^0)$ *such that* $W(\phi_h) = w$*, we have* $\delta S_U[\phi_h] \cdot w_{in} = 0$ *and hence,*

$$\begin{aligned} \Theta_U^h(\phi_h) \cdot W &= \delta S_U[\phi_h] \cdot w = \delta S_U[\phi_h] \cdot (w - w_{in}) = \delta S_U[\phi_h] \cdot w_\partial \\ &= \sum_{T \in \mathcal{T}[\partial U]} \int_T \left[d\phi_h \wedge *dw_\partial - N'(\phi_h) \wedge *w_\partial \right]. \end{aligned} \tag{2.14}$$

As discussed above, the discrete Cartan form reproduces the discrete Cartan form in Marsden et al. [29] and Chen [13]. However, we will now explicitly show this for this example. We express the action as a function of the components ϕ_h^{ij}*:*

$$S_U[\{\phi_h^{ij}\}] = \int_U [d\phi_h \wedge *d\phi_h - N(\phi_h) dt \wedge dx] = \int_U \left[\frac{1}{2} \sum_{i,j} \sum_{k,l} \phi_h^{ij}\phi_h^{kl} d\psi_{ij} \wedge *d\psi_{kl} - N(\sum_{k,l} \phi_h^{kl}\psi_{kl}) dt \wedge dx \right].$$

Let $ij \in \mathcal{I}[\partial U]$*, i.e., the index corresponds to a node on* ∂U *(consisting of indices* ij *such that either* $i = 0$ *or* M *or* $j = 0$ *or* N*). Marsden et al. [29] and Chen [13] define the discrete Cartan form associated to this node as*

$$\frac{\partial S_U[\{\phi_h^{kl}\}]}{\partial \phi_h^{ij}} d\phi_h^{ij} \tag{2.15}$$

(where here d *is the vertical exterior derivative along the fiber and not the exterior derivative on the base space). Compute*

$$\frac{\partial S_U[\{\phi_h^{kl}\}]}{\partial \phi_h^{ij}} = \int_U \left[\sum_{k,l} \phi_h^{kl} d\psi_{ij} \wedge *d\psi_{kl} - N'(\sum_{k,l} \phi_h^{kl}\psi_{kl})\psi_{ij} dt \wedge dx \right].$$

With coordinate ϕ_h^{ij} *on* Λ_h^0*, we can express the vector field* $W = \sum_{k,l} W^{kl}\partial/\partial\phi_h^{kl}$ *and hence* $W^{kl}(\phi_h) = w^{kl}$*. Pairing (2.15) with* W *and summing over all* $ij \in \mathcal{I}[\partial U]$*, we see that this gives (2.14), since* $w_\partial = \sum_{ij \in \mathcal{I}[\partial U]} w^{ij}\psi_{ij}$ *and* ψ_{ij} *for* $ij \in \mathcal{I}[\partial U]$ *are supported on* $\cup_{T \in \mathcal{T}[\partial U]} T$*.*

Finally, we now discuss in what sense the discrete Cartan form for this example converges to the continuum Cartan form. Consider a node $ij \in \mathcal{I}[\partial U]$ *along, say, the* $\{t = t_0\}$ *edge of* ∂U*, so that* $i = 0$*. We compute part of the discrete Cartan form for a boundary variation* w^{0j} *associated to this node; namely, we compute the part associated to the derivative in the* t *direction, since this is the normal direction along this edge. This is given by*

$$\begin{aligned} \int_U \sum_{k,l} \phi_h^{kl}\chi_k'(t)\xi_l(x) w^{0j}\chi_0'(t)\xi_j(x) dt \wedge dx &= \int_U \sum_{k=0}^{1} \sum_{l=j-1}^{j+1} \phi_h^{kl}\chi_k'(t)\xi_l(x) w^{0j}\chi_0'(t)\xi_j(x) dt \wedge dx \\ &= \sum_{l=j-1}^{j+1} \frac{\phi_h^{0l} - \phi_h^{1l}}{\Delta t} (\xi_l, \xi_j)_{L^2} w^{0j}. \end{aligned}$$

Since $(\xi_l, \xi_j)_{L^2}$ for $l = j-1, j, j+1$ has total mass Δx, this formally converges to $\int \frac{\partial \phi}{\partial n} w dx$ (noting that the normal vector on this edge is $-\hat{t}$). Repeating this over all nodes on ∂U, this part of the discrete Cartan form formally converges to

$$\int_{\partial U} \frac{\partial \phi}{\partial n} w dS$$

(where dS is the codimension one measure on ∂U), which is the continuum Cartan form. Furthermore, the other terms in the discrete Cartan form which we did not explicitly write formally converge to zero; this follows since the remaining terms formally vanish by the weak Euler–Lagrange equations in the continuum limit. To be more rigorous, for the Poisson equation, we should express the above as

$$\int_{\partial U} \partial_n(\phi)\, Tr(w) dS,$$

where $Tr : H\Lambda^0(X) \to H^{1/2}\Lambda^0(\partial X)$ (recall that we define $H\Lambda^0(X)$ as the Hilbert space of square integrable functions with square integrable derivative) and $\partial_n : \{u \in H\Lambda^0(X) : \Delta u \in L^2(X)\} \to H^{-1/2}(\partial X)$ are bounded operators. In particular, if $N'(\phi)$ is square integrable, then ∂_n is appropriately bounded for this problem; for example, this holds if $N(\phi)$ is polynomial in ϕ with degree $p \geq 2$ (since X is compact) and hence the nonlinearity $N'(\phi)$ can be polynomial with degree $p \geq 1$ (with $p = 1$ corresponding to the linear Helmholtz equation). In this case, convergence in $H\Lambda^0$ of the discrete solution ϕ_h to a solution ϕ of the weak Euler–Lagrange equation gives weak convergence of the discrete Cartan form to the continuum Cartan form. For the wave equation, due to the metric signature, one has to be more careful regarding the definition of the relevant Sobolev spaces although the discrete Cartan form formally converges in the sense above; we aim to pursue rigorous convergence of the discrete Cartan form for evolution equations in future work.

In the next two sections, we will utilize the discrete Cartan form to state discrete analogues of multisymplecticity and Noether's theorem. We will see that these statements, involving Θ_U^h, will be in direct analogy to the continuum theorems, involving Θ_U.

2.2. **Discrete Multisymplectic Form Formula.** We now state a discrete analogue of the multisymplectic form formula, which generalizes the preservation of the symplectic form under the flow of a symplectic vector field. Namely, if ϕ is a solution to the Euler–Lagrange equations and V, W are first variations at ϕ (their flow on ϕ is still a solution), then

$$\int_{\partial U} (j^1\phi)^* \left(j^1 V \lrcorner\, j^1 W \lrcorner\, \Omega_{\mathcal{L}} \right) = 0, \tag{2.16}$$

where $U \subset X$ is a submanifold with smooth closed boundary (Marsden et al. [29]). The multisymplectic form formula encompasses many physical conservation laws appearing in Lagrangian field theories. For example, viewing a Lagrangian field theory in the instantaneous canonical formulation, multisymplecticity gives rise to the usual field-theoretic notion of symplecticity (Marsden et al. [29]). Furthermore, multisymplecticity encompasses the notion of reciprocity in many physical systems, relating the infinitesimal perturbation of a system by a source and the associated infinitesimal perturbation of the response by the system (see, for example, Vankerschaver et al. [35] for Lorenz reciprocity in electromagnetism and McLachlan and Stern [31] for reciprocity in semilinear elliptic PDEs, within the context of multisymplecticity). Additionally, for wave propagation problems, multisymplecticity provides a geometric formulation for the conservation of wave action (Bridges [10, 11]). Since multisymplecticity is an important property of Lagrangian field theories encompassing many natural physical conservation laws, we will investigate multisymplecticity within our discretization framework.

In the literature, integrators which admit a discrete analogue of this formula are referred to as "multisymplectic integrators". We show that our discrete system (2.4) admits a discrete multisymplectic form formula. The main idea of the derivation for the multisymplectic form formula is to look at second variations of the action at ϕ with respect to first variations V and W, $d^2S[\phi]\cdot(V,W)=0$. More specifically, one decomposes the variation of the action into two functionals, corresponding to interior and boundary variations:

$$dS[\phi]\cdot V = \underbrace{-\int_U (j^1\phi)^*(j^1V \lrcorner\, \Omega_{\mathcal{L}})}_{\equiv \alpha_U(\phi)\cdot V} + \underbrace{\int_{\partial U} (j^1\phi)^*(j^1V \lrcorner\, \Theta_{\mathcal{L}})}_{=\Theta_U(\phi)\cdot V}.$$

Then, $0 = d^2S[\phi]\cdot(V,W) = d\alpha_U(\phi)\cdot(V,W) + d\Theta_U(\phi)\cdot(V,W)$. The term $d\alpha_U(\phi)\cdot(V,W)$ vanishes from the first variation property, so the multisymplectic form formula can be expressed

$$d\Theta_U(\phi)\cdot(V,W) = 0,$$

which is equivalent to equation (2.16).

In our construction, the main impediment for a discrete analogue of the multisymplectic form formula is that a solution of the discrete equation (2.4) does not in general satisfy an Euler–Lagrange equation locally (i.e., for arbitrary U) but rather integrated over a regular region U. Additionally, there is an additional contribution from the boundary components of the variation in the elements neighboring the boundary $T\in\mathcal{T}[\partial U]$. It is in this restricted setting that we have a discrete multisymplectic form formula.

Theorem 2.1. ***(Discrete Multisymplectic Form Formula)*** *Let U be a regular region and let ϕ_h be a solution of the local DEL (2.5) and $V,W\in\mathfrak{X}(\Lambda^k_h)$ be first variations for ϕ_h (i.e., their flow on ϕ_h still satisfies the DEL, but for arbitrary boundary variations), then*

$$d\Theta^h_U(\phi_h)\cdot(V,W) = 0. \tag{2.17}$$

Proof. Decompose the variation of the action into interior and boundary variations,

$$dS[\phi_h]\cdot V = \underbrace{-\int_U (j^1\phi_h)^*(j^1V_{in} \lrcorner\, \Omega_{\mathcal{L}})}_{\equiv \alpha^h_U(\phi_h)\cdot V_{in}} \underbrace{- \sum_{T\in\mathcal{T}[\partial U]}\int_T (j^1\phi_h)^*(j^1V_{\partial} \lrcorner\, \Omega_{\mathcal{L}}) + \int_{\partial U} (j^1\phi_h)^*(j^1V \lrcorner\, \Theta_{\mathcal{L}})}_{=\Theta^h_U(\phi_h)\cdot V},$$

so that $dS[\phi_h]\cdot V = \alpha^h_U(\phi_h)\cdot V + \Theta^h_U(\phi_h)\cdot V$ and hence,

$$0 = d^2S[\phi_h]\cdot(V,W) = d\alpha^h_U(\phi_h)\cdot(V,W) + d\Theta^h_U(\phi_h)\cdot(V,W).$$

Define a discrete first variation as a vector field B whose flow preserves the discrete Euler–Lagrange equation (2.3), i.e., $d(\delta S[\phi_h]\cdot a)\cdot B = 0$ for any $a\in\mathring{\Lambda}^k_h$; equivalently, this can be expressed as $d(\alpha^h_U(\phi_h)\cdot A)\cdot B$ for any $A\in\mathfrak{X}(\mathring{\Lambda}^k_h)$. Then, express

$$d\alpha_U(\phi_h)\cdot(V,W) = d(\alpha_U(\phi_h)\cdot V_{in})\cdot W - d(\alpha_U(\phi_h)\cdot W_{in})\cdot V - \alpha_U(\phi_h)\cdot[V_{in},W_{in}].$$

The first two terms on the right hand side of the above equation vanish by the definition of first variation; furthermore, the third term vanishes by the fact that $V_{in},W_{in}\in\mathfrak{X}(\mathring{\Lambda}^k_h)$ implies $[V_{in},W_{in}]\in\mathfrak{X}(\mathring{\Lambda}^k_h)$ and the fact that $\alpha_U(\phi_h)$ contains $\mathfrak{X}(\mathring{\Lambda}^k_h)$ in its kernel (by the discrete Euler–Lagrange equation). Hence, $d\alpha^h_U(\phi_h)\cdot(V,W)=0$. Thus, we have

$$d\Theta^h_U(\phi_h)\cdot(V,W) = d^2S[\phi_h]\cdot(V,W) = 0.$$

□

Remark 2.3. *Although we immediately see that the discrete multisymplectic form formula* $d\Theta_U^h(\phi_h)\cdot(V,W)=0$ *is in direct analogy with the continuum multisymplectic form formula* $d\Theta_U(\phi)\cdot(V,W)=0$*, if we write the discrete formula using the definition of the discrete Cartan form, we see that there is an additional contribution corresponding to the integration over elements* $T\in\mathcal{T}[\partial U]$*. Although we will not write this out explicitly, we see that this additional contribution involves the linearization of the quantity* $\sum_{T\in\mathcal{T}[\partial U]}\int_T(j^1\phi_h)^*(j^1V_\partial\lrcorner\,\Omega_\mathcal{L})$ *by* W *(and, vice versa, the linearization of* $\sum_{T\in\mathcal{T}[\partial U]}\int_T(j^1\phi_h)^*(j^1W_\partial\lrcorner\,\Omega_\mathcal{L})$ *by* V*). Since,* $\sum_{T\in\mathcal{T}[\partial U]}\int_T$ *is* $O(h)$ *as discussed previously, we only need control of the residual associated to the linearized equations to formally show convergence of the discrete multisymplectic form formula to the continuum multisymplectic form formula.*

We note that the aforementioned convergence is formal since it must also be combined appropriately with convergence of the discrete solution to a continuum weak solution using bounds on the projection. One possible method for combining these is the following observation. Since, by assumption, the projections are cochain projections, we have

$$d^2S_h[\phi]\cdot(V,W)=d^2(\pi_h^*S)[\phi]\cdot(V,W)=d^2S[\pi_h\phi]\cdot(T\pi_h\cdot V,T\pi_h\cdot W).$$

In particular, for first variations $V,W\in\mathfrak{X}(Y)$ *for the degenerate action,* $T\pi_h\cdot V,T\pi_h\cdot W$ *correspond to first variations of the discrete Euler–Lagrange equations, and the discrete multisymplectic form formula can be reinterpreted as the multisymplectic form formula for the degenerate action. Note also that for cochain projections, a simple calculation shows that* $j^1(T\pi_h\cdot V)=T(\pi_h^k\times\pi_h^{k+1})\cdot j^1V$*, so that the terms in the integrand of the discrete multisymplectic form formula, (2.17), are in the image of the (tangent) projections. This allows us to formulate the discrete multisymplectic formula in terms of the projection and its tangent lift, and hence more directly determine in what sense the discrete multisymplectic form formula converges as* $h\to 0$*.*

Of course, without specifying a particular field theory and finite element spaces, we cannot proceed further to show convergence. We aim to investigate more rigorous convergence results for particular field theories in future work. See also the discussion below regarding convergence of the discrete Noether theorem to its continuum analogue.

Remark 2.4. *As noted before, the discrete Cartan form, in the case of nodal interpolating shape functions, gives precisely the discrete notion of Cartan form introduced in Marsden et al. [29]. In this case, our discrete multisymplectic form formula* $d\Theta_U^h(\phi_h)\cdot(V,W)=0$ *(for first variations* V,W*) gives precisely the discrete multisymplectic form formula derived in Marsden et al. [29].*

2.3. Discrete Noether's Theorem.

In this section, we will discuss the covariant momentum map structure associated to the discrete Lagrangian structure and establish discrete analogues of Noether's theorem.

Let G be a Lie group with $\mathfrak{g}:=T_eG$ its Lie algebra. Suppose G has a smooth group action on Y via vertical bundle automorphisms; this induces a lifted action of G on the associated jet bundle space. Associated to $\eta\in G$ are its actions η_Y,η_{J^1Y} on the respective spaces (see Gotay et al. [15]). We say that a Lagrangian density $\mathcal{L}$ is G-invariant if

$$\mathcal{L}(\eta_{J^1Y}\gamma)=\mathcal{L}(\gamma), \tag{2.18}$$

for each $\eta\in G$, $\gamma\in J^1_xY$ (for every $x\in X$). Infinitesimally, $\delta\mathcal{L}(\gamma)\cdot\xi=0$, for all $\xi\in\mathfrak{g}$.

Given such a G-invariant Lagrangian, there exists a covariant momentum map $J^\mathcal{L}:J^1Y\to\mathfrak{g}^*\otimes\Lambda^n(J^1Y)$ given by

$$\langle J^\mathcal{L},\xi\rangle=\xi_{J^1Y}\lrcorner\,\Theta_\mathcal{L},$$

where the duality pairing is between $\mathfrak{g}$ and its dual, and for $\xi\in\mathfrak{g}$, ξ_{J^1Y} is the associated infinitesimal generator. This covariant momentum map satisfies Noether's theorem, which implies a divergence form conservation law

$$d[(j^1\phi)^*\langle J^{\mathcal{L}}, \xi\rangle] = 0, \tag{2.19}$$

for sections ϕ of the configuration bundle satisfying the associated Euler–Lagrange equations. We can express equation (2.19) in integral form as

$$\Theta_U(\phi) \cdot \xi_Y = \int_{\partial U} (j^1\phi)^*(\xi_{J^1Y} \lrcorner\, \Theta_{\mathcal{L}}) = 0. \tag{2.20}$$

Remark 2.5. *More generally, one can look at symmetries of the action up to a boundary term. In this context, we say that a group G is a symmetry for the theory if there exists $K : J^1(Y) \to \mathfrak{g}^* \otimes \Lambda^n J^1(Y)$ such that $\delta\mathcal{L}(\gamma) \cdot \xi = \langle K, \xi\rangle$ for all $\xi \in \mathfrak{g}$; i.e., the action transforms infinitesimally up to a total derivative. In this case, there exists an inhomogeneous momentum map $\langle J^{\mathcal{L}}, \xi\rangle = \xi_{J^1Y} \lrcorner\, \Theta_{\mathcal{L}} - \langle K, \xi\rangle$. Analogous statements of the following discussion follow in this case.*

In the discrete setting, we would like to consider to what extent equations (2.19) and (2.20) hold. To find a discrete analogue, note that

$$d[(j^1\phi)^*\langle J^{\mathcal{L}}, \xi\rangle] = d[(j^1\phi)^*(\xi_{J^1Y} \lrcorner\, \Theta_{\mathcal{L}})] = \Big(\frac{\delta L}{\delta \phi^A}(\pounds_\xi \phi)^A + \delta_\xi L\Big)(j^1\phi) d^{n+1}x,$$

where $\pounds_\xi\phi = T\phi \circ \xi_X - \xi_Y \circ \phi \in \mathfrak{X}(Y)$ (note $\xi_X = 0$ since we assume G acts vertically, but we write the general definition since Proposition 2.3 still holds in this setting), $\delta L/\delta\phi^A$ is the total variation of the Lagrangian with respect to ϕ, which vanishes when ϕ is a solution of the Euler–Lagrange equation, and $\delta_\xi L$ is the variation induced by the infinitesimal action which vanishes when $\mathcal{L}$ is section-equivariant. We see that the main obstacle is that for a solution $\phi \in \Lambda^k_h$ of the discrete equations (2.4), the right hand side does not vanish unless integrated over U and unless the variation field $\pounds_\xi\phi \in \Lambda^k_h$. Keeping this in mind, we state the following discrete analogue of Noether's theorem.

Theorem 2.2 (Discrete Noether's Theorem)**.** *Let U be a regular region. Let $\mathcal{L}$ be G-invariant. Then, for a discrete solution $\phi_h \in \Lambda^k_h$ of (2.4) and $\xi \in \mathfrak{g}$,*

$$\begin{aligned}\int_U \Big(EL(j^1\phi_h) \wedge \star(T\pi_h \cdot \pounds_\xi\phi_h)(\phi_h) + \delta_\xi L(j^1\phi_h)\Big) d^{n+1}x \\ - \sum_{T \in \mathcal{T}[\partial U]} \int_T EL(j^1\phi_h) \wedge \star(T\pi_h \cdot \pounds_\xi\phi_h)(\phi_h)_\partial = 0,\end{aligned} \tag{2.21a}$$

where $EL := \partial_2\mathcal{L} + d^*\partial_3\mathcal{L}$ and $T\pi_h$ is the lift of the projection. Furthermore, if the vertical component of $\pounds_\xi\phi_h$ is in Λ^k_h, then

$$\int_U d[(j^1\phi_h)^*\langle J^{\mathcal{L}}, \xi\rangle] - \sum_{T \in \mathcal{T}[\partial U]} \int_T EL(j^1\phi) \wedge \star(\pounds_\xi\phi_h)(\phi_h)_\partial = 0. \tag{2.21b}$$

Proof. The first equation is simply re-expressing the DEL $dS[\phi_h] \cdot V_{in} = 0$ (where $V = T\pi_h \cdot \pounds_\xi\phi_h$) in terms of V and V_∂, via $V_{in} = V - V_\partial$.

The second equation (2.21*b*) follows by the above discussion (expressing $d[(j^1\phi)^*\langle J^{\mathcal{L}}, \xi\rangle]$ in terms of the variations of L) and (2.21*a*) since the projection leaves the subspace invariant. □

Of course, the stronger statement (2.21b) resembles the divergence form of Noether's theorem more closely than (2.21a), which requires the vertical component of $\pounds_\xi\phi_h$ to be in Λ^k_h. The next proposition gives a sufficient condition for when this requirement holds.

Proposition 2.3. *Let π_h be an equivariant map with respect to the action of G on the configuration bundle. Then, for any section ϕ of the configuration bundle and $\xi \in \mathfrak{g}$,*

$$\pounds_\xi(\pi_h\phi) = T\pi_h \cdot \pounds_\xi\phi. \tag{2.22}$$

In particular, for $\phi_h \in \Lambda^k_h$, $\pounds_\xi\phi_h = T\pi_h \cdot \pounds_\xi\phi_h$ and so in this case, (2.21a) is equivalent to (2.21b).

Proof. For the first statement, compute

$$T\pi_h \cdot \pounds_\xi\phi = T\pi_h \cdot (T\phi \circ \xi_X) - T\pi_h \cdot (\xi_Y \circ \phi).$$

Note $T\pi_h \cdot (T\phi \circ \xi_X) = T(\pi_h\phi) \circ \xi_X$. Furthermore, by the equivariance of π_h,

$$T\pi_h \cdot (\xi_Y \circ \phi) = \frac{d}{dt}\Big|_{t=0} \pi_h(e^{t\xi} \cdot \phi) = \frac{d}{dt}\Big|_{t=0} e^{t\xi} \cdot (\pi_h\phi) = \xi_Y \circ (\pi_h\phi).$$

Hence, $T\pi_h \cdot \pounds_\xi\phi = T(\pi_h\phi) \circ X - \xi_Y \circ \pi_h\phi = \pounds_\xi(\pi_h\phi)$. The statement for $\phi_h \in \Lambda^k_h$ follows immediately since the projection π_h acts invariantly on Λ^k_h. □

The above proposition shows that the projection being G-equivariant fits naturally into the variational principle, since the variation induced by the Lie algebra action is associated to a discrete variation (in other words, since $g \cdot (\pi_h Y) = \pi_h(g \cdot Y)$, the group action restricts to the discrete field space). To see this naturality with the variational principle more explicitly, one can derive a discrete Noether theorem in the case of a G-equivariant projection as follows. G-invariance of the Lagrangian implies G-invariance of the action, $S_U[G \cdot \phi] = S_U[\phi]$. Infinitesimally, $dS_U[\phi] \cdot \xi_Y = 0$. Assuming π_h is G-equivariant, the flow of ξ_Y appropriately restricts to Λ^k_h, so we can view $\xi_Y \in \mathfrak{X}(\Lambda^k_h)$. Hence, computing the variation (using equation (2.10a)),

$$\begin{aligned} 0 = dS_U[\phi_h] \cdot \xi_Y &= \int_U (j^1\phi_h)^*(j^1\xi_Y \lrcorner\, \Omega_\mathcal{L}) + \int_{\partial U} (j^1\phi_h)^*(j^1\xi_Y \lrcorner\, \Theta_\mathcal{L}) \\ &= \sum_{T \in \mathcal{T}[\partial U]} \int_T (j^1\phi_h)^*((j^1\xi_Y)_\partial \lrcorner\, \Omega_\mathcal{L}) + \int_{\partial U} (j^1\phi_h)^*(j^1\xi_Y \lrcorner\, \Theta_\mathcal{L}) = \Theta^h_U(\phi_h) \cdot \xi_Y, \end{aligned}$$

Thus, in the case of a G-equivariant projection, the discrete Noether conservation law (2.21b) can be expressed in terms of the discrete Cartan form as $\Theta^h_U(\phi_h) \cdot \xi_Y = 0$, in analogy with the continuum Noether conservation law (2.20).

This form of the discrete Noether theorem reproduces the discrete Noether theorem of Marsden et al. [29], in the case of nodal interpolating shape functions. Note, however, that Marsden et al. [29] make the assumption that the group G acts on the nodal field values in such a way that the Lagrangian is G-invariant; from our perspective, this is just a particular case of G-equivariance of the projection arising from constructing the projection by using G-equivariant nodal interpolants.

Remark 2.6. *One can weaken the assumption that the projection is equivariant. First, note that the projection does not need to be fully equivariant for Proposition 2.3 to hold. Rather, the projection only needs to be infinitesimally equivariant, $\pi_h(e^{t\xi} \cdot \phi) - e^{t\xi} \cdot \pi_h\phi = o(t)$ for all $\xi \in \mathfrak{g}$. This weakened condition may be useful for constructing equivariant projections, since the equivariance only needs to hold to a sufficient order.*

Furthermore, the group equivariance $\pi_h(g \cdot \phi) = g \cdot \pi_h\phi$ can be weakened to $\pi_h(g \cdot \phi) = \psi_h(g) \cdot \pi_h\phi$ where $\psi_h : G \to G$ is a differentiable group homomorphism. Since ψ_h is differentiable, it induces a Lie algebra homomorphism $\tilde{\psi}_h$, and the above discrete Noether theorem $\Theta^h_U(j^1\phi_h) \cdot \xi_Y = 0$ can be replaced by $\Theta^h_U(j^1\phi_h) \cdot \tilde{\psi}_h(\xi)_Y = 0$. As with the other remark on weakening the notion of equivariance, this weakened notion also allows one to construct more general equivariant projections.

We give two simple examples of group-equivariant cochain projections and subsequently remark on how one might construct more general group-equivariant cochain projections.

Example 2.3 (Global Linear Group Action)**.** *First, note that although we took our field configuration bundle to be $\Lambda^k(X)$, we could have more generally taken our fields to be vector-valued forms, corresponding to the bundle $\Lambda^k(X) \otimes V$ for some finite-dimensional vector space V. With a basis $\{e_i\}$ for V, the only modification to the discrete Euler–Lagrange (2.4) equation is that there are $\dim(V)$ equations corresponding to each component of the field $\phi^i \in \Lambda^k(X)$ in the expansion $\phi(x) = \sum_i \phi_i(x) \otimes e_i$.*

Suppose a Lagrangian with such a configuration bundle is invariant under the global action by a group representation $D : G \to GL(V)$. That is, D acts on $\phi \in \Lambda^k(X) \otimes V$ as $1_{\Lambda^k(X)} \otimes D$:

$$D(g)\phi(x) = \sum_i \phi_i(x) \otimes (D(g)e_i),$$

where $D(g)$ is independent of x.

Let $\pi_h^k : H\Lambda^k \to \Lambda_h^k$ and $\pi_h^{k+1} : H\Lambda^{k+1} \to \Lambda_h^k$ be cochain projections, i.e., satisfying $\pi_h^{k+1} d = d\pi_h^k$. We can extend these to cochain projections on vector-valued forms by $\tilde{\pi}_h = \pi_h \otimes 1_V$. Furthermore, group-equivariance follows from linearity of the group action and the above definitions,

$$\begin{aligned} D(g)\tilde{\pi}_h\phi = D(g)\tilde{\pi}_h \left(\sum_i \phi_i \otimes e_i \right) &= D(g) \sum_i \pi_h(\phi_i) \otimes e_i = \sum_i \pi_h(\phi_i) \otimes D(g)e_i \\ = \tilde{\pi}_h \left(\sum_i \phi_i \otimes D(g)e_i \right) &= \tilde{\pi}_h \left(D(g) \sum_i \phi_i \otimes e_i \right) = \tilde{\pi}_h D(g)\phi. \end{aligned}$$

A simple example of such a theory is the Schrödinger equation with $V = \mathbb{C}$, $G = U(1)$, and the group representation given by the fundamental representation of $U(1)$ in $GL(\mathbb{C})$. The corresponding conservation law is conservation of mass in the L^2 norm.

Example 2.4 (Yang–Mills Theory)**.** *As an example of a non-global (but still linear) group action, consider Yang–Mills theories with a structure group G. In this setting, the field $A \in \Lambda^1(X) \otimes \mathfrak{g}$; i.e., A is valued in the Lie algebra $\mathfrak{g}$ associated to G (more precisely, the field is valued in the adjoint representation of the Lie algebra). This class of theories is invariant under the linear action of $\Lambda^0(X) \otimes \mathfrak{g}$ (viewed as a group under addition) on $\Lambda^1(X) \otimes \mathfrak{g}$ given by*

$$D(\alpha)A = A + d\alpha,$$

for any $\alpha \in \Lambda^0(X) \otimes \mathfrak{g}$. Unlike the previous example, this action is local in the sense that $D(\alpha)$ depends on the position in spacetime.

Now, suppose that we have cochain projections for the sequence $H\Lambda^0 \xrightarrow{d} H\Lambda^1 \xrightarrow{d} H\Lambda^2$; that is, $\pi_h^2 d = d\pi_h^1, \pi_h^1 d = d\pi_h^0$. Extend these to projections $\tilde{\pi}_h$ on $H\Lambda \otimes \mathfrak{g}$ as in the previous example. The relation $\tilde{\pi}_h^2 d = d\tilde{\pi}_h^1$ is required for naturality of the variational structure. On the other hand, the relation $\tilde{\pi}_h^1 d = d\tilde{\pi}_h^0$ gives (weakened) group equivariance, in the following sense:

$$\tilde{\pi}_h^1(D(\alpha)A) = \tilde{\pi}_h^1(A + d\alpha) = \tilde{\pi}_h^1 A + \tilde{\pi}_h^1 d\alpha = \tilde{\pi}_h^1 A + d\tilde{\pi}_h^0 \alpha = D(\tilde{\pi}_h^0 \alpha)\tilde{\pi}_h^1 A.$$

Thus, as discussed in Remark 2.6, this is an example of weakened group equivariance where, rather than full group equivariance, there is an intertwining homomorphism ψ_h from the acting group to itself. In this case, the intertwining homomorphism is $\psi_h = \tilde{\pi}_h^0$.

In the continuum Hilbert space setting, the associated conservation law is the weak Gauss' law, where Gauss' law holds tested against any element of the Hilbert space. In the discrete setting, the *discrete Noether's theorem gives a discrete Gauss' law, where Gauss' law holds tested against any element of the finite-dimensional subspace.*

The previous two examples were simple in the sense that they had a linear or global group action. Although the second example was local, the acting group is contained in the Hilbert complex of forms and group-equivariance arose from having cochain projections.

To construct group-equivariant cochain projections for more general actions, one possible method would be to utilize group-equivariant interpolation [14; 25] in constructing the projection. One method to construct cochain projections from interpolants is to place an intermediate sequence between the sequence of Hilbert spaces and the sequence of finite-dimensional subspaces,

$$\begin{array}{ccc} H\Lambda^k & \xrightarrow{\;d\;} & H\Lambda^{k+1} \\ \Big\downarrow{\scriptstyle \sigma^k} & & \Big\downarrow{\scriptstyle \sigma^{k+1}} \\ C^k & \xrightarrow{\;D\;} & C^{k+1} \\ \Big\downarrow{\scriptstyle \mathcal{I}^k} & & \Big\downarrow{\scriptstyle \mathcal{I}^{k+1}} \\ \Lambda^k_h & \xrightarrow{\;d\;} & \Lambda^{k+1}_h, \end{array}$$

where $\{\sigma^m\}$ are the degrees of freedom mapping into the coefficient spaces $\{C^m\}$, $\{\mathcal{I}^m\}$ are interpolants from the coefficient spaces into the finite-dimensional subspaces, D realizes d in the coefficient space, and the projections are defined by $\pi_h = \mathcal{I} \circ \sigma$. The degrees of freedom must be unisolvent when restricted to the image of the interpolants. Constructing cochain projections amounts to ensuring that the top diagram commutes. Then, fixing group-equivariant interpolants $\mathcal{I}^k, \mathcal{I}^{k+1}$, group-equivariant cochain projections could be achieved by choosing the degrees of freedom such that they are unisolvent for this choice of interpolants and ensuring that the top diagram commutes. We will pursue such a construction in future work.

To conclude this section, we expand on the discussion in Section 2.1 regarding how the discrete Cartan form converges to the continuum Cartan form. Of course, without specifying a specific theory (i.e., Lagrangian), we cannot completely derive rigorous bounds, but we will sketch the process. Suppose we have a solution ϕ_h of the DEL converging to a solution ϕ of the Euler–Lagrange equations. As before, denote $EL = \partial_2\mathcal{L} + d^*\partial_3\mathcal{L}$. Then, the terms involving integration over $T \in \mathcal{T}[\partial U]$ can be bounded as

$$\begin{aligned} \sum_{T\in\mathcal{T}[\partial U]} \Big| \int_T (\partial_2\mathcal{L}(j^1\phi_h) + d^*\partial_3\mathcal{L}(j^1\phi_h)) \wedge \star V(\phi_h)_\partial \Big| &\leq \sum_{T\in\mathcal{T}[\partial U]} \|EL(j^1\phi_h)\|_{L^2(T)} \|V(\phi_h)_\partial\|_{L^2(T)} \\ &= \sum_{T\in\mathcal{T}[\partial U]} \|EL(j^1\phi_h) - EL(j^1\phi)\|_{L^2(T)} \|V(\phi_h)_\partial\|_{L^2(T)} \\ &\leq \underbrace{\sum_{T\in\mathcal{T}[\partial U]}}_{\lesssim h^{-n}} \underbrace{\|EL(j^1\phi_h) - EL(j^1\phi)\|_{L^2(T)}}_{\lesssim h^{n+1}\|EL(j^1\phi_h)-EL(j^1\phi)\|_{L^2(X)}} \underbrace{\Big(\|V(\phi_h)_\partial - V(\phi)_\partial\|_{L^2(T)} + \|V(\phi)_\partial\|_{L^2(T)}\Big)}_{\lesssim \|V(\phi_h)-V(\phi)\|_{L^2(X)} + \|V(\phi)\|_{L^2(X)} \sim O(1)}, \end{aligned}$$

so we see this contribution converges to zero at least linearly in h, assuming that the residuals $\|EL(j^1\phi_h) - EL(j^1\phi)\|_{L^2(X)}$ and $\|V(\phi_h) - V(\phi)\|_{L^2(X)}$ are at least $O(1)$, although it is often the case that the residuals will be $O(h^s)$ for some $s > 0$, in which case we get $O(h^{1+s})$ convergence. In particular, given a specific theory and symmetry group, one can apply this to the discrete Noether's theorem with $V = \xi_Y$ to show that the discrete conservation law converges to the continuum conservation law (involving only an integral over ∂U).

2.3.1. *Bounds for Noether's Theorem.* In the previous part, we showed that solutions to the discrete Euler–Lagrange equations admit a discrete analogue of Noether's theorem, which holds over regular regions U. Of course, this discrete law cannot be localized smaller than a single element. So, in this part, we determine a bound on the conservation law, which will allow us to establish the sense in which $\|d(j^1\phi_h)^*\langle J^{\mathcal{L}}, \xi\rangle\| \approx 0$ throughout the whole domain. We defer the proofs of the following statements to Appendix A.

So far, we have only seen that the Noether current evaluated on a discrete solution satisfies a global divergence-like equality (which, as previously remarked, can be localized down to the size of a single element). However, there should be some sense in which

$$\|d(j^1\phi)^*\langle J^{\mathcal{L}}, \xi\rangle\| \approx 0.$$

To arrive at this conclusion, we determine a bound on the divergence conservation law of the Noether current of the action S for a discrete solution. To answer this, we could directly measure $d[(j^1\phi_h)^*\langle J^{\mathcal{L}}, \xi\rangle]$ in $L^2\Lambda^{n+1}(X)$ (where ϕ_h satisfies the discrete Euler–Lagrange equations), but due to the derivative, this requires assuming $\phi^\xi, (\phi_h)^\xi \in H\Lambda^k$, and $(j^1\phi)^*\partial_3\mathcal{L}, (j^1\phi_h)^*\partial_3\mathcal{L} \in H^*\Lambda^{k+1}$ (H^* denotes square integrable with square integrable coderivative). Instead, by duality, we could weakly measure the conservation law by its action on forms with square integrable coderivatives via the following operator norm, which only requires the aforementioned terms to be square integrable. Let $\omega \in H\Lambda^n$; we can view $d\omega \in L^2\Lambda^{n+1}$ as a linear functional on $H^*\Lambda^{n+1}$ via the pairing

$$(d\omega, \alpha)_{L^2\Lambda^{n+1}(X)} = (\omega, d^*\alpha)_{L^2\Lambda^n(X)} + \int_{\partial X} \omega \wedge \star\alpha,$$

for $\alpha \in H^*\Lambda^{n+1}X$. Then, we define the following operator norms

$$\|d\omega\|^* = \sup_{\alpha \in H^*\Lambda^{n+1}\setminus\{0\}} \frac{|(d\omega, \alpha)_{L^2\Lambda^{n+1}}|}{\|\alpha\|_{H^*\Lambda^{n+1}}},$$

$$\|d\omega\|_0^* = \sup_{\alpha \in H_0^*\Lambda^{n+1}\setminus\{0\}} \frac{|(d\omega, \alpha)_{L^2\Lambda^{n+1}}|}{\|\alpha\|_{H^*\Lambda^{n+1}}},$$

where the supremum in the second norm is restricted to forms which vanish on the boundary. These gives the following set of bounds, where we view the solution ϕ of the full Euler–Lagrange equation as a constant and the arbitrary field ψ as varying. For simplicity, we denote $J(\psi, \xi) := (j^1\psi)^*\langle J^{\mathcal{L}}, \xi\rangle$ for a section ψ of the configuration bundle.

Proposition 2.4. *Let J be the covariant momentum map associated to the symmetry group G for the Lagrangian structure defined by $\mathcal{L}$, and suppose $J(\psi, \xi) \in H\Lambda^n$ for arbitrary ψ,ξ. Let ϕ be a solution to the Euler–Lagrange equation for $\mathcal{L}$ and for each $h > 0$, let ϕ_h be a solution of the discrete Euler–Lagrange equation such that $\phi_h \to \phi$ in $H\Lambda^k$. Then, for each $h > 0, \xi \in \mathfrak{g}$,*

$$\begin{aligned} &\|J(\phi_h, \xi) - J(\phi, \xi)\|_{L^2\Lambda^n} + \|dJ(\phi_h, \xi)\|_0^* \\ &\quad\lesssim \|\xi_X\|_{L^2\mathfrak{X}} \|\mathcal{L}(j^1\phi_h) - \mathcal{L}(j^1\phi)\|_{L^2\Lambda^{n+1}} \\ &\qquad + \|(\phi_h)^\xi - \phi^\xi\|_{L^2\Lambda^k} + \|\partial_3\mathcal{L}(j^1\phi_h) - \partial_3\mathcal{L}(j^1\phi)\|_{L^2\Lambda^{k+1}} \end{aligned} \tag{2.23a}$$

and

$$\lim_{h\to 0}\left(\|J(\phi_h, \xi) - J(\phi, \xi)\|_{L^2\Lambda^n} + \|dJ(\phi_h, \xi)\|_0^*\right) = 0. \tag{2.23b}$$

Remark 2.7. *In the above proposition, we assumed explicitly that ϕ_h converged to ϕ, which allows more generally for the Lagrangian to be degenerate (e.g. due to gauge freedom). .*

The above bound uses the $\|\cdot\|_0^*$ norm for estimating $dJ(\phi_h, \xi)$ and hence does not give information about the size of $J(\phi_h, \xi)$ at the boundary (an approximate solution should have $J(\phi_h, \xi) \approx J(\phi, \xi)$ along ∂X and hence the total flux of $J(\phi_h, \xi)$ through the boundary is approximately zero). In the subsequent bound, we show that the norm of the boundary terms can be bounded with norm of fractionally many derivatives of $J(\phi_h, \xi) - J(\phi, \xi)$ on the interior.

Proposition 2.5. *Assume as in Proposition 2.4. Additionally assume X is a Lipschitz domain and let $\delta \in (0, 1/2]$. Then,*

$$\| TrJ(\phi_h, \xi) - TrJ(\phi, \xi)\|_{H^\delta \Lambda^n(\partial X)} + \|dJ(\phi_h, \xi)\|^* \lesssim \|J(\phi_h, \xi) - J(\phi, \xi)\|_{H^{\frac{1}{2}+\delta}\Lambda^n(X)} \tag{2.24}$$

2.4. **A Discrete Variational Complex.** The variational bicomplex is a double complex on the spaces of differential forms over the jet bundle of a configuration bundle used to study the variational structures of Lagrangian field theories defined on this bundle (see, for example, Anderson [3]). The differential forms arising in Lagrangian field theory, such as the Lagrangian density, the Cartan form, and the multisymplectic form, can be interpreted as elements of this variational bicomplex. The cochain maps in this double complex are the horizontal and vertical exterior derivatives on the jet bundle, which give a geometric interpretation to the variations encountered in Lagrangian field theories. The variational bicomplex has also been extended to problems with symmetry in Kogan and Olver [23], and to the discrete setting for difference equations corresponding to discretizing Lagrangian field theories on a lattice in Hydon and Mansfield [22].

In this section, we interpret and summarize the results from the previous sections in terms of a discrete variational complex which arises naturally in our discrete construction and, in a sense, resembles the vertical direction of the variational bicomplex.

In our previous discussion, we saw a complex which arises from the space of discrete forms,

$$\Lambda_h^0 \xrightarrow{d} \Lambda_h^1 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda_h^n \xrightarrow{d} \Lambda_h^{n+1},$$

which forms a complex due to the cochain projection property. Now, consider instead the following "vertical" complex; consider the spaces of smooth forms on Λ_h^k, which we denote $\Omega(\Lambda_h^k)$, with the "vertical" exterior derivative $d_v : \Omega^m(\Lambda_h^k) \to \Omega^{m+1}(\Lambda_h^k)$ being the usual exterior derivative over the base manifold Λ_h^k (which is a vector space). This gives a discrete variational complex:

$$\begin{array}{c} \Omega^{\dim(\Lambda_h^k)}(\Lambda_h^k) \\ d_v \big\uparrow \\ \vdots \\ d_v \big\uparrow \\ \Omega^1(\Lambda_h^k) \\ d_v \big\uparrow \\ \Omega^0(\Lambda_h^k) \ . \end{array}$$

Note that in the previous sections, we used d to denote both the exterior derivative corresponding to the de Rham complex and the vertical exterior derivative (e.g., the multisymplectic form formula

$d\Theta_U^h(V,W) = 0$ is more precisely $d_v\Theta_U^h(V,W) = 0$), where it was understood which was meant by the spaces where the relevant quantities were defined; however, we will distinguish the two in this section to be more precise. We call the above a vertical complex for two reasons: first, the vertical exterior derivative corresponds to differentiation with respect to the fiber values (as we will see below); furthermore, it resembles the vertical direction of the variational bicomplex. However, in our construction, there is no horizontal direction, since in the discrete setting, we are considering transgressed forms (forms integrated over a region, e.g., in the definition of the action, discrete Cartan form, and discrete multisymplectic form), so the horizontal direction collapses.

Examples of forms in the discrete variational complex include the action (restricted to Λ_h^k) $S \in \Omega^0(\Lambda_h^k)$, the discrete Cartan form $\Theta^h \in \Omega^1(\Lambda_h^k)$, and the discrete multisymplectic form $d_v\Theta^h \in \Omega^2(\Lambda_h^k)$. Let $\{v_i\}$ be a basis for Λ_h^k; we then coordinatize the vector space Λ_h^k by the components of the expansion of any $\phi = \sum_i \phi^i v_i \in \Lambda_h^k$, which we denote as a vector $(\phi^i) = (\phi^0, \dots, \phi^{\dim(\Lambda_h^k)}) \in \Lambda_h^k$. For example, the vertical exterior derivative of the action is

$$d_v S[\phi] = \sum_j \frac{\partial S[(\phi^i)]}{\partial \phi^j} d_v \phi^j.$$

The naturality of the variational principle and the interpretation of the weak Euler–Lagrange equations as a Galerkin variational integrator, discussed in Section 2.1, relate the vertical exterior derivative of S to the variation of the degenerate action S_h. Now, let Π_i be the projection onto the i^{th} coordinate ϕ^i and let $\mathcal{I}[\partial U]$ denote the set of indices i such that v_i has nonvanishing trace on ∂U. Then, for $v = (v^i) \in \Lambda_h^k$, we have

$$\begin{aligned} v_\partial &= \sum_{i \in \mathcal{I}[\partial U]} \Pi_i(v), \\ v_{in} &= v - v_\partial = \sum_{i \notin \mathcal{I}[\partial U]} \Pi_i(v). \end{aligned}$$

Recall that we can view vector fields $V \in \mathfrak{X}(\Lambda_h^k)$ as maps $V : \Lambda_h^k \to \Lambda_h^k$, and we extend this to the vector fields $V_\partial(\phi) \equiv (V(\phi))_\partial$ and $V_{in}(\phi) \equiv (V(\phi))_{in}$. In particular, the discrete Cartan form in this notation is

$$\Theta^h(\phi) \cdot V = d_v S[\phi] \cdot V_\partial.$$

The variation of the action can then be expressed as

$$d_v S[\phi] \cdot V = \mathrm{EL}(\phi) \cdot V + \Theta^h(\phi) \cdot V,$$

where the Euler–Lagrange one-form is defined by $\mathrm{EL}(\phi) \cdot V = d_v S[\phi] \cdot V_{in}$. More explicitly, these can be expressed as

$$\begin{aligned} \Theta^h(\phi) &= \sum_{j \in \mathcal{I}[\partial U]} \frac{\partial S[(\phi^i)]}{\partial \phi^j} d_v \phi^j, \\ \mathrm{EL}(\phi) &= \sum_{j \notin \mathcal{I}[\partial U]} \frac{\partial S[(\phi^i)]}{\partial \phi^j} d_v \phi^j. \end{aligned}$$

In particular, the discrete Euler–Lagrange equations are given by null Euler–Lagrange condition, $\mathrm{EL}(\phi) = 0$ (that is, $\mathrm{EL}(\phi) \cdot V = 0$ for all V). Assuming a solution ϕ of the null Euler–Lagrange condition, we immediately see that

$$d_v S[\phi] \cdot V = \Theta^h(\phi) \cdot V,$$

and in particular, for a symmetry of the action $d_v S[\phi] \cdot \tilde{\xi} = 0$, we have the discrete Noether's theorem $\Theta^h(\phi) \cdot \tilde{\xi} = 0$. By taking the second exterior derivative of the action, we have

$$0 = d_v^2 S[\phi] = d_v \mathrm{EL}(\phi) + d_v \Theta^h(\phi).$$

The space of first variations (at ϕ) is precisely the kernel of the quadratic form $d_v \mathrm{EL}(\phi)$, so this gives the discrete multisymplectic form formula $d_v \Theta^h(\phi)(\cdot, \cdot) = 0$ when evaluated on first variations. Thus, the results of the previous sections can be concisely summarized in terms of the structure given by the discrete variational complex.

Furthermore, this framework also encompasses the discrete variational principle with quadrature, as discussed in Remark 2.2. Namely, from the discrete viewpoint, a discrete action is an element of $\Omega^0(\Lambda^k_h)$ and in particular, the discrete action with quadrature $\mathbb{S}$ from (2.12) is an element of $\Omega^0(\Lambda^k_h)$. Then, the variation of $\mathbb{S}$ can be decomposed into interior and boundary one-forms as before,

$$\begin{aligned} d_v \mathbb{S}[(\phi)] &= \mathbb{EL}(\phi) + \Theta^h(\phi), \\ \Theta^h(\phi) &= \sum_{j \in \mathcal{I}[\partial U]} \frac{\partial \mathbb{S}[(\phi^i)]}{\partial \phi^j} d_v \phi^j, \\ \mathbb{EL}(\phi) &= \sum_{j \notin \mathcal{I}[\partial U]} \frac{\partial \mathbb{S}[(\phi^i)]}{\partial \phi^j} d_v \phi^j. \end{aligned}$$

The discrete Euler–Lagrange equations with quadrature are given by the null Euler–Lagrange condition $\mathbb{EL}(\phi) = 0$, and subsequently, the discrete Noether's theorem and discrete multisymplectic form formula (in the case of quadrature) then follow analogously to before (where symmetries are with respect to $\mathbb{S}$ and the space of first variations at ϕ is the kernel of the quadratic form $d_v \mathbb{EL}(\phi)$).

3. Canonical Semi-discretization of Lagrangian Field Theories

Turning now to the canonical formalism of field theories, we assume that our $(n+1)$-dimensional spacetime X is globally hyperbolic; i.e., X contains a smooth Cauchy hypersurface Σ such that every infinite causal curve intersects Σ exactly once. It was shown in Bernal and Sánchez [9] that a globally hyperbolic spacetime is diffeomorphic to the product, $X \cong \mathbb{R} \times \Sigma$. Identifying X with the product, we have a slicing of the spacetime. Taking an interval $I \subseteq \mathbb{R}$, we have the spacelike embeddings

$$i_t : \Sigma \to X$$

for each $t \in I$, such that the images $\{\Sigma_t := i_t(\Sigma)\}_{t \in I}$ form a foliation of X.

We will assume our Lagrangian depends on time-dependent fields as $\mathcal{L}(x^\mu, \varphi, \dot{\varphi}, d\varphi)$, where the field $\varphi(t) \in H\Lambda^k(\Sigma_t)$ (denoted as φ as opposed to the full field ϕ) and the exterior derivative acts on $\Lambda^k(\Sigma_t)$ for each t.

We will discuss how a semi-discretization of the variational principle gives rise to finite-dimensional Lagrangian and Hamiltonian dynamical systems (see, for example, Abraham and Marsden [1]) and subsequently discuss how the energy-momentum map structure of a canonical field theory (see Gotay et al. [16]) is affected by semi-discretization.

3.1. Semi-discrete Euler–Lagrange Equations.

In this section, we formally derive the semi-discrete Euler–Lagrange equations. Given our $\Lambda^{n+1}(X)$-valued Lagrangian density, we can produce an instantaneous density by contracting with the generator of the slicing (the vector field whose flow advances time) and pulling back by the inclusion of Σ_t into X, which gives a $\Lambda^n(\Sigma_t)$-valued density, which we will still call $\mathcal{L}$. Alternatively, in coordinates where the density is $L\, dt \wedge V(t)$ and

$V(t)$ restricts to a volume form on Σ_t, then $\mathcal{L} = i_t^* L V(t)$. The action in the canonical framework is given by

$$S[\varphi] = \int_I dt \int_{\Sigma_t} \mathcal{L}(x^\mu, \varphi, \dot{\varphi}, d\varphi), \tag{3.1}$$

where $(x^\mu) = (t, x^1, \dots, x^n) = (t, x)$, and $x = (x^i)$ denotes spatial coordinates.

To derive a semi-discrete formulation of the Euler–Lagrange equations, instead of looking at arbitrary variations of the form $v(t,x)$, we instead consider variations of the form $u(t)v(x)$ where $v \in H\Lambda^k(\Sigma)$ and $u \in C_0^2(I, \mathbb{R})$. The basic idea of the semi-discrete formulation is to allow u to be arbitrary but restrict v to a finite-dimensional subspace Λ_h^k. As in the covariant case, in order to compute the variations formally without going through the Hamilton–Pontryagin principle, we will assume that the projections are cochain projections (with respect to the spatial exterior derivative d on Σ).

Assumption 3.1. *The projections $\pi_h^m : H\Lambda^m(\Sigma) \to \Lambda_h^m(\Sigma)$ are cochain projections; i.e., that $\pi_h^{k+1} d = d\pi_h^k$, with respect to $d : \Lambda^m(\Sigma) \to \Lambda^{m+1}(\Sigma)$.*

Remark 3.1. *Note that we assume a finite element discretization Λ_h^k of the fields on the reference space $H\Lambda^k(\Sigma)$ (with associated projection π_h). There are two ways to view the variations with respect to our slicing $\{\Sigma_t\}$. In one way, the field variation on the reference space $v \in \Lambda_h^k \subset H\Lambda^k(\Sigma)$ is pulled back to a field variation on a time slice $(i_t^{-1})^* v \in H\Lambda^k(\Sigma_t)$, where we restrict the embedding to its image $i_t : \Sigma \to \Sigma_t$. Alternatively, we can pull back forms on Σ_t to forms on Σ via i^* (e.g. the Lagrangian density and its derivatives) and perform any relevant integration over the reference space Σ. We will utilize the latter since in computation we prefer to work on one reference space. For simplicity, we will not explicitly write the pullbacks i_t^* but rather implicitly incorporate it into the spacetime dependence of the Lagrangian.*

Theorem 3.1. *The (semi-discrete) Euler–Lagrange equations corresponding to the variational principle $\delta S[\varphi] \cdot (uv) = 0$ for all $v \in \Lambda_h^k$ and $u \in C_0^2(I, \mathbb{R})$ are given by*

$$\frac{d}{dt}(\partial_3 \mathcal{L}, v)_{L^2\Lambda^k(\Sigma)} - (\partial_2 \mathcal{L}, v)_{L^2\Lambda^k(\Sigma)} - (\partial_4 \mathcal{L}, dv)_{L^2\Lambda^{k+1}(\Sigma)} = 0, \text{ for all } v \in \Lambda_h^k \text{ and } t \in I, \tag{3.2}$$

where $\mathcal{L}$ is evaluated at $(x^\mu, \varphi, \dot{\varphi}, d\varphi)$.

Proof. With $\mathcal{L}$ evaluated at $(x^\mu, \varphi, \dot{\varphi}, d\varphi)$ (and the integration pulled back to Σ), compute

$$\begin{aligned} 0 = \delta S[\varphi] \cdot (uv) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} S[\phi + \epsilon uv] \\ &= \int_I dt \int_\Sigma \Big[\partial_2 \mathcal{L} \wedge \star u(t) v + \partial_3 \mathcal{L} \wedge \star \dot{u}(t) v + \partial_4 \mathcal{L} \wedge \star u(t) dv\Big] \\ &= \int_I dt \Big[\int_\Sigma \Big(\partial_3 \mathcal{L} \wedge \star v\Big) \dot{u}(t) + \int_\Sigma \Big(\partial_2 \mathcal{L} \wedge \star v + \partial_4 \mathcal{L} \wedge \star dv\Big) u(t)\Big] \\ &= \int_I dt \Big[(\partial_3 \mathcal{L}, v)_{L^2} \dot{u}(t) + (\partial_2 \mathcal{L}, v)_{L^2} u(t) + (\partial_4 \mathcal{L}, dv)_{L^2} u(t)\Big] \\ &= -\int_I dt \Big[\frac{d}{dt}(\partial_3 \mathcal{L}, v)_{L^2} - (\partial_2 \mathcal{L}, v)_{L^2} - (\partial_4 \mathcal{L}, dv)_{L^2}\Big] u(t). \end{aligned}$$

Since $u \in C_0^2(I, \mathbb{R})$ is arbitrary, the terms in the brackets vanish, which gives (3.2). □

Remark 3.2. *Similar to our discussion of the covariant case, there is a naturality relation in the variational principle when using spatial cochain projections for the semi-discrete theory. In* *particular,*

$$S[\pi_h\varphi] = \int_I dt \int_\Sigma \mathcal{L}(x^\mu, \pi_h\varphi, \pi_h\dot{\varphi}, d\pi_h\varphi) = \int_I dt \int_\Sigma \mathcal{L}(x^\mu, \pi_h\varphi, \pi_h\dot{\varphi}, \pi_h d\varphi) =: S_h[\varphi],$$

so that the restricted variational principle can be realized as a full variational principle on a degenerate action, $\delta S[\pi_h\phi]\cdot(u\ \pi_h v) = \delta S_h[\phi]\cdot(uv)$. Analogous to the discussion in the covariant case, the cochain property additionally removes the ambiguity of how one should discretize the spatial derivative of the field (i.e., whether one should project before or after taking the spatial derivative).

We now show that the semi-discrete Euler–Lagrange equation (3.2) arises from an instantaneous Lagrangian. To do this, let $\{v_i\}$ be a basis for Λ^k_h. We define the instantaneous (semi-discrete) Lagrangian to be

$$L_h(t,\varphi^i,\dot{\varphi}^i) = \int_\Sigma \mathcal{L}(x^\mu, \varphi^i v_i, \dot{\varphi}^i v_i, \varphi^i dv_i), \tag{3.3}$$

where $\varphi = \varphi^i(t)v_i \in C^2(I,\Lambda^k_h)$ and the associated action $S_h[\{\varphi^i\}] = \int_I dt L_h(t,\varphi^i,\dot{\varphi}^i)$. We enforce the variational principle over curves $u = u^i(t)v_i \in C^2_0(I,\Lambda^k_h)$. The variational principle yields

$$\begin{aligned} 0 = dS_h[\{\varphi^i\}]\cdot\{u^j\} &= \frac{d}{d\epsilon}\Big|_0 S_h[\{\varphi^i+\epsilon u^j\}] = \sum_j \int_I dt\Big(\frac{\partial L_h}{\partial\varphi^j}(t,\varphi^i,\dot{\varphi}^i)u^j + \frac{\partial L_h}{\partial\dot{\varphi}^j}(t,\varphi^i,\dot{\varphi}^i)\dot{u}^j\Big) \\ &= \sum_j \int_I dt\Big[\frac{\partial L_h}{\partial\varphi^j}(t,\varphi^i,\dot{\varphi}^i) - \frac{d}{dt}\frac{\partial L_h}{\partial\dot{\varphi}^j}(t,\varphi^i,\dot{\varphi}^i)\Big]u^j. \end{aligned}$$

This holds for arbitrary $u^j \in C^2_0(I,\mathbb{R})$, so the term in the brackets vanishes for each j,

$$\frac{\partial L_h}{\partial\varphi^j}(t,\varphi^i,\dot{\varphi}^i) - \frac{d}{dt}\frac{\partial L_h}{\partial\dot{\varphi}^j}(t,\varphi^i,\dot{\varphi}^i) = 0, \tag{3.4}$$

by the fundamental lemma of the calculus of variations. Expressing the derivatives of L_h in terms of $\mathcal{L}$,

$$\frac{\partial L_h}{\partial\varphi^j} = (\partial_2\mathcal{L}, v_j)_{L^2\Lambda^k(\Sigma)} + (\partial_2\mathcal{L}, dv_j)_{L^2\Lambda^{k+1}(\Sigma)}, \tag{3.5a}$$

$$\frac{\partial L_h}{\partial\dot{\varphi}^j} = (\partial_3\mathcal{L}, v_j)_{L^2\Lambda^k(\Sigma)}. \tag{3.5b}$$

Substituting these expressions into equation (3.4), we see that this is equation (3.2) with the choice $v = v_j$. This holds for each basis form v_j and hence for arbitrary $v \in \Lambda^k_h$.

We will now introduce a Hamiltonian structure associated with the semi-discretization and show that, in the hyperregular case, this instantaneous Lagrangian system is equivalent to an instantaneous Hamiltonian system.

3.2. **Symplectic Structure of Semi-discrete Dynamics and Hamiltonian Formulation.**

Having derived the semi-discrete Euler–Lagrange equation (3.2), we now relate the symplectic structure on the cotangent space of the full field space $T^*H\Lambda^k(\Sigma)$ to a symplectic structure on the discretized space $T^*\Lambda^k_h$, and show that the semi-discrete Euler–Lagrange equations are equivalent to a Hamiltonian flow on $T^*\Lambda^k_h$ if the Lagrangian is hyperregular.

We work with the reference space Σ, since via the diffeomorphism $i_t : \Sigma \to \Sigma_t$, we can pullback forms on Σ to Σ_t or vice versa (or forms on iterated exterior bundles, such as the symplectic form which is an element of $\Lambda^2(T^*H\Lambda^k(\Sigma)))$. On the full phase space $T^*H\Lambda^k(\Sigma)$, the canonical one-form $\theta \in \Lambda^1(T^*H\Lambda^k(\Sigma))$ is given in coordinates by

$$\theta\big|_{(\varphi,\pi)} = \int_\Sigma \pi_A d\varphi^A \otimes d^n x_0 \tag{3.6}$$

and the corresponding symplectic form $\omega = -d\theta$ is given by

$$\omega\big|_{(\varphi,\pi)} = \int_\Sigma (d\varphi^A \wedge d\pi_A) \otimes d^n x_0.$$

Using the projection map $\pi_h : H\Lambda^k(\Sigma) \to \Lambda^k_h$, we have the pullback $\pi_h^* : T^*\Lambda^k_h \to T^*H\Lambda^k(\Sigma)$ and the twice iterated pullback $\pi_h^{**} : \Lambda^p(T^*H\Lambda^k(\Sigma)) \to \Lambda^p(T^*\Lambda^k_h)$ for any p. We define $\theta_h \equiv \pi_h^{**}\theta$ and $\omega_h \equiv \pi_h^{**}\omega = -d\theta_h \in \Lambda^2(T^*\Lambda^k_h)$. To find an expression for θ_h and ω_h, we will introduce global coordinates on $T^*\Lambda^k_h$. Let $\{v_i\}$ be a finite element basis for Λ^k_h; we will use the components φ^i of the basis expansion $\varphi = \varphi^i v_i$ as the coordinates on Λ^k_h; similarly, if we identify $T\Lambda^k_h \cong \Lambda^k_h \times \Lambda^k_h$, then we have a basis for $T^*_\varphi\Lambda^k_h$ being $v^i := (\cdot, v_i)_{L^2}$. This gives the trivialization $T^*\Lambda^k_h \cong \Lambda^k_h \times (\Lambda^k_h)^*$ with global coordinates $(\varphi,\pi) \sim (\varphi^i, \pi_i)$ where $\varphi = \varphi^i v_i$ and $\pi = \pi_i v^i$. We will denote these coordinates using vector notation $\vec{\varphi} = (\varphi^i)$, $\vec{\pi} = (\pi_i)$.

Proposition 3.1. *The 1-form θ_h is given in the above coordinates by*

$$\theta_h = v^j(v_i)\pi_j d\varphi^i = d\vec{\varphi}^T M\vec{\pi}, \tag{3.7}$$

where the mass matrix M has components $M_i{}^j := v^j(v_i) = \int_\Sigma v_i v_j d^n x_0$. Furthermore, the 2-form $\omega_h = -d\theta_h$ is a symplectic form on $T^*\Lambda^k_h$ with coordinate expression

$$\omega_h = d\varphi^i \wedge v^j(v_i) d\pi_j = d\vec{\varphi}^T \wedge M d\vec{\pi}. \tag{3.8}$$

Proof. Let $(\varphi,\pi) \in T^*\Lambda^k_h$ and $U \in T_{(\varphi,\pi)}(T^*\Lambda^k_h)$, with coordinate expression

$$U(\varphi,\pi) = \Phi^i \frac{\partial}{\partial\varphi^i} + \Pi_i \frac{\partial}{\partial\pi_i}.$$

Note that $\theta|_{(\varphi',\pi')}(V)$ gives the canonical pairing between the $\partial/\partial\varphi'$ component of V and π' by equation (3.6). Then, since π_h^* is an inclusion $T^*\Lambda^k_h \hookrightarrow T^*H\Lambda^k(\Sigma)$ and hence $T\pi_h^*$ is an inclusion on the corresponding tangent space, this gives

$$\theta_h|_{(\varphi,\pi)}(U) = \theta|_{\pi_h^*(\varphi,\pi)}(T\pi_h^* U) = \langle \Phi, \pi\rangle = \Phi^i \pi_j \int_\Sigma v_i v_j d^n x_0 = v^j(v_i)\pi_j \Phi^i = v^j(v_i)\pi_j d\varphi^i(U).$$

Equation (3.8) then follows from taking (minus) the exterior derivative of equation (3.7).

The nondegeneracy and closedness of ω_h clearly follow from the (global) coordinate expression (3.8) above. In particular, since the mass matrix M is invertible (hence nondegenerate), ω_h is nondegenerate. Closedness follows from

$$d\omega_h = d^2\vec{\varphi}^T \wedge M d\vec{\pi} - d\vec{\varphi}^T \wedge dM \wedge d\vec{\pi} - d\vec{\varphi}^T \wedge M d^2\vec{\pi} = 0.$$

Alternatively, ω_h is closed as the pullback of a closed form ω. □

Remark 3.3. *The above defines a standard symplectic form on $T^*\Lambda^k_h$ corresponding to the polarization $T^*\Lambda^k_h = \Lambda^k_h \times (\Lambda^k_h)^*$. To see ω_h in standard form, we can change basis. Let Q be an orthogonal matrix which diagonalizes M (recall that the mass matrix is symmetric), i.e., $QMQ^T = D$. Define coordinates $\vec{q} = Q\vec{\varphi}$ and $\vec{p} = DQ\vec{\pi}$; then*

$$\omega_h = d\vec{\varphi}^T \wedge M d\vec{\pi} = d\vec{\varphi}^T \wedge Q^T DQ d\vec{\pi} = d(Q\vec{\varphi})^T \wedge d(DQ\vec{\pi}) = d\vec{q}^T \wedge d\vec{p}.$$

However, we will work with the form of ω_h corresponding to the finite element basis (3.8) since it is more directly applicable to our discretization. Also, if we chose the dual basis l^j to be different *from the basis $v^j = (\cdot, v_j)$, M would not necessarily be symmetric but would still define a symplectic form. This is because for our finite element method to be consistent, we require that the matrix with components $l^j(v_i)$ is invertible. Hence, it is more natural to work with the coordinates $(\vec{\varphi}, \vec{\pi})$.*

Let $H : T^*\Lambda^k_h \to \mathbb{R}$ be the Hamiltonian of our theory, expressed in our global coordinates as $H = H(\vec{\varphi}, \vec{\pi})$. The dynamics of the Hamiltonian system (ω_h, H) is the flow generated by the Hamiltonian vector field X_H satisfying $X_H \lrcorner\, \omega_h = dH$, or with vector field components $X_H = (\dot{\varphi}^i, \dot{\pi}_i)$,

$$\begin{cases} M^j{}_k \dot{\varphi}^k = \frac{\partial H}{\partial \pi_j}, \\ M_j{}^k \dot{\pi}_k = -\frac{\partial H}{\partial \varphi^j}. \end{cases} \tag{3.9}$$

Remark 3.4. *In the above, we denote row j and column k of M as $M_j{}^k$ and for M^T as $M^j{}_k$, which allows more generally for M to not be symmetric as discussed previously. If we define $\vec{z}$ as the concatenation of $\vec{\varphi}$ and $\vec{\pi}$, the equations (3.9) can be written in skew-symmetric form*

$$\frac{d}{dt}\vec{z} = J_M \nabla_{\vec{z}} H,$$

where $J_M = \begin{pmatrix} 0 & (M^{-1})^T \\ -M^{-1} & 0 \end{pmatrix}$.

Remark 3.5. *In our discussion of the covariant discretization of Lagrangian field theories, we saw that the variation of the discretized action on the discrete space can be naturally related to the variation of a degenerate action on the full space. In the semi-discrete setting, an analogous statement can be made in terms of the semi-discrete symplectic structure and a presymplectic structure on the full space. Namely, we have the symplectic form $\omega_h \in \Lambda^2(T^*\Lambda^k_h)$. Now, consider the presymplectic form $\tilde{\omega}_h \in \Lambda^2(T^*H\Lambda^k)$ defined by $\tilde{\omega}_h = i_h^{**}\omega_h$ where $i_h = (\pi_h)^\dagger : \Lambda^k_h \hookrightarrow H\Lambda^k$ is the inclusion. Clearly, $\tilde{\omega}_h$ is closed as the pullback of a closed form. To see that it is degenerate, for any $V, W \in \mathfrak{X}(T^*H\Lambda^k)$,*

$$\tilde{\omega}_h(V,W) = (i_h^{**}\pi_h^{**}\omega)(V,W) = \omega(T(\pi_h^* i_h^*)V, T(\pi_h^* i_h^*)W).$$

Since $i_h\pi_h$ has a nontrivial kernel, so does $T(\pi_h^* i_h^*) = T(i_h\pi_h)^*$ and hence $\tilde{\omega}_h$ is degenerate. The flow of a vector field in the kernel of $\tilde{\omega}_h$, projected back to the semi-discrete space, corresponds to equivalent states in the semi-discrete setting. Quotienting the presymplectic manifold $(T^*H\Lambda^k, \tilde{\omega}_h)$ by the orbits of the flow of vector fields in the kernel of $\tilde{\omega}_h$ gives the symplectic manifold $(T^*\Lambda^k_h, \omega_h)$. This relates a symplectic flow on $(T^*\Lambda^k_h, \omega_h)$ to an equivalence class of presymplectic flows on $(T^*H\Lambda^k, \tilde{\omega}_h)$, where the equivalence class is formed by orbits of the flow of vector fields in the kernel of $\tilde{\omega}_h$.

We also allow our Hamiltonian to explicitly depend on time, $H : I \times T^*\Lambda^k_h \to \mathbb{R}$, i.e., the domain of H is the extended phase space $I \times T^*\Lambda^k_h$. The dynamics are now given by any vector field X_H on the extended phase space such that $X_H \lrcorner\, (\omega_h + dH \wedge dt) = 0$ (here, ω_h is extended to the full phase space by acting trivially in the temporal direction). If we consider the component X_H^V of X_H over the field space $T^*\Lambda^k_h$, then the above is equivalent to $X_H^V \lrcorner\, \omega_h = d_v H$ holding for all times; this is given again by equation (3.9) but with explicit time dependence in H (note that $d_v H$ is the vertical exterior derivative; in coordinates, $d_v H(t,\varphi,\pi) = \frac{\partial H}{\partial \varphi^i} d\varphi^i + \frac{\partial H}{\partial \pi_j} d\pi_j$). We could also allow explicit time dependence in M, but since we pullback our integration to Σ, we view M as constant and absorb the time dependence into H. For our setup, explicit time dependence is generally necessary since our foliation is not necessarily trivial and the Hamiltonian may be time-dependent.

Now, we would like to relate the semi-discrete Euler–Lagrange equations (3.2) to the Hamiltonian dynamics of ω_h. The first step is to produce a Hamiltonian associated to the instantaneous

Lagrangian

$$L(t,\varphi,\dot\varphi) = \int_\Sigma \mathcal{L}(x^\mu,\varphi,\dot\varphi,d\varphi).$$

To do this, we use the Legendre transform, which takes the form $\pi = \partial L/\partial\dot\varphi$. The pairing of π with a tangent vector field with components (φ, v) is given by computing the variation

$$\langle \pi, v\rangle = \left\langle \frac{\partial L}{\partial\dot\varphi}, v\right\rangle = \frac{d}{d\epsilon}\Big|_{\epsilon=0} L(t,\varphi,\dot\varphi+\epsilon v) = (\partial_3\mathcal{L}, v)_{L^2\Lambda^k}.$$

The instantaneous Hamiltonian is given by

$$H(t,\varphi,\pi) = \langle \pi,\dot\varphi\rangle - L(t,\varphi,\dot\varphi)$$

(where the $\dot\varphi$ dependence is removed either by extremizing over $\dot\varphi$ or, assuming L is hyperregular, by inverting the Legendre transform to get $\dot\varphi$ as a function of (φ,π)). Restricting to our finite element space $T^*\Lambda^k_h$ gives our discrete Hamiltonian H_h defined by

$$H_h(t,\varphi^i,\pi_i) = H(t,\varphi^i v_i,\pi_i v^j) = \langle \pi_j v^j, \dot\varphi^i v_i\rangle - L(t,\varphi^i v_i,\dot\varphi^i v_i) = M_i{}^j \pi_j\dot\varphi^i - L(t,\varphi^i v_i,\dot\varphi^i v_i).$$

Note that H_h corresponds to the Legendre transform of the semi-discrete Lagrangian (3.3), where we recall the duality pairing between $(\varphi^j,\pi_j)\in T^*\Lambda^k_h$ and $(\varphi^i,\dot\varphi^i)\in T\Lambda^k_h$ is given by $M_i{}^j\pi_j\dot\varphi^i$.

Proposition 3.2. *Assume that L_h is hyperregular, then the dynamics associated with the Hamiltonian system (ω_h, H_h) is equivalent to the semi-discrete Euler–Lagrange equations (3.2).*

Proof. Since we assumed that L_h is hyperregular, i.e., that the associated Legendre transform is a diffeomorphism $T\Lambda^k_h \to T^*\Lambda^k_h$, we have $\dot\varphi^i$ as a function of (φ^j,π_j). To verify the equivalence, we compute the equations (3.9) for our given system. Compute for L evaluated at $(t,\varphi^i v_i,\dot\varphi^i v_i)$,

$$\begin{aligned}
M_j{}^k\dot\pi_k &= -\frac{\partial H_h}{\partial\varphi^j} = -\frac{\partial}{\partial\varphi^j}\Big(M_i{}^k\pi_k\dot\varphi^i - L\Big)\\
&= -M_i{}^k\pi_k\frac{\partial\dot\varphi^i}{\partial\varphi^j} + \frac{\partial}{\partial\varphi^j}\int_{\Sigma_t}\mathcal{L}(x^\mu,\varphi^i v_i,\dot\varphi^i v_i,\varphi^i dv_i)\\
&= -M_i{}^k\pi_k\frac{\partial\dot\varphi^i}{\partial\varphi^j} + \int_{\Sigma_t}\Big[\partial_2\mathcal{L}\wedge\star\frac{\partial(\varphi^i v_i)}{\partial\varphi^j} + \partial_3\mathcal{L}\wedge\star\frac{\partial(\dot\varphi^i v_i)}{\partial\varphi^j} + \partial_4\mathcal{L}\wedge\star\frac{\partial(\varphi^i dv_i)}{\partial\varphi^j}\Big]\\
&= -M_i{}^k\pi_k\frac{\partial\dot\varphi^i}{\partial\varphi^j} + \int_{\Sigma_t}\Big[\partial_2\mathcal{L}\wedge\star v_j + \partial_3\mathcal{L}\wedge\star v_i\frac{\partial\dot\varphi^i}{\partial\varphi^j} + \partial_4\mathcal{L}\wedge\star dv_j\Big]\\
&= -M_i{}^k\pi_k\frac{\partial\dot\varphi^i}{\partial\varphi^j} + (\partial_3\mathcal{L}, v_i)\frac{\partial\dot\varphi^i}{\partial\varphi^j} + (\partial_2\mathcal{L}, v_j)_{L^2} + (\partial_4\mathcal{L}, dv_j)_{L^2}\\
&= (\partial_2\mathcal{L}, v_j)_{L^2} + (\partial_4\mathcal{L}, dv_j)_{L^2},
\end{aligned}$$

where in the second to last line, the first two terms cancel since $(\partial_3\mathcal{L}, v_i) = \langle\pi, v_i\rangle = \langle \pi_k v^k, v_i\rangle = M_i{}^k\pi_k$. Then, note the left hand side is equivalently given by

$$M_j{}^k\dot\pi_k = M_j{}^k\frac{d}{dt}\pi_k = \frac{d}{dt}\big(M_j{}^k\pi_k\big) = \frac{d}{dt}\big(\langle v^k, v_j\rangle\pi_k\big) = \frac{d}{dt}\langle\pi, v_j\rangle = \frac{d}{dt}(\partial_3\mathcal{L}, v_j)_{L^2}.$$

Thus,

$$\frac{d}{dt}(\partial_3\mathcal{L}, v_j)_{L^2} = (\partial_2\mathcal{L}, v_j)_{L^2} + (\partial_4\mathcal{L}, dv_j)_{L^2}$$

which holds for each j and hence is equivalent to (3.2). □

Remark 3.6. *In the above proposition, we assumed that L_h was hyperregular for the equivalence. If L_h is not hyperregular (e.g. corresponding to a degenerate field theory), the dynamics associated to H_h evolve over a primary constraint surface. In this case, the dynamics of H_h on the constraint surface corresponds to a (not necessarily unique) solution of the semi-discrete Euler–Lagrange equation. In this setting, the dynamics are associated to the modified Hamiltonian $\bar{H}(\vec{\varphi},\vec{\pi},\lambda) = H(\vec{\varphi},\vec{\pi}) + \lambda^A \Phi_A(\vec{\varphi},\vec{\pi})$.*

The above also shows that, in the hyperregular case, the semi-discrete Euler–Lagrange equations correspond to a symplectic flow. The associated symplectic form is the pullback of ω_h by the Legendre transform $\mathbb{F}L_h : T\Lambda^k_h \to T^*\Lambda^k_h$. In the non-regular case, the semi-discrete Euler–Lagrange equations correspond to a presymplectic flow.

To summarize, in this section, we have pulled back the symplectic structure on $T^*H\Lambda^k$ to $T^*\Lambda^k_h$ and showed that the dynamics of the Hamiltonian system (ω_h, H_h) is equivalent (in the hyperregular case) to the semi-discrete Euler–Lagrange equations of the corresponding Lagrangian system. By applying a numerical integrator for the finite-dimensional Hamiltonian system associated to H_h, we obtain a full discretization of the evolution problem of a field theory.

3.3. **Energy-Momentum Map.** In this section, we examine how symmetries in the canonical formulation are affected by the semi-discretization of the field theory. In the canonical setting, the manifestation of the covariant momentum map is the energy-momentum map. If a vector in the Lie algebra of the symmetry group gives rise to an infinitesimal generator on X which is transverse to the foliation, its pairing with the energy-momentum map equals the instantaneous Hamiltonian defined by that generator (the "energy" component). On the other hand, if the corresponding generator is tangent to the foliation, the pairing is given by the usual momentum map of the instantaneous Hamiltonian theory, corresponding to the canonical form (3.6) (the "momentum" component). We will see that, in the case of an equivariant discretization, the iterated pullback of the energy-momentum map provides the natural energy-momentum structure of the semi-discrete theory.

We start by investigating the momentum map structure of the semi-discrete theory. Let K be a Lie group (with $\mathfrak{k} := T_eK$) acting on $H\Lambda^k$; for $\eta \in K$, we denote the group action $\overline{\eta}\varphi := \eta \cdot \varphi$ and the associated cotangent action is given by $\widetilde{\eta} := (\overline{\eta^{-1}})^*$ (we use the same notation for these actions restricted to Λ^k_h and $T^*\Lambda^k_h$, where the restriction is well-defined if the projection is group-equivariant).

Proposition 3.3. *Assume that K acts by symplectomorphisms on $(T^*H\Lambda^k, \omega)$; since K acts by cotangent lifts on $T^*H\Lambda^k$, it admits a canonical momentum map $J : T^*H\Lambda^k \to \mathfrak{k}^*$. Furthermore, assume that the projection map π_h is equivariant with respect to the K-action on $H\Lambda^k$ and Λ^k_h; i.e., $\pi_h\bar{\eta}\varphi = \bar{\eta}\pi_h\varphi$. Then, K acts by cotangent-lifted symplectomorphisms on $(T^*\Lambda^k_h, \omega_h)$ and the canonical momentum map for this action J_h is given by $J_h = \pi_h^{**}J = J \circ \pi_h^*$.*

Proof. To see that K preserves ω_h, for any $\eta \in K$, by equivariance, we have that

$$\widetilde{\eta}^*\omega_h = (\overline{\eta^{-1}})^{**}\pi_h^{**}\omega = (\overline{\eta^{-1}}\pi_h)^{**}\omega = (\pi_h\overline{\eta^{-1}})^{**}\omega = \pi_h^{**}(\overline{\eta^{-1}})^{**}\omega = \pi_h^{**}\omega = \omega_h.$$

A similar result holds for θ_h, since K preserves θ by virtue of the fact that it acts by cotangent lifted actions.

The canonical momentum map J is given by $\langle J(\varphi,\pi), \xi\rangle = \xi_{T^*H\Lambda^k}(\varphi,\pi) \lrcorner\, \theta|_{(\varphi,\pi)}$ for $(\varphi,\pi) \in T^*H\Lambda^k$ whereas $\langle J_h(\varphi,\pi), \xi\rangle = \xi_{T^*\Lambda^k_h}(\varphi,\pi) \lrcorner\, \theta_h|_{(\varphi,\pi)}$ for $(\varphi,\pi) \in T^*\Lambda^k_h$. These are both momentum maps for their respective actions since K acts by cotangent lifts. Then,

$$\begin{aligned}\langle J_h(\varphi,\pi),\xi\rangle &= \xi_{T^*\Lambda^k_h}(\varphi,\pi)\lrcorner\,\theta_h = \xi_{T^*\Lambda^k_h}(\varphi,\pi)\lrcorner\,\pi_h^{**}\theta \\ &= [T\pi_h^*\xi_{T^*\Lambda^k_h}(\varphi,\pi)]\lrcorner\,\theta = \Big[T\pi_h^*\frac{d}{dt}\Big|_{t=0}\widetilde{e^{t\xi}}(\varphi,\pi)\Big]\lrcorner\,\theta \\ &= \Big[\frac{d}{dt}\Big|_{t=0}\pi_h^*(\overline{e^{-t\xi}})^*(\varphi,\pi)\Big]\lrcorner\,\theta \\ &= \Big[\frac{d}{dt}\Big|_{t=0}(\overline{e^{-t\xi}}\pi_h)^*(\varphi,\pi)\Big]\lrcorner\,\theta \\ &= \Big[\frac{d}{dt}\Big|_{t=0}(\pi_h\overline{e^{-t\xi}})^*(\varphi,\pi)\Big]\lrcorner\,\theta \\ &= \Big[\frac{d}{dt}\Big|_{t=0}(\overline{e^{-t\xi}})^*\pi_h^*(\varphi,\pi)\Big]\lrcorner\,\theta \\ &= \xi_{T^*H\Lambda^k}(\pi_h^*(\varphi,\pi))\lrcorner\,\theta = \langle (J\circ\pi_h^*)(\varphi,\pi),\xi\rangle.\end{aligned}$$

where we have implicitly evaluated θ_h at (φ,π) and θ at $\pi_h^*(\varphi,\pi)$. Hence, $J_h = J\circ\pi_h^*$ or, equivalently, $J_h = \pi_h^{**}J$. □

Remark 3.7. *As can be seen in the proof, one does not need full K-equivariance of the projection, but only infinitesimal equivariance, i.e., $\pi_h(e^{t\xi}\varphi) - e^{t\xi}\pi_h\varphi = o(t)$.*

Furthermore, one can weaken the notion of equivariance to $\pi_h\overline{\eta} = \overline{\psi_h(\eta)}\pi_h$, where $\psi_h : K \to K$ is a differentiable group homomorphism. In this case, if $\tilde{\psi}_h$ denotes the induced Lie algebra homomorphism, we can see from the above proof that the semi-discrete momentum map is related to the original momentum map via $\langle J_h,\xi\rangle = \langle J\circ\pi_h^*, \tilde{\psi}_h(\xi)\rangle$.

As discussed in the covariant case, the weakening of this condition can allow us to construct more general projections.

Corollary 3.1. *Assuming as in the proposition, if J is Ad*-equivariant, then so is J_h.*

Proof. This follows immediately from $J_h = J\circ\pi_h^*$, K-equivariance of π_h, and the Ad*-equivariance $J\circ\widetilde{\eta} = \mathrm{Ad}^*_\eta J$ (where $\mathrm{Ad}^*_\eta := (\mathrm{Ad}(\eta^{-1}))^*$):

$$\begin{aligned}J_h\circ\widetilde{\eta} &= J\circ\pi_h^*\circ(\overline{\eta^{-1}})^* = J\circ(\overline{\eta^{-1}})^*\circ\pi_h^* \\ &= J\circ\widetilde{\eta}\circ\pi_h^* = (\mathrm{Ad}^*_\eta J)\circ\pi_h^* = \mathrm{Ad}^*_\eta(J\circ\pi_h^*) = \mathrm{Ad}^*_\eta J_h,\end{aligned}$$

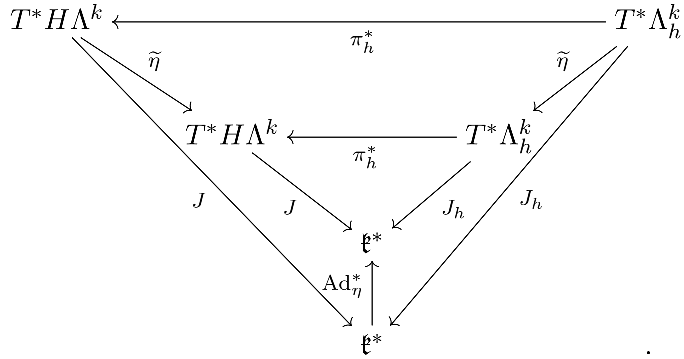
where the equality $(\mathrm{Ad}^*_\eta J)\circ\pi_h^* = \mathrm{Ad}^*_\eta(J\circ\pi_h^*)$ holds since the coadjoint action acts on J after it is evaluated on its input (since then it is an element of $\mathfrak{k}^*$). In particular, $(\mathrm{Ad}^*_\eta J)(\varphi,\pi) := \mathrm{Ad}^*_\eta(J(\varphi,\pi))$, so that

$$((\mathrm{Ad}^*_\eta J)\circ\pi_h^*)(\varphi,\pi) = (\mathrm{Ad}^*_\eta J)(\pi_h^*(\varphi,\pi)) = \mathrm{Ad}^*_\eta(J(\pi_h^*(\varphi,\pi))) = \mathrm{Ad}^*_\eta((J\circ\pi_h^*)(\varphi,\pi))).$$

Stated another way, this follows from associativity of the composition of functions, viewing Ad^*_η as a function $\mathfrak{k}^*\to\mathfrak{k}^*$. □

Remark 3.8. *Of course, since K acts by cotangent lifts and hence by canonical symplectomorphisms, J is an Ad*-equivariant momentum map, and the corollary tells us that J_h is as well. However, as we remark below, one may consider more general actions which admit momentum maps, and it is not necessarily the case that those momentum maps are Ad*-equivariant. The result of the previous corollary still holds in this more general setting.*

The naturality of the momentum map structures from the previous proposition and corollary can be summarized via the following commuting diagram; for any $\eta\in K$,



.

Remark 3.9. *In the above proposition, we only assumed that π_h was equivariant with respect to the K-action on the configuration space, and it follows that π_h^* is equivariant with respect to the lifted action on the cotangent space. However, for more general actions on the cotangent space (not arising from a cotangent lift), one must instead assume π_h^* is equivariant with respect to this action. In this case, if the K-action on $T^*H\Lambda^k$ admits a momentum map J, then $J_h = \pi_h^{**}J$ is a momentum map for the action on $T^*\Lambda_h^k$. To verify this, let $(\varphi,\pi) \in T^*\Lambda_h^k$. We know that $d\langle J,\xi\rangle = i_{\xi_{T^*H\Lambda^k}}\omega$. Thus,*

$$d\langle J_h,\xi\rangle = \pi_h^{**}d\langle J,\xi\rangle = \pi_h^{**}(i_{\xi_{T^*H\Lambda^k}}\omega).$$

Then, observe that by equivariance, $\xi_{T^*H\Lambda^k}(\varphi,\pi) = T\pi_h^*\xi_{T^*\Lambda_h^k}(\varphi,\pi)$. Then, for any $X \in TT^*_{(\varphi,\pi)}\Lambda_h^k$,

$$d\langle J_h,\xi\rangle(X) = (i_{\xi_{T^*H\Lambda^k}}\omega)(T\pi_h^*X) = \omega(T\pi_h^*\xi_{T^*\Lambda_h^k}, T\pi_h^*X) = (\pi_h^{**}\omega)(\xi_{T^*\Lambda_h^k}, X) = (i_{\xi_{T^*\Lambda_h^k}}\omega_h)(X),$$

where the above is evaluated at (φ,π), which verifies that J_h is a momentum map. For the subsequent discussion, we will assume that K acts by cotangent lifts.

We now define the energy-momentum map and its semi-discrete counterpart. We consider vectors on Σ_t with both tangent components in $T\Sigma_t$ and components transverse to the foliation, which in our adapted coordinates are in the span of $\partial/\partial t$. We extend the canonical form θ to act on vector fields on the extended phase space (in the same way we extended ω_h in our previous discussion of time-dependence). Letting $\tilde{\mathcal{L}}$ denote the Lagrangian density on the full spacetime (related to the spatial density by $\mathcal{L} = i_t^*\partial_t \lrcorner \tilde{\mathcal{L}}$), define the map $\mathcal{J}$ from $I \times T^*H\Lambda^k$ to the dual of the space of vector fields on the extended phase space, via

$$\langle \mathfrak{J}(t,\varphi,\pi), V\rangle = (V \lrcorner\, \theta)(t,\varphi,\pi) - \int_\Sigma i_t^* V_t \lrcorner\, \tilde{\mathcal{L}}(x^\mu,\varphi,\dot{\varphi},d\varphi), \tag{3.10}$$

where we view $\dot{\varphi}$ as a function of (φ,π), and where V_t is the tangent-lift of the bundle projection $I \times T^*H\Lambda^k(\Sigma_t) \to I$ applied to V.

Proposition 3.4. *$\mathfrak{J}$ is the energy-momentum map, in the following sense:*

(i) ***(Energy)*** *Let Φ_t^H denote the Hamiltonian flow of H and X_H be the associated generator on the extended phase space; then,*

$$\langle \mathfrak{J}(t,\varphi,\pi), X_H\rangle = H(t,\varphi,\pi).$$

(ii) ***(Momentum)*** *If V is tangent to the foliation, then,*

$$\langle \mathfrak{J}(t,\varphi,\pi), V\rangle = (V \lrcorner\, \theta)(t,\varphi,\pi),$$

and in particular, if there is a K-action as in Proposition (3.3) on the phase space over Σ_t, its momentum map J is given by

$$\langle J(t,\varphi,\pi),\xi\rangle = \langle \mathfrak{J}(t,\varphi,\pi), \xi_{T^*H\Lambda^k}\rangle,$$

such that (for each fixed t) $d\langle J(t,\varphi,\pi),\xi\rangle = \xi_{T^*H\Lambda^k} \lrcorner\, \omega(t,\varphi,\pi)$.

Proof. For the proof of (i), in local coordinates, we have

$$X_H = \frac{d}{dt}\Big|_{t=0} \Phi_t^H(t',\varphi,\pi) = \frac{\partial}{\partial t} + \dot{\varphi}^A \frac{\partial}{\partial \varphi^A} + \dot{\pi}_B \frac{\partial}{\partial \pi_B},$$

and $(X_H)_t = \partial/\partial t$. Using expressions (3.6) and (3.10) and the definition of the instantaneous Lagrangian density $\mathcal{L} = i_t^* \partial_t \lrcorner\, \widetilde{\mathcal{L}}$,

$$\begin{aligned}
\langle \mathfrak{J}(t,\varphi,\pi), X_H\rangle &= (X_H \lrcorner\, \theta)(t,\varphi,\pi) - \int_\Sigma i_t^*(X_H)_t \lrcorner\, \widetilde{\mathcal{L}}(x^\mu,\varphi,\dot{\varphi},d\varphi) \\
&= \dot{\varphi}^A \frac{\partial}{\partial \varphi^A} \lrcorner \Big(\int_{\Sigma_t} \pi_A d\varphi^A \otimes d^n x_0 \Big) - \int_\Sigma i_t^* \frac{\partial}{\partial t} \lrcorner\, \widetilde{\mathcal{L}}(x^\mu,\varphi,\dot{\varphi},d\varphi) \\
&= \int_{\Sigma_t} \pi_A \dot{\varphi}^A d^n x_0 - \int_\Sigma \mathcal{L}(t,x^i,\varphi,\dot{\varphi},d\varphi) \\
&= \langle \pi, \dot{\varphi}\rangle - L(t,\varphi,\dot{\varphi}) = H(t,\varphi,\pi).
\end{aligned}$$

For the proof of (ii), note that for V tangent to the foliation, $V_t = 0$, which immediately gives the first equation of (ii). Setting the vector field to an infinitesimal generator of a K-action gives the momentum map

$$\langle \mathfrak{J}(t,\varphi,\pi), \xi_{T^*H\Lambda^k}\rangle = (\xi_{T^*H\Lambda^k} \lrcorner\, \theta)(t,\varphi,\pi).$$

□

We now define the semi-discrete analogue of the energy-momentum map (3.10). Define the semi-discrete energy-momentum map $\mathcal{J}_h$ from $I \times T^*\Lambda_h^k$ to the dual of vector fields on the extended discrete phase space, via

$$\langle \mathfrak{J}_h(t,\varphi,\pi), V\rangle = (V \lrcorner\, \theta_h)(t,\varphi,\pi) - \int_\Sigma i_t^* V_{t,h} \lrcorner\, \widetilde{\mathcal{L}}_h(x^\mu,\varphi,\dot{\varphi},d\varphi), \tag{3.11}$$

where $V_{t,h} = (T\pi_h^* V)_t$ and $\widetilde{\mathcal{L}}_h$ is the restriction of $\widetilde{\mathcal{L}}$ via precomposition with π_h^*. Of course, the analogous statement of the previous proposition holds for the semi-discrete energy-momentum map. Furthermore, $\mathcal{J}_h$ is the restriction of $\mathcal{J}$ in the following sense.

Proposition 3.5. *For (t,φ,π) in the extended discrete phase space and V a vector field over this space,*

$$\langle \mathfrak{J}_h(t,\varphi,\pi), V\rangle = \langle \mathfrak{J}(t,\pi_h^*(\varphi,\pi)), T\pi_h^* V\rangle.$$

Proof. This follows directly from the definitions;

$$\begin{aligned}
\langle \mathfrak{J}(t,\pi_h^*(\varphi,\pi)), T\pi_h^* V\rangle &= (T\pi_h^* V \lrcorner\, \theta)(t,\pi_h^*(\varphi,\pi)) - \int_\Sigma i_t^*(T\pi_h^* V)_t \lrcorner\, \widetilde{\mathcal{L}}(t,\pi_h^*[(\varphi,\dot{\varphi},d\varphi)|_{(\varphi,\pi)}]) \\
&= (V \lrcorner\, \pi_h^{**}\theta)(t,\varphi,\pi) - \int_\Sigma i_t^* V_{t,h} \lrcorner\, \widetilde{\mathcal{L}}_h(t,\varphi,\dot{\varphi},d\varphi) \\
&= (V \lrcorner\, \theta_h)(t,\varphi,\pi) - \int_\Sigma i_t^* V_{t,h} \lrcorner\, \widetilde{\mathcal{L}}_h(t,\varphi,\dot{\varphi},d\varphi) = \langle \mathfrak{J}_h(t,\varphi,\pi), V\rangle.
\end{aligned}$$

□

The significance of this definition of the semi-discrete energy-momentum map is that it recovers the properties of Proposition 3.4 in the semi-discrete setting.

Proposition 3.6.

(i) ***(Semi-discrete Energy)*** *For (t, φ, π) in the extended discrete phase space,*

$$\langle \mathfrak{J}_h(t, \varphi, \pi), X_{H_h} \rangle = H_h(t, \varphi, \pi).$$

(ii) ***(Semi-discrete Momentum)*** *If there is a K-action on the discrete phase space, then the momentum map J_h is given by*

$$\langle J_h(t, \varphi, \pi), \xi \rangle = \langle \mathfrak{J}_h(t, \varphi, \pi), \xi_{T^*\Lambda^k_h} \rangle.$$

Furthermore, if the K-action on the discrete space arises from an action on the full space such that π_h is K-equivariant, then for any $\xi \in \mathfrak{k}$,

$$\langle \mathfrak{J}_h(t, \varphi, \pi), \xi_{T^*\Lambda^k_h} \rangle = \langle \mathfrak{J}(t, \pi_h^*(\varphi, \pi)), \xi_{T^*H\Lambda^k} \rangle.$$

Proof. The first two equations follow from analogous computations to Proposition 3.4. The last equation follows from the equivariance of π_h,

$$T\pi_h^* \xi_{T^*\Lambda^k_h}(\varphi, \pi) = \xi_{T^*H\Lambda^k}(\pi_h^*(\varphi, \pi)),$$

and Proposition 3.5. □

The significance of a semi-discrete analogue of the energy-momentum map, aside from extending the semi-discrete momentum map structure (discussed in Proposition 3.3), is in determining semi-discrete analogues of Noether's second theorem, which we will pursue in subsequent work.

3.4. **Temporal Discretization of the Semi-Discrete Theory.** To complete the discussion of the semi-discrete theory, we must of course discretize in time. We obtain a full discretization of the semi-discrete theory by discretizing the semi-discrete Euler–Lagrange equation (3.2) in time via a Galerkin Lagrangian variational integrator applied to the instantaneous semi-discrete Lagrangian (3.3), and show that this is equivalent to the full spacetime DEL (2.3) with tensor product elements. The associated finite element on the full spacetime is a tensor product mesh, obtained by discretizing the space Σ and extendeding these elements in time by a partition of I. Of course, this is not the most general setup for a spacetime discretization, but often one wishes to discretize in time separately; for example, by choosing the appropriate temporal basis functions, the computation becomes local in time so that one can time march the solution from the initial data, instead of solving the entire DEL on the spacetime grid. Furthermore, there are constructions of cochain projections for tensor product elements (Arnold [5]) so that with these finite element spaces, the naturality of the variational principle discussed in Section 2 carries over in the tensor product setting.

Remark 3.10. *There is a slight subtlety here when comparing to the covariant theory on the full spacetime X. In the covariant theory, we consider k-forms on X, $\Lambda^k X$, whereas here we are considering k-forms on Σ, $\Lambda^k(\Sigma)$. Letting $\pi_1 : \mathbb{R} \times \Sigma \to \mathbb{R}, \pi_2 : \mathbb{R} \times \Sigma \to \Sigma$ be the projections, we have pointwise,*

$$\wedge^k(T^*X) = \wedge^k T^*(\mathbb{R} \times \Sigma) \cong \Big(\pi_1^*(\wedge^0 T^*\mathbb{R}) \wedge \pi_2^*(\wedge^k T^*\Sigma)\Big) \oplus \Big(\pi_1^*(\wedge^1 T^*\mathbb{R}) \wedge \pi_2^*(\wedge^{k-1} T^*\Sigma)\Big).$$

This congruence does not hold at the level of sections: to see this in coordinates (t, x) on $\mathbb{R} \times \Sigma$, we have forms which look like $f(t)g(x)dx^{j_1} \wedge \cdots \wedge dx^{j_k}, f(t)dt \wedge g(x)dx^{j_1} \wedge \cdots \wedge dx^{j_{k-1}}$ which cannot give a form which looks like e.g. $h(t, x)dt \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{k-1}}$ where h is some function that cannot be expressed as a product $f(t)g(x)$. However, we are assuming time-dependent fields $\varphi : t \mapsto H\Lambda^k(\Sigma)$ so we do have the forms which look like $\varphi(t) = g(t, x)dx^{j_1} \wedge \cdots \wedge dx^{j_k}$. Thus, we only need to consider multiple fields to obtain full generality $\varphi_1 : t \mapsto H\Lambda^k(\Sigma), \varphi_2 : t \mapsto H\Lambda^{k-1}(\Sigma)$ (here we are identifying $C^\infty\mathbb{R} = \Lambda^0\mathbb{R} \cong \Lambda^1\mathbb{R}$, so by $\varphi_2(t)$ we really mean $\varphi_2(t)dt$). Of course, this issue does not arise for scalar functions, so for simplicity, in this section, we will consider scalar functions (i.e.,

$k = 0$), although the discussion generalizes to the case of arbitrary k; one just needs to consider multiple fields.

Assume the setup as in the discussion of the semi-discrete theory. Furthermore, assume that we have a finite element discretization of $H_0(I)$ (the space of square integrable functions in time with square integrable derivative, which vanish on ∂I) with basis functions $\{w_\alpha\}$. Recall the instantaneous semi-discrete Lagrangian (3.3) is a function of the curves $\varphi^i(t), \dot{\varphi}^i(t)$ which are the coefficients of the expansions of $\varphi(t), \dot{\varphi}(t) \in \Lambda^0_h$ relative to the basis $\{v_i\}$ of Λ^0_h. Using the basis $\{w_\alpha\}$, we discretize these curves as

$$\varphi^i(t) = (\varphi^i)^\alpha w_\alpha(t),$$

where $\varphi(t,x) = (\varphi^i)^\alpha w_\alpha(t) v_i(x)$ in this notation. We consider the associated fully discrete action as a function of the coefficients,

$$S[\{(\varphi^i)^\alpha\}] = \int_I dt L_h(t, \varphi^i(t), \dot{\varphi}^i(t)) = \int_I dt L_h(t, (\varphi^i)^\alpha w_\alpha, (\varphi^i)^\alpha \dot{w}_\alpha).$$

Enforcing the discrete variational principle in time gives the weak form of the Euler–Lagrange equations,

$$0 = \frac{\delta S}{\delta(\varphi^i)^\alpha} = \left(\frac{\partial L_h}{\partial \varphi^i}, w_\alpha\right)_{L^2(I)} + \left(\frac{\partial L_h}{\partial \dot{\varphi}^i}, \dot{w}_\alpha\right)_{L^2(I)}.$$

Substituting equations (3.5a) and (3.5b) gives

$$\begin{aligned} 0 &= -((\partial_3 \mathcal{L}, v_i)_{L^2(\Sigma)}, \dot{w}_\alpha)_{L^2(I)} - ((\partial_2 \mathcal{L}, v_i)_{L^2(\Sigma)}, w_\alpha)_{L^2(I)} - ((\partial_4 \mathcal{L}, dv_i)_{L^2\Lambda^1(\Sigma)}, w_\alpha)_{L^2(I)} \\ &= -(\partial_3 \mathcal{L}, v_i \dot{w}_\alpha)_{L^2(\Sigma\times I)} - (\partial_2 \mathcal{L}, v_i w_\alpha)_{L^2(\Sigma\times I)} - (\partial_4 \mathcal{L}, (dv_i) w_\alpha)_{L^2\Lambda^1(\Sigma)\times L^2(I)} \\ &= -\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}, v_i \dot{w}_\alpha\right)_{L^2(\Sigma\times I)} - \left(\frac{\partial \mathcal{L}}{\partial \varphi}, v_i w_\alpha\right)_{L^2(\Sigma\times I)} - \left(\frac{\partial \mathcal{L}}{\partial (d\varphi)}, (dv_i) w_\alpha\right)_{L^2\Lambda^1(\Sigma)\times L^2(I)}. \end{aligned}$$

Note that these equations can also be obtained directly from the semi-discrete Euler–Lagrange equations (3.2) by applying the Galerkin method in time with respect to the basis $\{w_\alpha\}$. Here, d denotes the spatial exterior derivative on Σ. If d_t denotes the temporal exterior derivative and we identity functions on I with one-forms on I, we have $\dot{w}_\alpha \cong d_t w_\alpha$. If $d_T = d + d_t$ denotes the total exterior derivative on $\Sigma \times I$, then $d_T(v_i w_\alpha) = (dv_i) w_\alpha + v_i d_t w_\alpha$. We now view the time-dependent function $\varphi : t \mapsto \varphi(t)$ as a function ϕ on spacetime, so the above can be written

$$\begin{aligned} 0 &= -\left(\frac{\partial \mathcal{L}}{\partial \phi}, v_i w_\alpha\right)_{L^2(\Sigma\times I)} - \left(\frac{\partial \mathcal{L}}{\partial (d\phi)}, (dv_i) w_\alpha\right)_{L^2\Lambda^1(\Sigma)\times L^2(I)} - \left(\frac{\partial \mathcal{L}}{\partial (d_t\phi)}, v_i d_t w_\alpha\right)_{L^2(\Sigma)\times L^2\Lambda^1(I)} \\ &= -\left(\frac{\partial \mathcal{L}}{\partial \phi}, v_i w_\alpha\right)_{L^2(\Sigma\times I)} - \left(\frac{\partial \mathcal{L}}{\partial (d_T\phi)}, d_T(v_i w_\alpha)\right)_{L^2\Lambda^1(\Sigma\times I)}, \end{aligned}$$

which is the DEL (2.3) with tensor product basis $\{v_i w_\alpha\}$.

Note that this result can also be obtained from the semi-discrete Hamiltonian setting, using the fact that the semi-discrete Hamiltonian and semi-discrete Lagrangian formulations are equivalent in the hyperregular case by Proposition 3.2, and the fact that generalized Lagrangian variational integrators and generalized Hamiltonian variational integrators are equivalent in the hyperregular case, as established in Leok and Zhang [26].

4. Conclusion and Future Directions

In this paper, we showed how discretizing the variational principle for Lagrangian field theories using finite element cochain projections naturally gives rise to a discrete variational structure which is analogous to the continuum variational structure; namely, the discrete variational structure is encoded by the discrete Cartan form. Our discrete Cartan form generalizes the discrete Cartan form introduced by Marsden et al. [29] to more general finite element spaces within the finite element exterior calculus framework. Using the discrete Cartan form, we expressed a discrete multisymplectic form formula and a discrete Noether theorem in direct analogy to their continuum counterparts. Furthermore, we studied semi-discretization of Lagrangian PDEs by spatial cochain projections, showing that such semi-discretization gives rise to semi-discrete symplectic, Hamiltonian, and energy-momentum map structures. Finally, we related the methods obtained by covariant discretization and canonical semi-discretization in the case of tensor product finite elements.

In the paper, we outlined several possible research directions, including studying particular field theories and showing rigorous convergence of the discrete Cartan form, constructing group-equivariant cochain projections, and establishing a discrete Noether's second theorem utilizing the semi-discrete energy-momentum map. Another natural research direction would be to extend the discrete variational structures presented here to the discontinuous Galerkin setting and compare them with the results obtained in the multisymplectic Hamiltonian setting by McLachlan and Stern [31]. In particular, we expect that in this setting, the discrete Cartan form would only involve integration over ∂U, since boundary variations can be localized to codimension-one simplices, unlike for conforming finite element spaces. Furthermore, we aim to investigate how the discrete variational structures presented in this paper (in the conforming setting and extended to the discontinuous Galerkin setting) can be used to provide a geometric variational framework for studying lattice field theories, building on the discrete variational framework for lattice field theories initiated in Arjang and Zapata [4].

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Appendix A. Proofs of the Bounds for Noether's Theorem

Proof of Proposition 2.4. Equation (2.23b) follows immediately from equation (2.23a) by the assumed convergence and the fact that the relevant operations are continuous, noting that $\phi_h \to \phi$ in $H\Lambda^k$ implies $j^1\phi_h = (\phi_h, d\phi_h) \to (\phi, d\phi) = j^1\phi$ (as a side note, of course the first term in the limit goes to zero since J is continuous as a map with inputs from $H\Lambda^k \times \mathfrak{g}$. However, we need not assume dJ is continuous on $H\Lambda^k \times \mathfrak{g}$ and the second term still goes to zero by the first bound).

For the first equation, note by definition of the norm

$$\|dJ(\phi_h,\xi)\|_0^* = \|dJ(\phi_h,\xi) - dJ(\phi,\xi)\|_0^* \leq \|J(\phi_h,\xi) - J(\phi,\xi)\|_{L^2\Lambda^n}.$$

Then, via $J(\psi,\xi) = \psi^\xi \wedge \star\partial_3\mathcal{L}(j^1\psi) + \xi_X \lrcorner\, \mathcal{L}(j^1\psi) - K(\psi,\xi)$,

$$\begin{aligned}\|J(\phi_h,\xi) - J(\phi,\xi)\| &\leq \|(\phi_h)^\xi \wedge \star\partial_3\mathcal{L}(j^1\phi_h) - \phi^\xi \wedge \star\partial_3\mathcal{L}(j^1\phi)\| \\ &\quad + \|\xi_X \lrcorner\, (\mathcal{L}(j^1\phi_h) - \mathcal{L}(j^1\phi))\| + \|K(\phi_h,\xi) - K(\phi,\xi)\|,\end{aligned}$$

where the norms are taken in the appropriate $L^2\Lambda^k$ spaces. To conclude, bound

$$\begin{aligned}&\|(\phi_h)^\xi \wedge \star\partial_3\mathcal{L}(j^1\phi_h) - \phi^\xi \wedge \star\partial_3\mathcal{L}(j^1\phi)\| \\ &\quad \leq \|(\phi_h)^\xi \wedge \star\partial_3\mathcal{L}(j^1\phi_h) - (\phi_h)^\xi \wedge \star\partial_3\mathcal{L}(j^1\phi)\| + \|(\phi_h)^\xi \wedge \star\partial_3\mathcal{L}(j^1\phi) - \phi^\xi \wedge \star\partial_3\mathcal{L}(j^1\phi)\|\end{aligned}$$

$$\begin{aligned}&\leq \|(\phi_h)^\xi\|\|\partial_3\mathcal{L}(j^1\phi_h)-\partial_3\mathcal{L}(j^1\phi)\|+\|(\phi_h)^\xi-\phi^\xi\|\|\partial_3\mathcal{L}(j^1\phi)\|\\ &\leq (\|\phi^\xi\|+\|\phi^\xi-(\phi_h)^\xi\|)\|\partial_3\mathcal{L}(j^1\phi_h)-\partial_3\mathcal{L}(j^1\phi)\|+\|(\phi_h)^\xi-\phi^\xi\|\|\partial_3\mathcal{L}(j^1\phi)\|\\ &\lesssim \|\partial_3\mathcal{L}(j^1\phi_h)-\partial_3\mathcal{L}(j^1\phi)\|+\|(\phi_h)^\xi-\phi^\xi\|,\end{aligned}$$

noting that we view ϕ as constant. □

Proof of Proposition 2.5. To bound the left hand side of equation (2.24) in terms of norms on X, note that the trace is a bounded operator $\mathrm{Tr}: H^{\frac{1}{2}+\delta}\Lambda^k(X)\to H^\delta\Lambda^k(\partial X)$, where the (fractional) derivatives in H^s include any possible derivatives of the forms, not just d and d^*. The boundedness of the trace immediately gives the bound on $\|\mathrm{Tr}J(\phi_h,\xi)-\mathrm{Tr}J(\phi,\xi)\|_{H^\delta\Lambda^n(\partial X)}$. For the $\|dJ(\phi_h,\xi)\|^*$ term, note the bound on the L^2 norm at the boundary given by the followning continuous embeddings,

$$H^{\frac{1}{2}+\delta}\Lambda^k(X)\xrightarrow{\mathrm{Tr}} H^\delta\Lambda^k(\partial X)\xhookrightarrow{i} L^2\Lambda^k(\partial X).$$

Then, since $dJ(\phi,\xi)=0$, we have to bound

$$\Big|\Big(d(J(\phi_h,\xi)-J(\phi,\xi)),v\Big)_{L^2\Lambda^{n+1}}\Big|=\Big|\int_{\partial X}\Big(J(\phi_h,\xi)-J(\phi,\xi)\Big)\wedge\star v+\Big(J(\phi_h,\xi)-J(\phi,\xi),d^*v\Big)_{L^2\Lambda^n}\Big|.$$

Dealing with the second term on the RHS is straightforward. For the first term on the RHS, note the abuse of notation where we should include the trace of the forms to the boundary. and note the wedge product is unnecessary since $\star v$ is a 0-form. Hence. the term reads

$$\int_{\partial X}\mathrm{Tr}\Big(J(\phi_h,\xi)-J(\phi,\xi)\Big)\mathrm{Tr}(\star v).$$

Utilizing the above boundedness of $i\circ\mathrm{Tr}: H^{\frac{1}{2}+\delta}\Lambda^k(X)\to L^2\Lambda^k(\partial X)$,

$$\begin{aligned}&\Big|\int_{\partial X}\mathrm{Tr}\Big(J(\phi_h,\xi)-J(\phi,\xi)\Big)\mathrm{Tr}(\star v)\Big|\\ &\quad\leq \|\mathrm{Tr}[J(\phi_h,\xi)-J(\phi,\xi)]\|_{L^2\Lambda^n(\partial X)}\|\mathrm{Tr}(\star v)\|_{L^2\Lambda^0(\partial X)}\\ &\quad\lesssim \|J(\phi_h,\xi)-J(\phi,\xi)]\|_{H^{\frac{1}{2}+\delta}\Lambda^n(X)}\|\star v\|_{H^{\frac{1}{2}+\delta}\Lambda^0(X)}\\ &\quad= \|J(\phi_h,\xi)-J(\phi,\xi)]\|_{H^{\frac{1}{2}+\delta}\Lambda^n(X)}\|v\|_{H^{\frac{1}{2}+\delta}\Lambda^{n+1}(X)}\\ &\quad\leq \|J(\phi_h,\xi)-J(\phi,\xi)]\|_{H^{\frac{1}{2}+\delta}\Lambda^n(X)}\|v\|_{H^1\Lambda^{n+1}(X)}.\end{aligned}$$

Finally, note for the space of top-degree forms, the norms on $H^1\Lambda^{n+1}(X)$ and $H^*\Lambda^{n+1}(X)$ are equivalent, since for $v\in\Lambda^{n+1}$, d^*v can be isometrically identified with the gradient $\nabla(\star v)$. By definition of the $\|\cdot\|^*$ norm, putting this all together gives the bound for the second term. □

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