GENERALIZED DONALDSON-THOMAS INVARIANTS VIA KIRWAN BLOWUPS

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

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Abstract

In this thesis, we develop a virtual cycle approach towards generalized Donaldson-Thomas theory of Calabi-Yau threefolds. Let σ be a stability condition on the bounded derived category $D^b(\operatorname{Coh} W)$ of a Calabi-Yau threefold W and $\mathcal{M}$ a moduli stack of σ-semistable objects of fixed topological type.

We construct an associated Deligne-Mumford stack $\widetilde{\mathcal{M}}$, called the Kirwan partial desingularization of $\mathcal{M}$, with an induced semi-perfect obstruction theory of virtual dimension zero, and define the generalized Donaldson-Thomas invariant via Kirwan blowups to be the degree of the virtual cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}}$. Examples of applications include Gieseker stability of coherent sheaves and Bridgeland and polynomial stability of perfect complexes.

When $\mathcal{M}$ is a moduli stack of Gieseker semistable sheaves, this is invariant under deformations of the complex structure of W. More generally, deformation invariance is true under appropriate properness assumptions which are expected to hold in all cases.

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Chapter 1

Introduction

1.1 Classical Donaldson-Thomas theory

We begin by giving a brief account of classical Donaldson-Thomas (DT) theory and its main features. Let W be a smooth, projective Calabi-Yau threefold (CY3) over $\mathbb{C}$, i.e. $K_W \simeq \mathcal{O}_W$ and $\dim_{\mathbb{C}} W = 3$.

Counting subvarieties of a given type inside an algebraic variety has been one of the important themes in algebraic geometry. For example, a smooth cubic surface contains 27 lines [Har77].

Significant advances in string theory in the 1990's motivated the development of several mathematical theories of enumerating curves in CY3's, most notably Gromov-Witten and DT invariants. These conjecturally capture the same information [MNOP06] and are also related to other enumerative theories, such as Stable Pair [PT09], BPS and Gopakumar-Vafa invariants [KL12, MT16].

DT theory is a sheaf theoretic technique of enumerating curves in W. DT invariants were first introduced by Thomas in his thesis [Tho00]. The necessary technical tool to achieve this was provided by the theory of virtual fundamental cycles and perfect obstruction theories, developed by Li-Tian [LT98] and Behrend-Fantechi [BF97]. One obtains the associated numerical invariant by integrating against a cycle in the moduli space parametrizing the geometric objects of interest, e.g. curves or bundles of given topological data.

More precisely, for $\gamma \in H^*(W, \mathbb{Q})$, let $\mathcal{M}^{ss}(\gamma)$ be the moduli stack parametrizing Gieseker semistable sheaves on W of fixed Chern character γ. We consider the following situation.

Assumption 1.1.1. *Every semistable sheaf in* $\mathcal{M}^{ss}(\gamma)$ *is stable, i.e.* $\mathcal{M}^{ss}(\gamma) = \mathcal{M}^{s}(\gamma)$.

We then have the following theorem.

Theorem 1.1.2. [Tho00, HT10] *Let* $\mathcal{M}^{ss}(\gamma) = \mathcal{M}^{s}(\gamma)$ *be as above such that Assumption 1.1.1 holds and denote by* $M^s(\gamma)$ *its coarse moduli space. Then* $M^s(\gamma)$ *admits a perfect obstruction theory of virtual dimension zero and hence a virtual cycle* $[M^s(\gamma)]^{\mathrm{vir}} \in A_0(M^s(\gamma))$. *The classical Donaldson-Thomas invariant is defined as*

$$\mathrm{DT}(\mathcal{M}^s(\gamma)) := \deg[M^s(\gamma)]^{\mathrm{vir}}.$$

It is invariant under deformation of the complex structure of W.

Example 1.1.3. *Let* $I_{\beta,n}$ *be the Hilbert scheme parametrizing subschemes* $C \subset W$ *such that* $[C] = \beta \in H_2(W,\mathbb{Z})$ *and* $\chi(\mathcal{O}_C) = n$. *Then every ideal sheaf is stable and hence we have an associated DT invariant. This is a "virtual" count of curves in* W *of topological type* (β, n).

The perfect obstruction theory is induced by the universal ideal sheaf $\mathcal{I}$. *Let* $\pi\colon W \times I_{\beta,n} \to I_{\beta,n}$ *be the projection morphism. The two-term perfect complex* $R\pi_* R\mathcal{H}om(\mathcal{I},\mathcal{I})_0[2]$ *gives a perfect obstruction theory on* $I_{\beta,n}$. *Here the subscript* 0 *denotes the traceless part of* $R\mathcal{H}om(\mathcal{I},\mathcal{I})$.

In general, $\mathcal{M}^{ss}(\gamma) = [Q^{ss}/G]$ is a global quotient stack obtained by Geometric Invariant Theory (GIT) [HL10, MFK94]. Q is an appropriate Quot scheme, hence projective, and $G = \mathrm{GL}(n,\mathbb{C})$ (for n large) is acting linearly on Q. We have then the following diagram

$$\begin{tikzcd}
\mathcal{M}^s(\gamma) = [Q^s/G] \arrow[r] \arrow[d] & {[Q^{ss}/G]} = \mathcal{M}^{ss}(\gamma)\ , \arrow[d] \\
M^s(\gamma) \arrow[r] & M^{ss}(\gamma)
\end{tikzcd}$$

where the horizontal arrows are open embeddings and moreover $M^{ss}(\gamma)$ is projective. Every stable sheaf E is simple, i.e. $\mathrm{End}(E) = \{\mathbb{C}\cdot \mathrm{id}\}$ given by scaling, and therefore $\mathcal{M}^s(\gamma) \to M^s(\gamma)$ is a coarse moduli space and a $\mathbb{C}^\times$-gerbe. Since the center $Z(G) = \mathbb{C}^\times$ acts trivially on Q and this coincides with the scaling action on each sheaf, we may ignore scaling by replacing G with $\mathrm{PGL}(n,\mathbb{C})$, which we do from now on. Then by abuse of notation $\mathcal{M}^s(\gamma) \cong M^s(\gamma)$ and stable sheaves have trivial automorphism groups.

Assumption 1.1.1 is important in multiple ways. If $\mathcal{M}^s(\gamma) = \mathcal{M}^{ss}(\gamma)$ then the sheaves in question have trivial automorphisms and in particular $\mathcal{M}^{ss}(\gamma)$ is a scheme or more generally a Deligne-Mumford stack, whereas in general it is only an Artin stack, since semistable sheaves can have positive dimensional automorphism groups (even after quotienting out scaling automorphisms). For example, if E is stable and $F = E \oplus E$ is semistable, then $\mathrm{Aut}(F) = \mathrm{GL}(2, \mathbb{C})$. $\mathcal{M}^{ss}(\gamma)$ is also proper, as $M^{ss}(\gamma)$ is projective. One is then able to apply the machinery of perfect obstruction theory [BF97, LT98], which requires a DM stack in order to produce a virtual cycle and properness so that one may take the degree thereof.

In fact, even more is true. In [Beh09], Behrend defined a canonical constructible function $\nu_{\mathcal{M}} \colon \mathcal{M} \to \mathbb{Z}$ for any stack $\mathcal{M}$. $\nu_{\mathcal{M}}$ is a measure of the singularity of $\mathcal{M}$. For example, if $\mathcal{M}$ is a smooth DM stack, then $\nu_{\mathcal{M}} \equiv (-1)^{\dim \mathcal{M}}$. Behrend further defined the notion of a symmetric obstruction theory and observed that the perfect obstruction theory of Theorem 1.1.2 is symmetric. We have the following remarkable theorem.

Theorem 1.1.4. [Beh09, MT11] *Let $\mathcal{M}$ be a proper Deligne-Mumford stack with a symmetric obstruction theory. Then*

$$\deg[\mathcal{M}]^{\mathrm{vir}} = \chi(\mathcal{M}, \nu_{\mathcal{M}}) := \sum_{n \in \mathbb{Z}} n \chi(\nu_{\mathcal{M}}^{-1}(n)) \in \mathbb{Z}. \tag{$\dagger$}$$

Example 1.1.5. *Suppose $M := M^s(\gamma)$ is as in Theorem 1.1.2. Then its perfect obstruction theory is symmetric and thus*

$$\mathrm{DT}(\mathcal{M}^s(\gamma)) = \deg[M]^{\mathrm{vir}} = \chi(M, \nu_M).$$

1. *This equality has the striking implications that the classical DT invariant is motivic, in the sense that it is a weighted Euler characteristic, and moreover it only depends on the scheme structure of M and not its obstruction theory, since the function $\nu_{\mathcal{M}}$ depends only on M itself.*

2. *If M is smooth, then ($\dagger$) reduces to the Gauss-Bonnet theorem*

$$c_{\mathrm{top}}(\Omega_M) = \deg[M]^{\mathrm{vir}} = \chi(M, \nu_M) = (-1)^{\dim M} \chi(M).$$

1.2 Main results

1.2.1 Statement of results and outline of approach

The main objective of this thesis is to give a definition of a generalized DT invariant, by which we mean a direct generalization of Theorem 1.1.2 when Assumption 1.1.1 fails, thereby obtaining a virtual count of semistable sheaves even when there exist ones which are semistable but not stable.

The first main result that we show is as follows.

Theorem 1.2.1. [KLS17, Kiem, Li, S.] *Let $\mathcal{M} = \mathcal{M}^{ss}(\gamma)$ be the moduli stack of Gieseker semistable sheaves on W of Chern character γ, where the $\mathbb{C}^\times$-scaling automorphisms have been rigidified. Then there exist:*

1. *A proper DM stack $\widetilde{\mathcal{M}}$ with a morphism $\widetilde{\mathcal{M}} \to \mathcal{M}$, which gives an isomorphism over the stable locus $\mathcal{M}^s$. $\widetilde{\mathcal{M}}$ is called the Kirwan partial desingularization of $\mathcal{M}$.*
2. *A semi-perfect obstruction theory of virtual dimension zero on $\widetilde{\mathcal{M}}$, which extends the symmetric obstruction theory of the stable locus $\mathcal{M}^s$, and thus a virtual cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}} \in A_0(\widetilde{\mathcal{M}})$.*

We define the generalized Donaldson-Thomas invariant via Kirwan blowups (also called the DTK invariant) as

$$\mathrm{DTK}(\mathcal{M}) := \deg[\widetilde{\mathcal{M}}]^{\mathrm{vir}}.$$

$\mathrm{DTK}(\mathcal{M})$ *is invariant under deformations of the complex structure of W.*

Remark 1.2.2. *When $\mathcal{M}^{ss}(\gamma) = \mathcal{M}^s(\gamma)$, i.e Assumption 1.1.1 holds, then $\widetilde{\mathcal{M}} = \mathcal{M}$ and the semi-perfect obstruction theory of $\widetilde{\mathcal{M}}$ is the symmetric obstruction theory of $\mathcal{M}$. So the DTK invariant is indeed a generalized DT invariant.*

As remarked in the preceding section, the main issue arising in the absence of Assumption 1.1.1 is that $\mathcal{M}$ is in general an Artin stack. Therefore, Theorem 1.2.1 says that we may replace $\mathcal{M}$ with a canonical DM stack and work there instead, preserving all the data on the stable locus $\mathcal{M}^s$.

We now briefly outline the construction. $\mathcal{M} = \mathcal{M}^{ss}(\gamma)$ has the following important properties:

1. $\mathcal{M}$ is a global quotient stack obtained by GIT. Therefore, we may write $\mathcal{M} = [X/G]$ with a sequence of closed embeddings $X \subset P \subset (\mathbb{P}^N)^{ss}$, where P is smooth and $G = \mathrm{PGL}(n, \mathbb{C})$ acts linearly on $\mathbb{P}^N$.

2. $\mathcal{M}$ is the truncation of a (-1)-shifted symplectic derived Artin stack [PTVV13] and in particular a d-critical Artin stack [BBBBJ15, Joy15].

To construct the Kirwan partial desingularization $\widetilde{\mathcal{M}}$, we adapt Kirwan's partial desingularization procedure. In [Kir85], Kirwan described a canonical blowup procedure to produce a partial desingularization $\widetilde{P} \to P$, yielding a proper DM stack $[\widetilde{P}/G]$ obtained by GIT and an isomorphism over the stable locus $[P^s/G]$.

By generalizing and adapting appropriately the notion of intrinsic blowup, introduced in [KL13b], we define a closed G-invariant subscheme $\widetilde{X} \subset \widetilde{P} \times_P X$, which is independent of all choices involved and hence canonical. We then define $\widetilde{\mathcal{M}} = [\widetilde{X}/G]$.

To obtain the obstruction theory on $\widetilde{\mathcal{M}}$, we use the d-critical structure of $\mathcal{M}$. $\mathcal{M}$ is d-critical and hence X is a G-invariant d-critical locus. Then for every $x \in X$ such that $G \cdot x$ is closed in X and H is the stabilizer of x in G (hence reductive), we have G-invariant affine Zariski open $x \in U \subset X, x \in V \subset P$ and locally closed affine subschemes $T \subset U,\ S \subset V$ granted by Luna's étale slice theorem such that we have a diagram

$$\begin{tikzcd}
{[T/H]} \arrow[r] \arrow[d] & {[U/G]} \arrow[d] \\
{[S/H]} \arrow[r] & {[V/G]}
\end{tikzcd} \tag{1.1}$$

with étale horizontal arrows and $T = (df = 0) \subseteq S$ for $f \colon S \to \mathbb{A}^1$ an H-invariant regular function on S.

Thus we have the following H-equivariant 4-term complex

$$\mathfrak{h} = \mathrm{Lie}(H) \longrightarrow T_S|_T \xrightarrow{d(df)^\vee} F_S|_T = \Omega_S|_T \longrightarrow \mathfrak{h}^\vee. \tag{1.2}$$

For $x \in T$ with finite stabilizer, this is quasi-isomorphic to a 2-term complex which provides a perfect obstruction theory of $[T/H]$ and thus of $[U/G]$ near x.

In general, let $\widetilde{x} \in \widetilde{X}$ be lying over $x \in X$ with stabilizer R. Then we can lift (1.2) canonically and find an étale neighborhood $[\widetilde{T}/R] \to [\widetilde{S}/R] \to [\widetilde{P}/G]$ of $\widetilde{x} \in \widetilde{P}$, a vector bundle $F_{\widetilde{S}}$ over $\widetilde{S}$ with an invariant section $\omega_{\widetilde{S}} \in H^0(\widetilde{S}, F_{\widetilde{S}})$ such that $\widetilde{T} = (\omega_{\widetilde{S}} = 0) \subset \widetilde{S}$, and a divisor $D_{\widetilde{S}}$, all R-equivariant, such that (1.2) lifts canonically to a sequence

$$\mathfrak{r} = \mathrm{Lie}(R) \longrightarrow T_{\widetilde{S}}|_{\widetilde{T}} \longrightarrow F_{\widetilde{S}}|_{\widetilde{T}} \longrightarrow \mathfrak{r}^\vee(-D_{\widetilde{S}}) \tag{1.3}$$

whose first arrow is injective and last arrow is surjective. Therefore, (1.3) is quasi-isomorphic to a 2-term complex

$$(d\omega_{\widetilde{S}}^\vee)^\vee : T_{[\widetilde{S}/R]}|_{\widetilde{T}} \longrightarrow F_{\widetilde{S}}^{\mathrm{red}}|_{\widetilde{T}}, \tag{1.4}$$

where $F_{\widetilde{S}}^{\mathrm{red}}$ is the kernel of the last arrow in (1.3). Quotienting by R, we get

$$(d(\omega_{\widetilde{S}}^{\mathrm{red}})^\vee)^\vee : T_{[\widetilde{S}/R]}|_{[\widetilde{T}/R]} \longrightarrow F_{[\widetilde{S}/R]}^{\mathrm{red}}|_{[\widetilde{T}/R]}. \tag{1.5}$$

We show that the collection of the cokernels $\mathrm{coker}(d(\omega_{\widetilde{S}}^{\mathrm{red}})^\vee)^\vee$ patch to a coherent sheaf $\mathcal{O}b_{\widetilde{\mathcal{M}}}$ of $\mathcal{O}_{\widetilde{\mathcal{M}}}$-modules, and that the symmetric obstruction theories of the various T defined by $(df = 0)$ induce a semi-perfect obstruction theory [CL11] on $\widetilde{\mathcal{M}} = [\widetilde{X}/G]$, with obstruction sheaf $\mathcal{O}b_{\widetilde{\mathcal{M}}}$.

The relative version of the above construction can be constructed along parallel lines, using the machinery of derived symplectic geometry. This implies the deformation invariance of the DTK invariant.

We may further generalize Theorem 1.1.2 and Theorem 1.2.1 to the case of semistable perfect complexes on W. The second main result that we show is the following.

Theorem 1.2.3. [Sav, S.] *Let $\mathcal{M} = \mathcal{M}^{\sigma-ss}(\gamma)$ be a moduli stack of semistable perfect complexes in $D^b(\mathrm{Coh}\, W)$, where σ is an appropriate stability condition (cf. Definition 6.3.7), $\gamma \in H^*(W, \mathbb{Q})$ and $\mathbb{C}^\times$-scaling automorphisms of complexes have been rigidified.*

Then there exists a Kirwan partial desingularization $\widetilde{\mathcal{M}} \to \mathcal{M}$, which is a proper DM stack and isomorphic to $\mathcal{M}$ over the stable locus $\mathcal{M}^s$. $\widetilde{\mathcal{M}}$ admits a semi-perfect obstruction theory of virtual dimension zero, extending the symmetric obstruction theory of $\mathcal{M}^s$, and thus a virtual cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}} \in A_0(\widetilde{\mathcal{M}})$.

We therefore may define the generalized Donaldson-Thomas invariant via Kirwan blowups as

$$\mathrm{DTK}(\mathcal{M}) := \deg[\widetilde{\mathcal{M}}]^{\mathrm{vir}}.$$

Examples of $\mathcal{M}^{\sigma-ss}(\gamma)$ where the Theorem applies include Bridgeland stability, as considered in [PT15], and polynomial stability, as considered in [Bay09, Lo11, Lo13].

The main difference with the case of sheaves is that, while still the truncation of a (-1)-shifted symplectic derived Artin stack, $\mathcal{M}$ is no longer a global quotient stack obtained by GIT. However, it has a similar structure as one can show that it has a good moduli space [Alp13, AHLH]. With a bit more care and using a Luna étale slice theorem for stacks with good moduli spaces [AHR15], we show that all the arguments for the construction of $\widetilde{\mathcal{M}}$ and its obstruction theory go through in this case as well.

Finally, we remark that even though the deformation invariance property is not stated in Theorem 1.2.3, it is true in all applications using upcoming work [BLM$^+$] regarding stability conditions in families and a relative Luna étale slice theorem, which also holds by upcoming work in [AHR].

1.2.2 Comparison with other works and further directions

Our construction of the DTK invariant fits naturally in the context of obtaining generalizations of Donaldson-Thomas invariants.

At the level of numbers, Joyce-Song [JS12] have also constructed generalized DT invariants. Their approach is motivic in nature, using Hall algebras and Behrend's constructible function. In that sense, they are directly generalizing the right hand side of (†), whereas we are generalizing the left hand side. We do expect that the DTK invariant is related to the Joyce-Song invariant via a universal formula which will provide a natural generalization of Theorem 1.1.4 and also a wall-crossing formula for the DTK invariant. These will be investigated in future work.

Kontsevich-Soibelman have also developed a generalized DT theory with an associated wall-crossing formula in [KS10]. We also refer the reader to the recent work of Behrend-Ronagh in [BR16a, BR16b].

Beyond numbers, one might also desire a more categorical invariant. Categorifications of DT theory have been developed in [BBD$^+$12, KL12], where the DT invariant is expressed as the Euler characteristic of a perverse sheaf on the moduli space. Davison-Meinhardt [DM16] have also made great progress in categorifying the Joyce-Song DT invariants. We expect that the methods in this thesis can be adapted to develop a theory of a K-theoretic

DTK invariant, which will also be the subject of further work.

Regarding the Kirwan partial desingularization $\widetilde{\mathcal{M}}$, Edidin-Rydh have also developed a desingularization procedure for stacks with good moduli spaces in [ER17]. For smooth stacks, our desingularization is the same as theirs. For singular stacks, our Kirwan blowups can be phrased in their language of saturated blowups, however the desingularization they obtain is a closed substack of ours. There might be a derived algebraic geometric reason behind this, which seems worth thinking about. Finally, a very interesting question is whether $\widetilde{\mathcal{M}}$ admits a moduli interpretation. This seems especially challenging, but, if successfully carried out, we believe that it will significantly deepen our understanding and thus hope that it will eventually be answered.

1.3 Overview of the thesis

In Chapter 1, we give an introduction to classical DT theory, followed by a statement of the main results of the thesis and a sketch of their proof. We then put the work in context compared to other works and exhibit possible future directions. We finally fix notation and conventions that are followed throughout.

In Chapter 2, we give some background on GIT and Kirwan's partial desingularization procedure. We then proceed to define Kirwan blowups and construct a Kirwan partial desingularization for singular quotient stacks obtained by GIT. After some background on stacks with good moduli spaces and their properties, we generalize the construction to those as well. The material in Sections 2.3, 2.4 and 2.6 is original, of which 2.3 is based on joint work with Young-Hoon Kiem and Jun Li.

Chapter 3 contains necessary material on perfect obstruction theory, including symmetric obstruction theory, and semi-perfect obstruction theory.

Chapter 4 contains original material consisting of local calculations, which form the necessary formalism and backbone of the construction of the obstruction theory of the Kirwan partial desingularization in our theorems. Part of the material in Subsections 4.1.1 and 4.1.2 is based on joint work with Young-Hoon Kiem and Jun Li.

In Chapter 5, we collect background material on d-critical loci and derived symplectic geometry. In turn, this is used in conjunction with the results of Chapter 4 and gives a sufficiently flexible framework in the relative case as well.

Finally, in Chapter 6, we define DTK invariants for sheaves and complexes and prove their deformation invariance in the former case. Most of the material is original, unless attributed to other authors.

1.4 Notation and conventions

The following tables summarize notation and abbreviations that are commonly used throughout the thesis.

Table 1.1: Summary of notation

Notation	**Explanation**
W	Smooth, projective Calabi-Yau threefold
C	Smooth, quasi-projective scheme, commonly a curve
$L_{U/C}^{\geq -1}$	Truncated cotangent complex of $U \to C$
γ	Element of $H^*(W, \mathbb{Q})$
$\mathcal{M}$	Artin stack
$\mathcal{M}^{ss}(\gamma)$	Stack of Gieseker semistable sheaves on W of Chern character γ
σ	Stability condition on $D^b(\operatorname{Coh} W)$
$\mathcal{M}^{\sigma-ss}(\gamma)$	Stack of σ-semistable objects in a heart $\mathcal{A} \subset D^b(\operatorname{Coh} W)$
G, H, R	Complex reductive groups, usually with $H \leq G$ or $R \leq G$
$\mathfrak{g}, \mathfrak{h}, \mathfrak{r}$	Lie algebras of G, H, R respectively
X, P	G-equivariant schemes, P smooth, projective, $X \subset P$ closed subscheme
V, U	G-equivariant schemes, V smooth, $U \subset V$ closed subscheme

Notation	**Explanation**
$G \cdot x, G_x$	Orbit and stabilizer of a closed point $x \in X$ in a G-scheme X
Z_R, X^R	Fixed locus of R-action on X
S, T	R-/H-equivariant schemes, S smooth, $T \subset S$ closed subscheme, usually occuring as étale slices at a closed orbit of a pair V, U
U^{intr}, V^{intr} etc.	Intrinsic blowups with respect to a group action
$X /\!\!/ G$	GIT quotient of a G-scheme X with linearized action
$\widehat{U}, \widehat{V}, \widehat{X}, \widehat{P}$ etc.	Kirwan blowups with respect to a group action
$\widetilde{U}, \widetilde{X}, \widetilde{\mathcal{M}}$ etc.	Kirwan partial desingularizations of schemes or stacks
F_V	Equivariant vector bundle on a G-scheme V
ω_V	Element of $H^0(V, F_V)^G$
$(A^\bullet, \delta)$	Commutative differential negatively graded algebra over a ring S
$\mathbf{Spec}(A^\bullet)$	Derived affine scheme
$\boldsymbol{U}, \boldsymbol{T}, \boldsymbol{\mathcal{M}}$ etc.	Derived schemes or Artin stacks
$\mathbb{L}_{A^\bullet}, \mathbb{L}_{\mathbf{Spec}\, A^\bullet}, \mathbb{L}_{\mathcal{M}}$	Derived cotangent complexes

Table 1.2: Summary of abbreviations

Abbreviation	**Explanation**
CY3	Calabi-Yau threefold
DT	Donaldson-Thomas
DTK	Donaldson-Thomas invariant via Kirwan blowups
DM	Deligne-Mumford
GIT	Geometric Invariant Theory

Additionally:

- Throughout, we work over the field of complex numbers $\mathbb{C}$.
- Unless otherwise stated, we consider connected reductive groups for simplicity of exposition.
- For a morphism $\rho\colon U \to V$ and a sheaf E on V, we often use $E|_U$ to denote ρ^*E. Typically ρ will be a locally closed embedding or unramified.

Chapter 2

Kirwan Partial Desingularization

2.1 Geometric Invariant Theory (GIT)

In this section, we give some brief background on GIT by stating the basic definitions and results we will need. A good reference for the subject is [MFK94].

2.1.1 The local case

Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over $\mathbb{C}$ with an action by a reductive group G. We have the following theorem due to Nagata.

Theorem 2.1.1. *The ring A^G of invariant elements of A under the action of G is finitely generated over $\mathbb{C}$.*

This lets us define the GIT quotient of X as follows.

Definition 2.1.2. *The GIT quotient of X is the affine scheme $X/\!\!/G := \operatorname{Spec} A^G$. The natural morphism $X \to X/\!\!/G$ is a good categorical quotient.*

One important feature of this setup is the existence of the Reynolds operator.

Theorem 2.1.3. *There exists an A^G-linear map $\mathcal{R} \colon A \to A^G$, which restricts to the identity on A^G. $\mathcal{R}$ is called the Reynolds operator.*

More generally, there exists a Reynolds operator $M \to M^G$ for any finitely generated G-equivariant A-module M.

Remark 2.1.4. *One may think of the Reynolds operator as an averaging operator. If G is the complexification of a compact Lie group K and μ is a bi-invariant Haar measure on K then $\mathcal{R}$ is the operation of averaging over the action of K with respect to μ. Since K is Zariski dense in G, the result is also G-invariant.*

We may take an étale slice for the action around a closed point with closed orbit, granted by the following theorem.

Theorem 2.1.5. (Luna's étale slice theorem) [Dré04, Theorem 5.3] *Let $x \in X$ be a closed point with closed orbit $G \cdot x$ and (thus reductive) stabilizer $H = G_x$. There exists a locally closed H-invariant affine subscheme $T \subset X$ containing x such that group multiplication $T \times G \to X$ induces an étale morphism $T \times_H G \to X$. Here $T \times_H G = (T \times G)/H$, where H acts freely on $T \times G$ by $(t, g)h = (th, h^{-1}g)$. Moreover, $T \times_H G \to X$ is strongly étale, meaning that the diagram*

$$\begin{tikzcd}
T \times_H G \arrow[r] \arrow[d] & X \arrow[d] \\
T /\!\!/ H \arrow[r] & X /\!\!/ G
\end{tikzcd}$$

is cartesian and the lower horizontal arrow is étale.

If X is normal or smooth, then T may be taken to be normal or smooth respectively.

2.1.2 The global case

Let X be a projective scheme over $\mathbb{C}$ with an ample line bundle L and an embedding

$$i\colon X \to \mathbb{P}^N = \mathbb{P}\left(H^0\left(X, L^{\otimes r}\right)\right)$$

such that $L^{\otimes r} = i^*\mathcal{O}_{\mathbb{P}^N}(1)$ for some sufficiently large r. Suppose that X admits an action by a reductive group G.

Definition 2.1.6. *We say that the G-action on X is linearized with respect to L or that L is G-linearized if the action of G on X lifts to an action on L. More precisely, if we denote the group action by $\sigma\colon X \times G \to X$, $\pi_1\colon X \times G \to X$ the projection to the first factor and $\pi_{12}\colon X \times G \times G \to X \times G$ the projection to the first two factors, a G-linearization is an isomorphism*

$$\Phi\colon \sigma^* L \longrightarrow \pi_1^* L$$

satisfying the cocycle condition

$$(id_X \times \mu)^*\Phi = \pi_{12}^*\Phi \circ (\sigma \times id_G)^*\Phi$$

where $\mu\colon G \times G \to G$ *is group multiplication.*

Once we have a linearization of the G-action on X, we can define the notions of stability and semistability.

Definition 2.1.7. *Let* $x \in X$*. We say that* x *is:*

1. *semistable, if there exists a* $d \in \mathbb{N}_+$ *and an invariant section* $s \in H^0(X, L^{\otimes d})^G$ *such that* $s(x) \neq 0$*.*
2. *stable, if it is semistable and moreover the orbit* $G \cdot x$ *of* x *is closed in the set of semistable points of* X *and the stabilizer* G_x *of* x *is finite.*
3. *unstable, if it is not semistable.*

We denote the loci of semistable and stable points of X *by* $X^{ss}(L)$ *and* $X^s(L)$ *respectively. If the choice of* L *is clear from context, we will omit it from the notation and merely write* X^{ss}, X^s *instead.*

Remark 2.1.8. *It is clear from the above definition that* $X^s(L) \subset X^{ss}(L) \subset X$*. Also, both inclusions are open embeddings.*

The main theorem of GIT is as follows.

Theorem 2.1.9. *Let* X *be a projective scheme with a* G*-linearized ample line bundle* L*. We define the GIT quotient of* X *as*

$$X^{ss}/\!\!/G := \operatorname{Proj}\left(\bigoplus_{d\geq 0} H^0\left(Y, L^{\otimes d}\right)^G\right).$$

$X^{ss}/\!\!/G$ *is projective. There is a natural quotient map* $X^{ss} \to X^{ss}/\!\!/G$ *which is a good categorical quotient. Moreover, there is an open subscheme* $X^s/G \subset X^{ss}/\!\!/G$ *such that* $\pi^{-1}(X^s/G) = X^s$ *and the induced morphism* $X^s \to X^s/G$ *is a good geometric quotient.*

2.2 Kirwan's partial desingularization procedure for smooth GIT quotients

In this section, we briefly recall Kirwan's desingularization procedure for smooth GIT quotients. We work with the following situation: Let G be a reductive group; G acts on a $\mathbb{P}^N$ via $G \to \mathrm{GL}_{N+1}$, and

$$X \subset P \subset (\mathbb{P}^N)^{ss} \tag{2.1}$$

are G-invariant closed subschemes where P is smooth. We also assume $P^s \neq \emptyset$.

2.2.1 Kirwan's blowup algorithm

We review Kirwan's partial desingularization $\widetilde{P}$ of P, as developed in [Kir85].

As the stabilizer groups of points in P^s are all finite, $P^s /\!\!/ G = P^s/G$ can have at worst finite quotient singularities. In stack language, this means that the quotient stack $[P^s/G]$ is a DM stack. When $P^s \neq P$, the GIT quotient $P /\!\!/ G$ will have worse than finite quotient singularities, and the quotient stack is not DM. In [Kir85], Kirwan produced a canonical procedure to blow up P in order to produce a DM stack out of the Artin stack $[P/G]$.

If the orbit of an $x \in P$ is closed in P, then the stabilizer G_x of x is a reductive subgroup of G. Let us fix a representative of each conjugacy class of subgroups R of G that appear as the identity component of the stabilizer G_x of an $x \in P$ with $G \cdot x$ closed in P. Let $\mathfrak{R}(P)$ denote the set of such representatives. By [Kir85], $\mathfrak{R}(P)$ is finite and $\mathfrak{R}(P) = \{1\}$ if and only if $P = P^s$.

Let $R \in \mathfrak{R}(P)$ be an element of maximal dimension and let Z_R be the fixed locus by the action of R, which is smooth. Then $GZ_R = G \times_{N^R} Z_R$ is smooth in P, where N^R is the normalizer of R in G.

We let $\pi : \mathrm{bl}_R(P) \to P$ be the blowup of P along $GZ_R = G \times_{N^R} Z_R$. Then $L = \pi^* \mathcal{O}_P(1)(-\epsilon E)$ is ample for $\epsilon > 0$ sufficiently small, where E denotes the exceptional divisor of π. The action of G on P induces a linear action of G on $\mathrm{bl}_R(P)$ with respect to L. We have the following theorem due to Reichstein.

Theorem 2.2.1. [Rei89] *Let $Z = GZ_R \subset P$ as above and denote $q \colon P \to P /\!\!/ G$. The unstable locus of $\mathrm{bl}_R(P)$ is the strict transform of the saturation $q^{-1}(q(Z))$ of Z.*

Therefore, the semistable points in the closure of $\mathrm{bl}_R(P)$ inside the projective space given by the embedding induced by the ample line bundle L all lie in $\mathrm{bl}_R(P)$. The unstable points in $\mathrm{bl}_R(P)$ are precisely those points whose orbit closure meets the unstable points in $E = \mathbb{P}N_{GZ_R/P}$, and the unstable points of $\mathrm{bl}_R(P)$ lying in the fiber $E|_x = \mathbb{P}N_{GZ_R/P}|_x$ for $x \in Z_R$ are precisely the unstable points of the projective space $\mathbb{P}N_{GZ_R/P}|_x$ with respect to the linear action of R. We define

$$\widehat{P} = (\mathrm{bl}_R(P))^{ss}.$$

By [Kir85], $\mathfrak{R}(\widehat{P}) = \mathfrak{R}(P) - \{R\}$.

Definition 2.2.2. *The scheme* $\widehat{P}$ *(resp.* $\widehat{P}/\!\!/G$*) is called the Kirwan blowup of* P *(resp.* $P/\!\!/G$*) with respect to the group* R*.*

Repeating the Kirwan blowup finitely many times, once for each element of $\mathfrak{R}(P)$ in order of decreasing dimension, we end up with a G-equivariant morphism

$$\widetilde{P} \longrightarrow P$$

which induces a projective morphism

$$\widetilde{P}/G \longrightarrow P/\!\!/G.$$

As $\widetilde{P}$ is smooth, $\widetilde{P}/G$ has at worst finite quotient singularities.

Definition 2.2.3. *The scheme* $\widetilde{P}$ *(resp.* $\widetilde{P}/G$*) is called the Kirwan partial desingularization of* P *(resp.* $P/\!\!/G$*).*

2.2.2 Kirwan blowup by slices

We show here how to perform Kirwan's algorithm by taking slices of closed points in P with closed G-orbit.

We remark that we can assume that the maximal dimension of elements in $\mathfrak{R}(P)$ is equal to the maximal dimension of elements of $\mathfrak{R}(X)$. Otherwise letting $R \in \mathfrak{R}(P)$ be an element of maximal dimension, the locus $GZ_R \subset P$ is disjoint from X. Thus we can replace P by $\widehat{P}$ and have $X \subset \widehat{P} \subset (\mathbb{P}^{N'})^{ss}$, for a different N', while having $\mathfrak{R}(\widehat{P}) = \mathfrak{R}(P) - \{R\}$.

Let $R \in \mathfrak{R}(X)$ be an element of maximal dimension. (By assumption, it is also an element in $\mathfrak{R}(P)$ of maximal dimension.) Let $x \in X$ be such that $G \cdot x$ is closed in X and the identity component of the stabilizer G_x is R. By Luna's étale slice theorem (cf. Theorem 2.1.5), there is a locally closed R-invariant smooth affine subvariety S' of P containing x such that the G-equivariant map

$$G \times_R S' \longrightarrow P, \quad (g, s) \mapsto gs,$$

is étale onto an open subset of P and the associated morphism of quotient stacks

$$[G \times_R S'/G] = [S'/R] \longrightarrow [P/G]$$

is étale.

We blow up S' along the fixed locus S'^R (of the action of R). Let $\widehat{S'}$ denote the semistable part in the blowup. (Thus $\widehat{S'}$ is the Kirwan blowup of S' associated with R.) Then we have an étale morphism

$$[\widehat{S'}/R] \longrightarrow [\widehat{P}/G]. \tag{2.2}$$

Similarly, $S = X \times_P S'$ is an étale slice of X at x, and $\widehat{S} = \widehat{S'} \times_{\widehat{P}} \widehat{X}$ is the Kirwan blowup of S associated with R, and

$$[\widehat{S}/R] \longrightarrow [\widehat{X}/G] \tag{2.3}$$

is étale.

In conclusion, the collection of (2.2) (resp. of (2.3)) together with $P - GZ_R$ (resp. with $X - GZ_R$) form an étale covering of $\widehat{P}$ (resp. of $\widehat{X}$).

Remark 2.2.4. *At this point we remark that we can follow the procedure outlined in the previous section more closely by taking a G_x-invariant slice S'' at x and then blowing it up along $G_x S''^R$. Then we would obtain a more refined version of* (2.2) *as a sequence of étale morphisms*

$$[\widehat{S''}/R] \longrightarrow [\widehat{S''}/G_x] \longrightarrow [\widehat{P}/G]$$

Moreover, the second arrow also induces an étale morphism at the level of GIT quotients

$$\widehat{S''} /\!\!/ G_x \longrightarrow \widehat{P} /\!\!/ G$$

by the properties of an étale slice (cf. [Dré04, Theorem 5.3]).

2.3 Kirwan partial desingularization of singular GIT quotients

In this section we generalize the results of the previous section to possibly singular GIT quotients, by adapting the intrinsic blowups introduced in [KL13b].

2.3.1 Intrinsic and Kirwan blowups

Suppose that U is a scheme with an action of a reductive group G. Let us assume that G is connected, as this will be the case when we take blowups throughout.

Suppose that we have an equivariant embedding $U \to V$ into a smooth G-scheme V and let I be the ideal defining U. Since $U \subset V$ is G-equivariant, G acts on I and we have a decomposition $I = I^{fix} \oplus I^{mv}$ into the fixed part of I and its complement as G-representations.

Let V^G be the fixed point locus of G inside V and $\pi\colon \mathrm{bl}_G(V) \to V$ the blowup of V along V^G. Let $E \subset \mathrm{bl}_G(V)$ be its exceptional divisor and $\xi \in \Gamma(\mathcal{O}_{\mathrm{bl}_G(V)}(E))$ the tautological defining equation of E. We claim that

$$\pi^{-1}(I^{mv}) \subset \xi \cdot \mathcal{O}_{\mathrm{bl}_G(V)}(-E) \subset \mathcal{O}_{\mathrm{bl}_G(V)}. \tag{2.4}$$

Let $\mathcal{R} : I \to I^{fix}$ be the Reynolds operator. Then for any $\zeta \in I^{mv}$, $\mathcal{R}(\zeta) = 0$. Let $x \in V^G$ be any closed point. Since $\mathcal{O}_x$ is fixed by G, we have $\zeta|_x = \mathcal{R}(\zeta)|_x = 0$. This proves that all elements in I^{mv} vanish along V^G, hence (2.4).

Consequently, $\xi^{-1}\pi^{-1}(I^{mv}) \subset \mathcal{O}_{\mathrm{bl}_G(V)}(-E) \subset \mathcal{O}_{\mathrm{bl}_G(V)}$. We define $I^{intr} \subset \mathcal{O}_{\mathrm{bl}_G(V)}$ to be

$$I^{intr} = \text{ideal generated by } \pi^{-1}(I^{fix}) \text{ and } \xi^{-1}\pi^{-1}(I^{mv}). \tag{2.5}$$

Definition 2.3.1. (Intrinsic blowup) *The G-intrinsic blowup of U is the subscheme $U^{intr} \subset \mathrm{bl}_G(V)$ defined by the ideal I^{intr}.*

Lemma 2.3.2. *The G-intrinsic blowup of U is independent of the choice of G-equivariant embedding $U \subset V$, and hence is canonical.*

Proof sketch. The proof is identical to [KL13b, Section 3.1]. We give a very brief account of the main steps involved.

One firstly establishes the claim in the case of U being a formal affine scheme such that the fixed locus U^G is an affine scheme which has the same support as U does.

For an affine scheme U with a G-action, one can then take the formal completion U^c of U along U^G and check that the G-intrinsic blowup of U^c glues naturally with $U - U^G$ to yield the G-intrinsic blowup of U.

Finally, one can show by taking a cover by (Zariski or étale) open affine schemes that the intrinsic blowup is well-defined for a general scheme or DM stack with a (representable) action of G. ☐

Remark 2.3.3. *The following remarks on the proof of Lemma 2.3.2 are in order:*

1. *If U is smooth, then the G-intrinsic blowup coincides with the blowup of U along U^G.*
2. *Since the core of the proof relies on working first in the formal completion of V^G inside V and proving the Lemma 2.3.2 in the case of formal schemes, this enables us to perform local calculations formally (or analytically) locally.*

Suppose U is an affine G-scheme, then we can think of all points of U as being semistable as in the local case for GIT in Subsection 2.1.1. We can also make sense of semistable points in U^{intr} without ambiguity.

In the Kirwan blowup, we can detect which points on the exceptional divisors that occur are unstable just by looking at the action of R on $\mathbb{P}N_{GZ_R/P}$. Furthermore, a point off the exceptional divisor is unstable if the closure of its orbit meets the unstable locus of the exceptional divisor. Here we are using again Theorem 2.2.1. Thus for any smooth affine G-scheme V, we can define its Kirwan blowup $\widehat{V}$ associated to any $R \in \mathfrak{R}(V)$ of maximal dimension.

It is not hard to see that if we have an equivariant embedding $V \to W$ between smooth schemes, then $(W^{intr})^{ss} \cap V^{intr} = (V^{intr})^{ss}$ based on our description. Hence, in the above situation where we consider $R = G$, we may define $(U^{intr})^{ss} := U^{intr} \cap (V^{intr})^{ss}$ for any equivariant embedding $U \to V$ into a smooth scheme V. This is independent of the choice of $U \to V$.

Definition 2.3.4. (Kirwan blowup) *We define the Kirwan blowup of a possibly singular affine G-scheme U associated with G to be $\widehat{U} = (U^{intr})^{ss}$.*

Example 2.3.5. *We give a few examples of Kirwan blowups when U is affine and $G = \mathbb{C}^\times$.*

1. *Suppose that U has a trivial G-action. Then $U^G = U$ and it is easy to check that $\widehat{U} = \emptyset$.*

2. *Let $V = \mathbb{C}^2_{x,y}$, where G acts on x with weight 1 and on y with weight -1. Then $V^{intr} = \mathrm{bl}_{V^G} V$ and $\widehat{V}$ is obtained by deleting the two punctured axes $(x = 0, y \neq 0)$ and $(x \neq 0, y = 0)$ as well as the points $0, \infty$ on the exceptional divisor $E = \mathbb{P}^1$.*

3. *Let $V = \mathbb{C}^2_{x,y}$ be as before and $U = (x^2y = 0,\ xy^2 = 0) \subset V$. Then $\widehat{U} \subset \widehat{V}$ is given by the vanishing of ξ^2, where ξ is the local equation defining the exceptional divisor of V^{intr}.*

Remark 2.3.6. *Suppose that $U \to V$ is a G-equivariant embedding into a smooth G-scheme V. Let, as before, I be the ideal of U in V.*

We now explain how one can proceed if G is not connected. Let G_0 be the connected component of the identity. This is a normal, connected subgroup of G of finite index. Let $I = I^{fix} \oplus I^{mv}$ be the decomposition of I into fixed and moving parts with respect to the action of G_0. Using the normality of G_0, we see that the fixed locus V^{G_0} is a closed, smooth G-invariant subscheme of V and also I^{fix}, I^{mv} are G-invariant.

Let $\pi\colon \mathrm{bl}_{V^{G_0}} V \to V$ be the blowup of V along V^{G_0} with exceptional divisor E and local defining equation ξ. Then we take I^{intr} to be the ideal generated by $\pi^{-1}(I^{fix})$ and $\xi^{-1}\pi^{-1}(I^{mv})$. Everything is G-equivariant and we define the U^{intr} as the subscheme of $\mathrm{bl}_{V^{G_0}} V$ defined by the ideal I^{intr}.

Finally, we need to delete unstable points. By the Hilbert-Mumford criterion (cf. [MFK94, Theorem 2.1]) it follows that semistability on E with respect to the action of G is the same as semistability with respect to the action of G_0, since every 1-parameter subgroup of G factors through G_0, and hence we may delete unstable points exactly as before, using the discussion in Subsection 2.2.1, and define the Kirwan blowup $\widehat{U}$.

One may check in a straightforwardly analogous way that this has the same properties (and intrinsic nature). It is obvious that if G is connected we obtain Definition 2.3.4.

2.3.2 Kirwan partial desingularization for quotient stacks

We continue working with the G-triple $X \subset P \subset (\mathbb{P}^N)^{ss}$ as in (2.1). We list

$$\mathfrak{R}(P^{ss}) = \{R_1, \ldots, R_m, \{1\}\}$$

in order of decreasing dimension.

We begin with $R = R_1 \in \mathfrak{R}(P^{ss})$. For any $x \in Z_R \subset P$, let S be an étale affine slice for x in P and let $T = S \times_P X$. (In case $x \notin X$, we can choose S so that $T = \emptyset$.)

As S is smooth, affine and R-invariant, we let $\widehat{S}$ be the Kirwan blowup of S associated with R. As $T \subset S$ is closed and R-invariant, we let $\widehat{T} \subset \widehat{S}$ be the Kirwan blowup of T associated with R. They fit into a commutative diagram

$$\begin{tikzcd}
G \times_R \widehat{T} \arrow[r] \arrow[d] & G \times_R \widehat{S} \arrow[r] \arrow[d] & \widehat{P} \arrow[d] \\
G \times_R T \arrow[r] \arrow[d] & G \times_R S \arrow[r] & P \\
X \arrow[urr] & &
\end{tikzcd} \tag{2.6}$$

This collection of étale maps $G \times_R S \to P$ cover the locus GZ_R inside P. Let $E \subset \widehat{P}$ be the exceptional divisor of $\widehat{P} \to P$. Because $P - GZ_R = \widehat{P} - E$, $P - GZ_R$ can be viewed as an open subscheme of $\widehat{P}$. Consequently, the collection of étale maps $G \times_R \widehat{S} \to \widehat{P}$ together with $P - E$ form an étale covering of $\widehat{P}$.

We next consider the collection of all possible $G \times_R \widehat{T} \subset G \times_R \widehat{S}$.

Proposition 2.3.7. *The collection of* $G \times_R \widehat{T} \to \widehat{P}$ *just mentioned together with* $X - GZ_R \subset \widehat{P} - E$ *form a closed subscheme* $\widehat{X} \subset \widehat{P}$*, called the Kirwan blowup of* X*. Further,* $\widehat{X}$ *is canonical in the sense that it is independent of the choice of slices or choice of projective embedding.*

Proof. We first show the independence from the particular choice of slices.

Let S_1, S_2 be two étale slices in P, such that $T_1 = S_1 \cap X$, $T_2 = S_2 \cap X$ are the induced slices for X. Let I_1, I_2 be the ideals of $T_1 \subset S_1$ and $T_2 \subset S_2$ respectively.

Near every point in P covered by S_1 and S_2, we can find a common étale refinement S_{12}. This can be seen as follows: Since $[P/G]$ has affine diagonal, the fiber product $S_1 \times_{[P/G]} S_2$ is an affine scheme with a $(R \times R)$-action. For any point z fixed by R we may take a slice S_{12} for z in $S_1 \times_{[P/G]} S_2$.

Consider the composition $p_i \colon S_{12} \to S_1 \times_{[P/G]} S_2 \to S_i$. Since S_{12} and S_i are smooth of the same dimension and p_i induces an isomorphism on tangent spaces at z it must be étale (up to shrinking). It is also evidently R-equivariant and hence is indeed an étale refinement of S_1 and S_2. It follows that $p_1^* I_1 = p_2^* I_2 = I_{12}$ as ideal sheaves on S_{12}, defining $T_{12} \subset S_{12}$.

Taking Kirwan blowups commutes with étale base change. We thus obtain induced étale maps $\widehat{p}_i\colon \widehat{S}_{12} \to \widehat{S}_i$, such that $\widehat{p}_1^* I_1^{intr} = \widehat{p}_2^* I_2^{intr} = I_{12}^{intr}$, defining the subscheme $\widehat{T}_{12} \subset \widehat{S}_{12}$.

Since we may cover $(G \times_R S_1) \times_P (G \times_R S_2)$ by étale opens of the form $G \times_R S_{12}$ around the fixed points of R, étale descent implies that we obtain a well-defined closed subscheme $\widehat{X} \subset \widehat{P}$.

Regarding the choice of projective embedding, we may cover X by G-invariant affine opens $U_\alpha \subset X$. Let $U_\alpha \to V_\alpha$ be equivariant embeddings into smooth G-schemes. Using those, we may define $\widehat{U}_\alpha \subset \widehat{V}_\alpha$ by first taking intrinsic blowups and restricting to semistable points. We observe that by Remark 2.1.1 the latter restriction is unambiguous, as the unstable points of the intrinsic blowup are the ones whose G-orbit closure intersects the unstable locus of the exceptional divisor. Therefore, by the canonical nature of intrinsic blowups, for each α the Kirwan blowup $\widehat{U}_\alpha$ is independent of the local embedding $U_\alpha \to V_\alpha$. Since we may choose those to come from a G-invariant open cover $V_\alpha \subset P$ of any projective embedding $X \subset P$, it follows that $\widehat{X}$ is independent of the choice of projective embedding. ☐

Remark 2.3.8. *When X^s is dense in X, then $\widehat{X} \to X$ is birational. The other extreme case is when $X^G = X$, then $\widehat{X} = \emptyset$. In general there are cases when $(X^G)_{red} = X_{red}$ and $\widehat{X} \neq \emptyset$.*

We let $P_1 = \widehat{P}$ and $X_1 = \widehat{X}$ be their respective Kirwan blowups associated with R_1. Then $X_1 \subset P_1$, and $\mathfrak{R}(P_1) = \{R_2, \cdots, R_m, \{1\}\}$. We let $X_2 \subset P_2$ be $\widehat{X_1} \subset \widehat{P_1}$, the Kirwan blowups associated with R_2, and so on, until we obtain $X_m \subset P_m$, having the property $\mathfrak{R}(P_m) = \{1\}$.

We denote

$$\widetilde{X} = X_m, \quad \widetilde{P} = P_m.$$

Definition 2.3.9. *We call $\widetilde{X}$ and $\widetilde{\mathcal{M}} = [\widetilde{X}/G]$ the Kirwan partial desingularization of X and the Artin stack $\mathcal{M} = [X/G]$, respectively.*

Remark 2.3.10. *When X^s is dense in X, $\widetilde{\mathcal{M}} \to \mathcal{M}$ is proper and birational.*

2.4 The relative case

Kirwan blowups behave well in families over a smooth curve.

Lemma 2.4.1. *Let G be a reductive group; let V be a smooth G-scheme, C a smooth curve and $\pi \colon V \to C$ a smooth G-equivariant morphism, where G acts on C trivially. Then V^G is smooth over C.*

Proof. This is a standard fact as we are working over $\mathbb{C}$. First, as V is smooth, V^G is smooth. To prove that V^G is smooth over C, we need to show that for any closed $x \in V^G$, the projection $d\pi(x) : T_x V^G \to T_{\pi(x)} C$ is surjective.

Indeed, as both V and V^G are smooth, $T_x V^G = (T_x V)^G$. Let $v \in T_x V$ so that $d\pi(x)(v) \neq 0$. Applying the Reynolds operator $\mathcal{R}$, we get that the G-invariant part of v, namely $\mathcal{R}(v) \in (T_x V)^G$, has $d\pi(x)(\mathcal{R}(v)) = d\pi(x)(v)$, thus $d\pi(x) : T_x V^G \to T_{\pi(x)} C$ is surjective. ☐

The same proof gives the following result on étale slices.

Lemma 2.4.2. *Let $\pi \colon V \to C$ be as in Lemma 2.4.1. If x is a closed point in V with reductive stabilizer H and S is an étale slice for x, then $\pi \colon S \to C$ is smooth.*

Corollary 2.4.3. *Let $\pi \colon V \to C$ be as in Lemma 2.4.1. Then for any point $c \in C$ we have a canonical isomorphism $(\widehat{V})_c \cong \widehat{V_c}$.*

Proof. This follows immediately from the fact that V^G is smooth over C. ☐

We obtain the following result on intrinsic partial desingularizations.

Proposition 2.4.4. *Let $X = (X^\dagger)^{ss}$ with $X \subset P \subset \mathbb{P}^N \times C$ be closed G-schemes as before (cf. (2.1)) except where C is a smooth curve and G acts on $\mathbb{P}^N$ via a homomorphism $G \to GL(N+1)$. Let $\widetilde{X}$ be the Kirwan partial desingularization of X. Then for any closed $c \in C$, $(\widetilde{X})_c = \widetilde{X_c}$.*

Proof. Let $\widehat{X}$ be the Kirwan blowup of X with respect to G. By the construction of Kirwan partial desingularization, the lemma follows from that $(\widehat{X})_c = \widehat{X_c}$.

We are considering the case where X comes with an equivariant C-embedding $X \subset V$ where $V \to C$ is smooth. Let $I \subset \mathcal{O}_V$ be the ideal sheaf of $X \subset V$. Then, applying the Kirwan blowup $\pi \colon \widehat{X} \to X$ we have a short exact sequence

$$0 \longrightarrow I^{intr} \longrightarrow \mathcal{O}_{\widehat{V}} \longrightarrow \mathcal{O}_{\widehat{X}} \longrightarrow 0.$$

Let $\mathbb{C}_c$ be the residue field at $c \in C$. We then have

$$I^{intr} \otimes_{\mathcal{O}_C} \mathbb{C}_c \longrightarrow \mathcal{O}_{(\widehat{V})_c} \longrightarrow \mathcal{O}_{(\widehat{X})_c} \longrightarrow 0.$$

This fits into a diagram of exact sequences

$$\begin{tikzcd}
I^{intr} \otimes_{\mathcal{O}_C} \mathbb{C}_c \arrow[r] \arrow[d] & \mathcal{O}_{(\widehat{V})_c} \arrow[r] \arrow[d] & \mathcal{O}_{(\widehat{X})_c} \arrow[r] \arrow[d] & 0 \\
(I \otimes_{\mathcal{O}_C} \mathbb{C}_c)^{intr} \arrow[r] & \mathcal{O}_{\widehat{V_c}} \arrow[r] & \mathcal{O}_{\widehat{X_c}} \arrow[r] & 0.
\end{tikzcd}$$

By the preceding corollary, the middle arrow is an isomorphism. Moreover, if $I = I^{fix} \oplus I^{mv}$ is the decomposition of I into its fixed and moving parts, since G is reductive,

$$I \otimes_{\mathcal{O}_C} \mathbb{C}_c = \left(I^{fix} \otimes_{\mathcal{O}_C} \mathbb{C}_c\right) \oplus (I^{mv} \otimes_{\mathcal{O}_C} \mathbb{C}_c)$$

is the decomposition into fixed and moving parts since the action of G on C is trivial. We conclude that the leftmost horizontal arrows have the same image under the identification $\mathcal{O}_{(\widehat{V})_c} \simeq \mathcal{O}_{\widehat{V_c}}$ and thus we have a natural isomorphism $(\widehat{X})_c \simeq \widehat{X_c}$. □

2.5 Stacks with good moduli spaces

Here we collect some useful results about the structure of a certain class of Artin stacks, namely those with affine diagonal admitting good moduli spaces, following the theory developed by Alper ét al. All the material of the section can be found in [Alp13] and [AHR15].

We have the following definition.

Definition 2.5.1. [Alp13, Definition 4.1] *A morphism $\pi\colon \mathcal{M} \to Y$, where $\mathcal{M}$ is an Artin stack and Y an algebraic space, is a good moduli space for $\mathcal{M}$ if the following hold:*

1. *π is quasi-compact and $\pi_*\colon \mathrm{QCoh}(\mathcal{M}) \to \mathrm{QCoh}(Y)$ is exact.*
2. *The natural map $\mathcal{O}_Y \to \pi_*\mathcal{O}_{\mathcal{M}}$ is an isomorphism.*

The intuition behind the introduction of the notion of good moduli space is that stacks $\mathcal{M}$ that admit good moduli spaces behave like quotient stacks $[X^{ss}/G]$ obtained from GIT with good moduli space given by the map $[X^{ss}/G] \to X^{ss}/\!\!/G$. In this sense, it is a generalization of GIT quotients for stacks.

We state the following properties of stacks with good moduli spaces.

Proposition 2.5.2. [Alp13, Proposition 4.7, Theorem 4.16, Proposition 9.1, Proposition 12.14] *Let $\mathcal{M}$ be locally noetherian and $\pi\colon \mathcal{M} \to Y$ be a good moduli space. Then:*

1. *π is surjective.*
2. *π is universally closed.*
3. *Two geometric points $x_1, x_2 \in \mathcal{M}(k)$ are identified in Y if and only if their closures $\overline{\{x_1\}}$ and $\overline{\{x_2\}}$ in $\mathcal{M}$ intersect.*
4. *Every closed point of $\mathcal{M}$ has reductive stabilizer.*
5. *Let $y \in |Y|$ be a closed point. Then there exists a unique closed point $x \in |\pi^{-1}(y)|$.*
6. *Suppose that*

$$\begin{tikzcd} \mathcal{M}' \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is a cartesian diagram of Artin stacks, with Y, Y' algebraic spaces.

   (a) *If $\mathcal{M} \to Y$ is a good moduli space, then $\mathcal{M}' \to Y'$ is a good moduli space.*

   (b) *If $Y' \to Y$ is fpqc and $\mathcal{M}' \to Y'$ is a good moduli space, then $\mathcal{M} \to Y$ is a good moduli space.*

7. *If $\mathcal{M}$ is of finite type, then Y is of finite type.*

Regarding the étale local structure of good moduli space morphisms for stacks with affine diagonal, we have the following theorem, which is a generalization of Luna's étale slice theorem for stacks.

Theorem-Definition 2.5.3. (Quotient chart) [AHR15, Theorem 2.9] *Let $\mathcal{M}$ be a locally noetherian Artin stack with a good moduli space $\pi\colon \mathcal{M} \to M$ such that π is of finite type with affine diagonal. If $x \in \mathcal{M}$ is a closed point, then there exists an affine scheme U with an action of G_x and a cartesian diagram*

$$\begin{tikzcd} {[U/G_x]} \arrow[r, "\Phi"] \arrow[d] & \mathcal{M} \arrow[d, "\pi"] \\ U/\!\!/G_x \arrow[r] & M \end{tikzcd} \tag{2.7}$$

such that Φ is étale, representable, affine and $U/\!\!/G_x$ is an étale neighbourhood of $\pi(x)$.

We refer to the data (U, Φ) as a quotient chart for $\mathcal{M}$ centered at x.

2.6 Kirwan partial desingularization for stacks with good moduli spaces

In Subsection 2.3.2, we constructed a Kirwan partial desingularization $\widetilde{\mathcal{M}}$ when $\mathcal{M}$ was a global quotient stack $[X/G]$ obtained by GIT. We also had the usual GIT morphism $\pi\colon [X/G] \to X/\!\!/G$.

In this section, we generalize the construction to the case of stacks with good moduli spaces.

Theorem 2.6.1. *Let $\mathcal{M}$ be an Artin stack of finite type over $\mathbb{C}$ with affine diagonal. Moreover, suppose that $\pi\colon \mathcal{M} \to M$ is a good moduli space morphism with π of finite type and with affine diagonal. Then there exists a canonical DM stack $\widetilde{\mathcal{M}}$, called the Kirwan partial desingularization of $\mathcal{M}$, together with a morphism $p\colon \widetilde{\mathcal{M}} \to \mathcal{M}$. Moreover, $\widetilde{\mathcal{M}}$ admits a good moduli space $\widetilde{M}$ and the induced morphism $\widetilde{M} \to M$ is proper.*

Proof. The proof is analogous to the arguments of section 2.3.2. The only difference of substance is the use of Theorem 2.5.3 instead of the usual Luna slice theorem.

Let $\mathcal{M}^{max}$ be the substack of $\mathcal{M}$ whose points have stabilizers of the maximum possible dimension. This is a closed substack of $\mathcal{M}$. For any closed point $x \in \mathcal{M}^{max}$, applying Theorem 2.5.3, we have a cartesian diagram

$$\begin{tikzcd}
{[U_x/G_x]} \arrow[r, "\Phi_x"] \arrow[d, "\pi_x"'] & \mathcal{M} \arrow[d, "\pi"] \\
U_x/\!\!/G_x \arrow[r] & M.
\end{tikzcd} \tag{2.8}$$

The morphisms Φ_x cover the locus $\mathcal{M}^{max}$. We may apply the Kirwan blowup to each quotient stack $[U_x/G_x]$ to obtain good moduli space morphisms $[\widehat{U}_x/G_x] \to \widehat{U}_x/\!\!/G_x$.

We need to check that these glue to give a stack $\mathcal{M}_1$ with a universally closed projection $\mathcal{M}_1 \to \mathcal{M}$ and a good moduli space $\mathcal{M}_1 \to M_1$. By the properties of the Kirwan blowup, the maximum stabilizer dimension of $\mathcal{M}_1$ will be lower than that of $\mathcal{M}$ and we may then repeat the procedure.

Suppose x, y are two closed points of $\mathcal{M}$ such that G_x, G_y are of maximum dimension.

We obtain a fiber diagram of stacks

$$\begin{tikzcd}
{[U_x \times_{\mathcal{M}} U_y/(G_x \times G_y)]} \arrow[r] \arrow[d] & {[U_y/G_y]} \arrow[d] \\
{[U_x/G_x]} \arrow[r] & \mathcal{M}
\end{tikzcd} \tag{2.9}$$

where $U_{xy} := U_x \times_{\mathcal{M}} U_y$ is an affine scheme. This is due to the cartesian diagram

$$\begin{tikzcd}
U_x \times_{\mathcal{M}} U_y \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\Delta_{\mathcal{M}}"] \\
U_x \times U_y \arrow[r] & \mathcal{M} \times \mathcal{M}
\end{tikzcd}$$

and the fact that $\mathcal{M}$ has affine diagonal.

By the intrinsic nature of the Kirwan blowup, one may easily verify that we obtain a diagram

$$\begin{tikzcd}
 & {[\widehat{U}_{xy}/(G_x \times G_y)]} \arrow[dl] \arrow[dr] & \\
{[\widehat{U}_x/G_x]} & & {[\widehat{U}_y/G_y]}
\end{tikzcd} \tag{2.10}$$

with affine, étale arrows and moreover there are canonical isomorphisms between $[\widehat{U}_{xy}/G_x \times G_y]$ and $[\widehat{U}_x/G_x] \times_{\mathcal{M}} [U_y/G_y]$ and $[U_x/G_x] \times_{\mathcal{M}} [\widehat{U}_y/G_y]$.

Using the charts $[\widehat{U}_x/G_x]$ together with a cover of $\mathcal{M} \setminus \mathcal{M}^{max}$, we therefore obtain an atlas for a stack $\mathcal{M}_1$ with a map to $\mathcal{M}$. By the canonical isomorphisms of the previous paragraph, $\mathcal{M}_1$ is independent of the particular choices of charts for $\mathcal{M}$.

Note that the morphisms Φ_x, Φ_y are strongly étale and hence stabilizer preserving. It follows that all arrows in (2.9) are stabilizer preserving and thus both arrows in diagram (2.10) are stabilizer preserving and étale and therefore must be strongly étale. We thus obtain a corresponding diagram of étale arrows at the level of good moduli spaces of the

Kirwan blowups

$$\begin{tikzcd}
 & & {[\widehat{U}_{xy}/(G_x \times G_y)]} \arrow[dll] \arrow[drr] \arrow[dd] & & \\
{[\widehat{U}_x/G_x]} \arrow[ddd] & & & & {[\widehat{U}_y/G_y]} \arrow[ddd] \\
 & & & & \\
 & & \widehat{U}_{xy}/\!\!/(G_x \times G_y) \arrow[dll] \arrow[drr] & & \\
\widehat{U}_x/\!\!/G_x & & & & \widehat{U}_y/\!\!/G_y
\end{tikzcd}$$

where both squares are cartesian. Hence the morphisms $[\widehat{U}_x/G_x] \to \widehat{U}_x/\!\!/G_x$ for all $x \in \mathcal{M}^{max}$ together with an atlas of $\mathcal{M} \setminus \mathcal{M}^{max}$ glue to give a morphism $\mathcal{M}_1 \to M_1$. By Proposition 2.5.2, this is a good moduli space morphism.

To see that $\mathcal{M}_1 \to M_1$ has affine diagonal, we may work étale locally. We consider the diagram

$$\begin{tikzcd}
{[U/G]} \arrow[r] \arrow[dr] & {[U/G] \times_{U/\!\!/G} [U/G]} \arrow[r] \arrow[d] & U/\!\!/G \arrow[d] \\
 & {[U/G] \times [U/G]} \arrow[r] & U/\!\!/G \times U/\!\!/G
\end{tikzcd} \tag{2.11}$$

where the right square is cartesian. If U is affine, then $[U/G]$ has affine diagonal by a standard argument (for example, see [Alp13, Example 12.10]), and $U/\!\!/G$ is also affine and thus its diagonal is a closed immersion. It follows from the diagram and the usual cancellation property that the diagonal of $[U/G] \to U/\!\!/G$ is affine. We can reduce to this case by using the above cover of $\mathcal{M}_1$ by quotient charts. This shows that $\mathcal{M}_1 \to M_1$ has indeed affine diagonal.

$\mathcal{M}_1$ also has affine diagonal since we have a cartesian diagram

$$\begin{tikzcd}
{[\widehat{U}_{xy}/(G_x \times G_y)]} \arrow[r] \arrow[d] & \mathcal{M}_1 \arrow[d] \\
{[\widehat{U}_x/G_x] \times [\widehat{U}_y/G_y]} \arrow[r] & \mathcal{M}_1 \times \mathcal{M}_1
\end{tikzcd}$$

where the lower horizontal arrows give an étale cover of $\mathcal{M}_1 \times \mathcal{M}_1$ and the left vertical arrow is affine.

$\mathcal{M}_1$, M_1 and the morphism $\mathcal{M}_1 \to M_1$ have the same properties as $\mathcal{M}$, M and $\mathcal{M} \to M$ and hence we may continue inductively to obtain the partial Kirwan desingularization $\widetilde{\mathcal{M}}$ and its good moduli space $\widetilde{\mathcal{M}} \to \widetilde{M}$.

Finally, if G is finite in (2.11), then the diagonal of $\mathcal{M} \to M$ is finite (cf. [MFK94, Proposition 0.8]). Thus $\widetilde{\mathcal{M}} \to \widetilde{M}$ is separated and by Proposition 2.5.2 also universally closed, hence proper. $\square$

Chapter 3

Semi-perfect Obstruction Theory

This chapter contains necessary material about semi-perfect obstruction theories, as developed in [CL11].

3.1 Perfect obstruction theory

Let $U \to C$ be a morphism, where U is a scheme of finite type and C a smooth quasi-projective scheme. We first recall the definition of perfect obstruction theory [BF97, LT98].

Definition 3.1.1. (Perfect obstruction theory [BF97]) *A (truncated) perfect (relative) obstruction theory consists of a morphism* $\phi\colon E \to L_{U/C}^{\geq -1}$ *in* $D^b(\operatorname{Coh} U)$ *such that*

1. *E is of perfect amplitude, contained in* $[-1, 0]$.

2. $h^0(\phi)$ *is an isomorphism and* $h^{-1}(\phi)$ *is surjective.*

We refer to $\mathcal{O}b_\phi := H^1(E^\vee)$ *as the obstruction sheaf of* ϕ.

Definition 3.1.2. (Infinitesimal lifting problem) *Let* $\iota\colon \Delta \to \bar{\Delta}$ *be an embedding with* $\bar{\Delta}$ *local Artinian, such that* $I \cdot \mathfrak{m} = 0$ *where I is the ideal of Δ and $\mathfrak{m}$ the closed point of* $\bar{\Delta}$. *We call* $(\Delta, \bar{\Delta}, \iota, \mathfrak{m})$ *a small extension. Given a commutative square*

$$\begin{tikzcd}
\Delta \arrow[r, "g"] \arrow[d, "\iota"'] & U \arrow[d] \\
\bar{\Delta} \arrow[r] \arrow[ur, dashed, "\bar{g}"'] & C
\end{tikzcd} \tag{3.1}$$

such that the image of g *contains a point* $p \in U$*, the problem of finding* $\bar{g}\colon \bar{\Delta} \to U$ *making the diagram commutative is the "infinitesimal lifting problem of* U/C *at* p*".*

Definition 3.1.3. (Obstruction space) *For a point* $p \in U$*, the intrinsic obstruction space to deforming* p *is* $T^1_{p,U/C} := H^1\left((L^{\geq -1}_{U/C})^{\vee}|_p\right)$*. The obstruction space with respect to a perfect obstruction theory* ϕ *is* $\mathrm{Ob}(\phi, p) := H^1(E^{\vee}|_p)$*.*

Given an infinitesimal lifting problem of U/C at a point p, there exists by the standard theory of the cotangent complex [Ill71] a canonical element

$$\omega\left(g, \Delta, \bar{\Delta}\right) \in \mathrm{Ext}^1\left(g^* L^{\geq -1}_{U/C}|_p, I\right) = T^1_{p,U/C} \otimes_{\mathbb{C}} I \tag{3.2}$$

whose vanishing is necessary and sufficient for the lift $\bar{g}$ to exist.

Definition 3.1.4. (Obstruction assignment) *For an infinitesimal lifting problem of* U/C *at* p *and a perfect obstruction theory* ϕ *the obstruction assignment at* p *is the element*

$$ob_U(\phi, g, \Delta, \bar{\Delta}) = h^1(\phi^{\vee})\left(\omega\left(g, \Delta, \bar{\Delta}\right)\right) \in \mathrm{Ob}(\phi, p) \otimes_{\mathbb{C}} I. \tag{3.3}$$

Suppose now that U is given by the vanishing of a global section $s \in \Gamma\left(V, F\right)$ where F is a vector bundle on a scheme V which is smooth over C. Let J denote the ideal sheaf of U in V and $j\colon U \to V$ the embedding. Then we have a perfect obstruction theory given by the diagram

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, "\phi"] & {[F^{\vee}|_U} \arrow[r, "d_{V/C}s^{\vee}"] \arrow[d, "s^{\vee}"] & {\Omega_{V/C}|_U]} \arrow[d, equal] \\ L^{\geq -1}_{U/\mathcal{M}} \arrow[r, equal] & {[J/J^2} \arrow[r, "d_{V/C}"] & {\Omega_{V/C}|_U].} \end{tikzcd} \tag{3.4}$$

Since V is smooth over C we can find a lift $g'\colon \bar{\Delta} \to V$ of the composition $j \circ g$. Composing with the section $s\colon V \to F$ we obtain a morphism $s \circ g'\colon \bar{\Delta} \to (g')^* F$. Since $g = g'|_{\Delta}$ factors through U, we must have $s \circ g' \in I \otimes_{\mathbb{C}} F|_p$.

Let $\rho\colon I \otimes_{\mathbb{C}} F|_p \to I \otimes_{\mathbb{C}} \mathcal{O}b_{\phi}|_p = I \otimes_{\mathbb{C}} \mathrm{Ob}(\phi, p)$ be the natural projection map.

Lemma 3.1.5. [KL12, Lemma 1.28] $ob_U(\phi, g, \Delta, \bar{\Delta}) = \rho\left(s \circ g'\right)$*.*

Proof. $ob_U(\phi, g, \Delta, \bar{\Delta})$ is given by the composition

$$g^*E \longrightarrow g^*L^{\geq -1}_{U/C} \longrightarrow L^{\geq -1}_{\Delta/\bar{\Delta}} \longrightarrow I[1]. \tag{3.5}$$

This fits into a commutative diagram

$$\begin{tikzcd}
g^*\Omega_{V/C} \arrow[r, equal] \arrow[d] & g^*\Omega_{V/C} \arrow[d] \arrow[dr, "ob_V"] & \\
g^*E \arrow[r, "\phi"] \arrow[d] & g^*L^{\geq -1}_{U/C} \arrow[r] & I[1] \\
g^*F^\vee[1] \arrow[urr, dashed] & &
\end{tikzcd} \tag{3.6}$$

Since V is smooth over C, the map ob_V must be zero. Using the distinguished triangle of the first column, we get a long exact sequence in cohomology

$$\begin{aligned}\operatorname{Hom}(g^*\Omega_{V/C}, I) = I \otimes_{\mathbb{C}} T_{V/C}|_p \xrightarrow{\left(d_{V/C}s^\vee\right)^\vee} \operatorname{Hom}(g^*F^\vee, I) = I \otimes_{\mathbb{C}} F|_p \\ \longrightarrow \operatorname{Ext}^1(g^*E, I) \longrightarrow \operatorname{Ext}^1(g^*\Omega_{V/C}, I).\end{aligned} \tag{3.7}$$

Now, the fact that ob_V is zero implies that $ob_U(\phi, g, \Delta, \bar{\Delta})$ lies in the cokernel $I \otimes_{\mathbb{C}} \operatorname{Ob}(\phi, p)$ of the map $(d_{V/C}s^\vee)^\vee$ in (3.7). It is now easy to see using the diagram

$$\begin{tikzcd}
g^*\Omega_{V/C}|_U \arrow[r] \arrow[d, equal] & g^*E \arrow[r] \arrow[d, "g^*\phi"] & g^*F^\vee|_U[1] \arrow[d] \arrow[dr, "(g')^*s^\vee|_\Delta"] & \\
g^*L^{\geq -1}_{V/C}|_U \arrow[r] \arrow[d] & g^*L^{\geq -1}_{U/C} \arrow[r] \arrow[d] & g^*L^{\geq -1}_{U/V} \arrow[r] \arrow[d] & g^*J/J^2[1] \arrow[d] \\
L^{\geq -1}_{\bar{\Delta}/C}|_\Delta \arrow[r] & L^{\geq -1}_{\Delta/C} \arrow[r] & L^{\geq -1}_{\Delta/\bar{\Delta}} \arrow[r] & I[1],
\end{tikzcd}$$

that indeed $ob_U(\phi, g, \Delta, \bar{\Delta}) = \rho(s \circ g')$. ☐

3.2 Symmetric obstruction theory

Symmetric obstruction theories are a special case of perfect obstruction theories. For simplicity, we take $C = \operatorname{Spec}\mathbb{C}$.

Theorem 3.2.1. [Beh09, Definition 3.5] *A perfect obstruction theory* $\phi\colon E \to L_U^{\geq -1}$ *is symmetric if* E *is endowed with an isomorphism* $\theta\colon E \to E^\vee[1]$ *satisfying* $\theta^\vee[1] = \theta$.

Remark 3.2.2. *The obstruction sheaf of a symmetric obstruction theory satisfies* $\mathcal{O}b_\phi = H^1(E^\vee) \simeq H^0(E) \simeq H^0(L_U^{\geq -1}) = \Omega_U$.

Example 3.2.3. *Suppose that* $U = (df = 0) \subset V$*, where* V *is smooth and* $f\colon V \to \mathbb{C}$ *is a regular function. Then the two-term complex*

$$E = [T_V|_U \xrightarrow{H_f} \Omega_V|_U],$$

where the arrow is given by the Hessian of f*, is a symmetric obstruction theory for* U.

However, there are symmetric obstruction theories which are not of this form, as was shown in [PT14].

3.3 Semi-perfect obstruction theory

Definition 3.3.1. *Let* $\phi\colon E \to L_{U/C}^{\geq -1}$ *and* $\phi'\colon E' \to L_{U/C}^{\geq -1}$ *be two perfect obstruction theories and* $\psi\colon \mathcal{O}b_\phi \to \mathcal{O}b_{\phi'}$ *be an isomorphism. We say that the obstruction theories give the same obstruction assignment via* ψ *if for any infinitesimal lifting problem of* U/C *at* p

$$\psi\left(ob_U(\phi, g, \Delta, \bar{\Delta})\right) = ob_U(\phi', g, \Delta, \bar{\Delta}) \in \mathrm{Ob}(\phi', p) \otimes_{\mathbb{C}} I. \tag{3.8}$$

We are now ready to give the definition of a semi-perfect obstruction theory.

Definition 3.3.2. (Semi-perfect obstruction theory [CL11]) *Let* $\mathcal{M} \to C$ *be a morphism, where* $\mathcal{M}$ *is a DM stack, proper over* C*, of finite presentation and* C *is a smooth quasi-projective scheme. A semi-perfect obstruction theory* ϕ *consists of an étale covering* $\{U_\alpha\}_{\alpha\in A}$ *of* $\mathcal{M}$ *and perfect obstruction theories* $\phi_\alpha\colon E_\alpha \to L_{U_\alpha/C}^{\geq -1}$ *such that*

1. *For each pair of indices* α, β*, there exists an isomorphism*

$$\psi_{\alpha\beta}\colon \mathcal{O}b_{\phi_\alpha}|_{U_{\alpha\beta}} \longrightarrow \mathcal{O}b_{\phi_\beta}|_{U_{\alpha\beta}}$$

so that the collection $(\mathcal{O}b_{\phi_\alpha}, \psi_{\alpha\beta})$ *gives descent data of a sheaf on* $\mathcal{M}$.

2. *For each pair of indices* α, β, *the obstruction theories* $E_\alpha|_{U_{\alpha\beta}}$ *and* $E_\beta|_{U_{\alpha\beta}}$ *give the same obstruction assignment via* $\psi_{\alpha\beta}$ *(as in Definition 3.3.1).*

Remark 3.3.3. *The obstruction sheaves* $\{\mathcal{O}b_{\phi_\alpha}\}_{\alpha\in A}$ *glue to define a sheaf* $\mathcal{O}b_\phi$ *on* $\mathcal{M}$. *This is the obstruction sheaf of the semi-perfect obstruction theory* ϕ.

Suppose now that $\mathcal{M} \to C$ is as above and admits a semi-perfect obstruction theory. Then, for each $\alpha \in A$, we have

$$\mathcal{C}_{U_\alpha/C} \subset N_{U_\alpha/C} = h^1/h^0((L^{\geq -1}_{U_\alpha/C})^\vee) \xhookrightarrow{h^1/h^0(\phi_\alpha^\vee)} h^1/h^0(E_\alpha^\vee) \subset h^1(E_\alpha^\vee),$$

where $\mathcal{C}_{U_\alpha/C}$ and $N_{U_\alpha/C}$ denote the intrinsic normal cone stack and intrinsic normal sheaf stack respectively, where by abuse of notation we identify a sheaf $\mathcal{F}$ on $\mathcal{M}$ with its sheaf stack.

We therefore obtain a cycle class $[\mathfrak{c}_{\phi_\alpha}] \in Z_*\mathcal{O}b_{\phi_\alpha}$ by taking the pushforward of the cycle $[\mathcal{C}_{U_\alpha/C}] \in Z_*N_{U_\alpha/C}$.

Theorem-Definition 3.3.4. [CL11, Theorem-Definition 3.7] *Let* $\mathcal{M}$ *be a DM stack, proper over* C, *of finite presentation and* C *a smooth quasi-projective scheme, such that* $\mathcal{M} \to C$ *admits a semi-perfect obstruction theory* ϕ. *The classes* $[\mathfrak{c}_{\phi_\alpha}] \in Z_*\mathcal{O}b_{\phi_\alpha}$ *glue to define an intrinsic normal cone cycle* $[\mathfrak{c}_\phi] \in Z_*\mathcal{O}b_\phi$. *Let* s *be the zero section of the sheaf stack* $\mathcal{O}b_\phi$. *The virtual cycle of* $\mathcal{M}$ *is defined to be*

$$[\mathcal{M}, \phi]^{\mathrm{vir}} := s^![\mathfrak{c}_\phi] \in A_*\mathcal{M},$$

where $s^! \colon Z_*\mathcal{O}b_\phi \to A_*\mathcal{M}$ *is the Gysin map. This virtual cycle satisfies all the usual properties, such as deformation invariance.*

Remark 3.3.5. *One can also consider étale covers of* $\mathcal{M}$ *by DM quotient stacks* $[U_\alpha/G_\alpha]$, *where* G_α *acts on* U_α *with finite stabilizers. Then there is a natural generalization of the notion of semi-perfect obstruction theory and Theorem 3.3.4 in this setting. This will be used in Chapter 6 in order to glue the intrinsic normal cone cycles obtained by perfect obstruction theories on a cover of this form.*

Chapter 4

Local Calculations

In this chapter, we collect a series of lemmas and propositions that will be useful in subsequent chapters. We encourage the reader to consult Subsection 5.1.1 and in particular Definition 5.1.1 and Definition 5.1.5 of a d-critical chart and an embedding of d-critical charts prior to reading the contents of this chapter, as we will be using this terminology when necessary.

4.1 Local models and standard forms

4.1.1 The absolute case

Let V be a smooth affine G-scheme. The action of G on V induces a morphism $\mathfrak{g}\otimes\mathcal{O}_V \to T_V$ and its dual $\sigma_V : \Omega_V \to \mathfrak{g}^\vee \otimes \mathcal{O}_V$.

We consider the following data on V.

Setup-Definition 4.1.1. *The quadruple (V, F_V, ω_V, D_V), where F_V is a G-equivariant vector bundle on V, ω_V an invariant section with zero locus $U = (\omega_V = 0) \subset V$ and $D_V \subset V$ an effective invariant divisor, satisfying:*

1. *$\sigma_V(-D_V) : \Omega_V(-D_V) \to \mathfrak{g}^\vee(-D_V)$ factors through a morphism ϕ_V as shown*

$$\Omega_V(-D_V) \longrightarrow F_V \xrightarrow{\phi_V} \mathfrak{g}^\vee(-D_V). \tag{4.1}$$

2. *The composition $\phi_V \circ \omega_V$ vanishes identically.*

3. *Let R be the identity component of the stabilizer group of a closed point in V with closed orbit. Let V^R denote the fixed point locus of R. Then $\phi_V|_{V^R}$ composed with the projection $\mathfrak{g}^\vee(-D_V) \to \mathfrak{r}^\vee(-D_V)$ is zero, where $\mathfrak{r}$ is the Lie algebra of R.*

gives rise to data

$$\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$$

on V. We say that these data give a weak local model structure for V. We also say that U is in weak standard form.

Remark 4.1.2. *Note that if $f\colon V \to \mathbb{A}^1$ is a G-invariant function on V, then $(U, V, \Omega_V, df, 0, \sigma_V)$ give a weak local model for V. This data is equivalent to an invariant d-critical chart (U, V, f, i) for U (see Definition 5.1.1 later). Therefore, an invariant d-critical locus (cf. Subsection 5.1.2) is a particular case of weak standard form.*

4.1.2 Blowup bundle and section

Let V be a smooth affine G-scheme, F_V a G-vector bundle on V and $\omega_V \in \Gamma(V, F_V)^G$ a G-invariant section. Then $U = (\omega_V = 0)$ is a G-invariant subscheme of V. Since G is reductive, we have a decomposition

$$F_V|_{V^G} = F_V|_{V^G}^{fix} \oplus F_V|_{V^G}^{mv}. \tag{4.2}$$

Definition 4.1.3. (Blowup bundle) *Let $\pi : \widehat{V} \to V$ be the Kirwan blowup of V associated with G. The blowup bundle of F_V, denoted by $F_{\widehat{V}}$, is defined as*

$$F_{\widehat{V}} := \ker\big(\pi^* F_V \longrightarrow \pi^*(F_V|_{V^G}) \longrightarrow \pi^*(F_V|_{V^G}^{mv})\big).$$

The blowup section

$$\omega_{\widehat{V}} \in \Gamma(\widehat{V}, F_{\widehat{V}}),$$

is the lift of ω_V, which exists since $\pi^*\omega_V$ maps to zero in $\pi^*(F_V|_{V^G}^{mv})$.

Proposition 4.1.4. *Let $U' \subset \widehat{V}$ be defined by the vanishing of $\omega_{\widehat{V}}$. Then U' is the Kirwan blowup $\widehat{U}$ of U.*

Proof. Let I be the ideal of U in V, generated by the section ω. We need to check that the ideal I^{intr} given by (2.5) coincides with the ideal generated by $\omega_{\widehat{V}}$. By the above, it suffices to work locally. But in local coordinates $\omega_{\widehat{V}}$ is obtained from $\pi^*\omega_V$ by multiplying the moving components with ξ^{-1}, where ξ is the local equation of the exceptional divisor, which immediately implies the claim. $\square$

The next lemma states that the structure of a weak local model behaves well under taking Kirwan blowups and slices thereof.

Lemma 4.1.5. *Let $\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$ be as in Setup 4.1.1. Let $\pi : \widehat{V} \to V$ be the Kirwan blowup of V associated with G. Then we have induced data $\Lambda_{\widehat{V}} = (\widehat{U}, \widehat{V}, F_{\widehat{V}}, \omega_{\widehat{V}}, D_{\widehat{V}}, \phi_{\widehat{V}})$, where $F_{\widehat{V}}$ is the blowup bundle of F_V, $\omega_{\widehat{V}}$ the blowup section and $D_{\widehat{V}} = \pi^* D_V + 2E$, that give a weak local model structure for $\widehat{V}$.*

Moreover, for a slice S of a closed point x in $\widehat{V}$ with closed G-orbit and stabilizer H, we obtain induced data $\Lambda_S = (T, S, F_S, \omega_S, D_S, \phi_S)$, where F_S is an H-equivariant bundle on S with a section ω_S and the conditions of Setup 4.1.1 are satisfied for S as well.

Proof. By pulling back via $\pi : \widehat{V} \to V$ the factorization in (4.1), we obtain

$$\pi^*\Omega_V(-D_V) \longrightarrow \pi^* F_V \xrightarrow{\pi^*\phi_V} \mathfrak{g}^\vee(-D_V).$$

Let $E \subset \widehat{V}$ be the exceptional divisor. By slight abuse of notation, we use D_V to also denote the pull-back of D_V to the blow-up. Then, applying (3) with $R = G$, $\pi^*\phi_V$ factors through $\mathfrak{g}^\vee(-E - D_V)$. Using the obvious inclusion $\Omega_{\widehat{V}}(-E) \to \pi^*\Omega_V$, we get that the morphism $\Omega_{\widehat{V}}(-E - D_V) \to \mathfrak{g}^\vee(-E - D_V)$ induced by the action of G, factors as

$$\Omega_{\widehat{V}}(-E - D_V) \longrightarrow \pi^*\Omega_V(-D_V) \longrightarrow \pi^* F_V \longrightarrow \mathfrak{g}^\vee(-E - D_V).$$

We have the following diagram

$$\begin{tikzcd}
\Omega_{\widehat{V}}(-E - D_V) \arrow[r] \arrow[d] & \pi^* F_V \arrow[r] \arrow[d] & \mathfrak{g}^\vee(-E - D_V) \arrow[d] \\
\Omega_{\widehat{V}}(-E - D_V)|_E \arrow[r] & (\pi^* F_V)\,|_E \arrow[r] \arrow[d] & \mathfrak{g}^\vee(-E - D_V)|_E \\
 & \pi^*\left(F_V|_{V^G}^{mv}\right) \arrow[ru] &
\end{tikzcd} \tag{4.3}$$

Note that

$$(\pi^* F_V)\,|_E \cong \pi^*\left(F_V|_{V^G}^{fix}\right) \oplus \pi^*\left(F_V|_{V^G}^{mv}\right) \tag{4.4}$$

and by equivariance we see that $\pi^*\phi_V|_E$ maps $\pi^*\left(F_V|_{V^G}^{fix}\right)$ to zero inside $\mathfrak{g}^\vee(-E-D_V)\,|_E$. This induces the last up-right arrow of the diagram. Therefore by taking kernels, we get a factorization

$$\sigma_{\widehat{V}} : \Omega_{\widehat{V}}(-D_{\widehat{V}}) \longrightarrow F_{\widehat{V}} \xrightarrow{\phi_{\widehat{V}}} \mathfrak{g}^\vee(-D_{\widehat{V}}), \tag{4.5}$$

where $D_{\widehat{V}} = 2E + D_V$. This shows (1) for $\widehat{V}$.

By looking at the diagram, we can easily see that it follows from the identical vanishing of $\phi_V \circ \omega_V$ on V that $\phi_{\widehat{V}} \circ \omega_{\widehat{V}}$ vanishes identically on $\widehat{V}$. This is (2).

Let us check (3) on $\widehat{V}$. Away from the exceptional divisor E, $\phi_{\widehat{V}}$ is the same as ϕ_V and hence we have the same vanishing on $\widehat{V}^R - E$. On the other hand, on $\widehat{V}$, Kirwan's general theory in [Kir85] guarantees that no new R can arise from the blow-up procedure and $(\widehat{V})^R$ is the proper transform of V^R. It readily follows that, since $\phi_{\widehat{V}}|_{(\widehat{V})^R}$ composed with the projection onto $\mathfrak{r}^\vee(-D_{\widehat{V}})$ is vanishing on $\widehat{V} - E$, it vanishes on $(\widehat{V})^R$ as desired.

Next we restrict (4.5) to a slice S in $\widehat{V}$. The fibration $G \times_H S \to G/H$ with fiber S gives an exact sequence

$$0 \longrightarrow (\mathfrak{g}/\mathfrak{h})^\vee \longrightarrow \Omega_{\widehat{V}}|_S \longrightarrow \Omega_S \longrightarrow 0,$$

where $\mathfrak{h}$ is the Lie algebra of H. The composition of the first arrow $(\mathfrak{g}/\mathfrak{h})^\vee \to \Omega_{\widehat{V}}|_S$ with the homomorphism $\sigma_{\widehat{V}} : \Omega_{\widehat{V}}|_S \to \mathfrak{g}^\vee$ induced by the action of G is the inclusion $(\mathfrak{g}/\mathfrak{h})^\vee \hookrightarrow \mathfrak{g}^\vee$. Therefore $(\mathfrak{g}/\mathfrak{h})^\vee(-D_{\widehat{V}})|_S$ is a subbundle of $\Omega_{\widehat{V}}(-D_{\widehat{V}})|_S$ and $\pi^*F_V|_S$ as well as $\mathfrak{g}^\vee(-D_{\widehat{V}})|_S$. If we take the quotient of (4.5) restricted to S by $(\mathfrak{g}/\mathfrak{h})^\vee(-D_{\widehat{V}})|_S$, we obtain a factorization

$$\sigma_S : \Omega_S(-D_S) \longrightarrow F_S \xrightarrow{\phi_S} \mathfrak{h}^\vee(-D_S) \tag{4.6}$$

of the morphism σ_S induced by the action of H on S, where D_S is the restriction of $D_{\widehat{V}}$ to S. This shows (1) for S. Finally, it is not hard to verify that (2) and (3) are also true. □

We may now give the following definition, which will be useful when we introduce the concept of Ω-compatibility later.

Definition 4.1.6. *We say that the data* $\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$ *give a local model for* V *if either they are the data of a d-critical chart* $\Lambda_V = (U, V, \Omega_V, 0, df)$ *or are obtained by such after a sequence of Kirwan blowups and/or taking slices of closed points with closed orbit. We also say that* U *is then in standard form.*

4.1.3 The relative case

In analogy with the absolute case, we give the following definition.

Definition 4.1.7. *We say that the tuple* $\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$ *gives a relative local model structure on* U *if* V *is a smooth* G*-equivariant scheme over* C*, in addition to the rest of the data satisfying Setup-Definition 4.1.1 and one of the following:*

1. $F_V = \Omega_{V/C}$*,* $D_V = 0$ *and* $\phi_V \colon \Omega_{V/C} \to \mathfrak{g}^\vee$ *is the dual of the* G*-action. In this case, we call* Λ_V *a quasi-critical chart on* V*.*
2. Λ_V *is obtained by a quasi-critical chart by a sequence of Kirwan blowups and taking étale slices of closed points with closed orbit.*

We then say that the C*-scheme* U *is in relative standard form.*

4.1.4 Obstruction theory of local model

Suppose that U is in weak standard form for data Λ_V of a local model on V.

Lemma 4.1.8. *The following sequence is a complex:*

$$K_V = [\mathfrak{g} \xrightarrow{\sigma_V^\vee} T_V|_U \xrightarrow{(d_V \omega_V^\vee)^\vee} F_V|_U \xrightarrow{\phi_V} \mathfrak{g}^\vee(-D_V)]. \tag{4.7}$$

Proof. Since ω_V is G-invariant, the composition $(d_V \omega_V^\vee)^\vee \circ \sigma_V = 0$. Moreover, since $\phi_V \circ \omega_V = 0$, by differentiating we obtain $\phi_V \circ (d_V \omega_V^\vee)^\vee = -\omega_V^\vee \circ d_V \phi_V$, which is zero when restricted to $U = (\omega_V = 0)$. This proves the lemma. □

Definition 4.1.9. (Reduced tangent and obstruction sheaf) *We define the reduced tangent sheaf and reduced obstruction sheaf of* V *to be*

$$T_V^{\mathrm{red}} := \operatorname{coker} \sigma_V^\vee, \quad \textit{and} \quad F_V^{\mathrm{red}} := \ker \phi_V.$$

The section ω_V *induces a section of* F_V^{red}*, denoted by* ω_V^{red}*.*

The restriction of the complex (4.7) to the stable part U^s of U gives rise (and is quasi-isomorphic) to a two-term complex

$$K_V^{\mathrm{red}} = [\big(d_V(\omega_V^{\mathrm{red}})^\vee\big)^\vee : T_V^{\mathrm{red}}|_{U^s} \longrightarrow F_V^{\mathrm{red}}|_{U^s}].$$

We denote $\operatorname{coker}\big(d_V(\omega_V^{\mathrm{red}})^\vee\big)^\vee|_{U^s}$ by $\mathcal{O}b_V^{\mathrm{red}}$. One can easily check that there are natural isomorphisms

$$H^1(K_V)|_{U^s} \cong H^1(K_V|_{U^s}) \cong \mathcal{O}b_V^{\mathrm{red}}. \tag{4.8}$$

We refer to $\mathcal{O}b_V^{\mathrm{red}}$ as the reduced obstruction sheaf. This is validated by the following proposition.

Proposition 4.1.10. (Reduced obstruction theory) *The dual of K_V^{red} induces a perfect obstruction theory on the DM stack $[U^s/G]$.*

Proof. On V^s, $\sigma_V^\vee$ is injective and ϕ_V is surjective. The latter follows from the fact that the surjective morphism $\sigma_V^\vee(-D_V)$ factors through ϕ_V. In particular, the two terms of K_V^{red} are bundles and U^s is the zero locus of $\omega_V^{\mathrm{red}}|_{V^s}$.

Let $q_V : V^s \to [V^s/G]$ be the quotient morphism. We have the exact triangle of truncated cotangent complexes

$$q_V^* L^{\geq -1}_{[V^s/G]} \longrightarrow L^{\geq -1}_{V^s} \longrightarrow \mathfrak{g}^\vee \longrightarrow q_V^* L^{\geq -1}_{[V^s/G]}[1],$$

from which we deduce that $T_V^{\mathrm{red}}|_{V^s} = q_V^* T^{\leq 1}_{[V^s/G]}$. Therefore, using the same triangle for U^s, we find that

$$q_U^* L^{\geq -1}_{[U^s/G]} = [I^s/(I^s)^2 \longrightarrow \Omega_V^{\mathrm{red}}|_{U^s}],$$

where I^s is the ideal of U^s in V^s, $\Omega_V^{\mathrm{red}}|_{V^s} = \big(T_V^{\mathrm{red}}|_{V^s}\big)^\vee = \ker \sigma_V|_{V^s} = q_V^* \Omega_{[V^s/G]} = q_V^* L^{\geq -1}_{[V^s/G]}$ and that there is an arrow

$$\begin{tikzcd}
K_V^{\mathrm{red}}|_{U^s}^\vee \arrow[r, equal] \arrow[d] & {[F_V^{\mathrm{red}}|_{U^s}^\vee} \arrow[r, "d_V(\omega_V^{\mathrm{red}})^\vee"] \arrow[d, "(\omega_V^{\mathrm{red}})^\vee"] & {\Omega_V^{\mathrm{red}}|_{U^s}]} \arrow[d, equal] \\
q_U^* L^{\geq -1}_{[U^s/G]} \arrow[r, equal] & {[I^s/(I^s)^2} \arrow[r, "d_V"] & {\Omega_V^{\mathrm{red}}|_{U^s}].}
\end{tikzcd}$$

Therefore $K_V^{\mathrm{red}}|_{U^s}$ descends to a perfect obstruction theory on $[U^s/G]$. □

Remark 4.1.11. *Since the rank of F_V is equal to $\dim V$, in order to obtain a zero-dimensional virtual cycle, we need to replace $F_V|_{U^s}$ by $F_V^{\mathrm{red}}|_{U^s}$, which has rank $\dim V - \dim G = \dim[V^s/G]$. Note that the surjective morphism $F_V|_{U^s} \xrightarrow{\phi_V} \mathfrak{g}^\vee(-D_V)$ induces a twisted cosection $\mathcal{O}b_{U^s} \to \mathfrak{g}^\vee(-D_V)$ (cf. [KL13a]), which enables us to make the perfect obstruction theory 0-dimensional.*

4.2 Ω-equivalence

We introduce the following definition.

Definition 4.2.1. *Let V be a smooth affine G-scheme and F_V a G-equivariant bundle on V. We say that two invariant sections $\omega_V, \bar{\omega}_V \in \Gamma(V, F_V)$ are Ω-equivalent if*

1. *$(\omega_V) = (\bar{\omega}_V) =: I_U$ as ideals in $\mathcal{O}_V$, and*
2. *there exist equivariant morphisms $A, B \colon F_V \to T_V$ such that*

$$\omega_V^\vee = \bar{\omega}_V^\vee + \bar{\omega}_V^\vee \circ A^\vee \circ (d\bar{\omega}_V^\vee) \ \left(mod\ I_U^2\right), \tag{4.9}$$
$$\bar{\omega}_V^\vee = \omega_V^\vee + \omega_V^\vee \circ B^\vee \circ (d\omega_V^\vee) \ \left(mod\ I_U^2\right). \tag{4.10}$$

The reason for introducing this notion is the following proposition.

Proposition 4.2.2. *Let U be an affine G-invariant d-critical scheme and (U, V, f, i) and (U, V, g, i) two invariant d-critical charts (see Definition 5.1.1) with V an affine, smooth G-scheme. Then $\omega_f = df$, $\omega_g = dg$ are Ω-equivalent sections of Ω_V.*

Proof. We may assume that $f - g \in I_U^2$, where I_U is the ideal of U in V.

Let $x_1, ..., x_n$ be étale coordinates on V. Let us write f_i for $\frac{\partial f}{\partial x_i}$, f_{ij} for $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and H_f for the Hessian of f for convenience (and similarly for g). Then we have $I_U = (f_i)_{i=1}^n = (g_i)_{i=1}^n$. Moreover $f - g \in I_U^2$ implies that

$$f = g + \sum_{k,l} a^{kl} \cdot g_k \cdot g_l,\ a^{kl} \in \Gamma(\mathcal{O}_V). \tag{4.11}$$

Differentiating, we obtain for any i and pair (i, j) the relations

$$\begin{aligned} f_i &= g_i + \sum_{k,l} a^{kl} \cdot g_{ki} \cdot g_l + \sum_{k,l} a^{kl} \cdot g_k \cdot g_{li} \bmod I_U^2, \\ f_{ij} &= g_{ij} + \sum_{k,l} a^{kl} \cdot g_{lj} \cdot g_{ki} + \sum_{k,l} a^{kl} \cdot g_{kj} \cdot g_{li} \bmod I_U. \end{aligned} \tag{4.12}$$

Note that we may re-write (4.12) as

$$\omega_f = \omega_g + H_g \circ A \circ dg \left(\operatorname{mod} I_U^2\right),$$

where A is a morphism $A\colon \Omega_V \to T_V$.

Since df, dg and H_g are invariant, applying the Reynolds operator we can assume that A is equivariant. Hence

$$\omega_f^\vee = \omega_g^\vee + \omega_g^\vee \circ A^\vee \circ (d_V \omega_g^\vee) \ \left(\operatorname{mod} I_U^2\right), \tag{4.13}$$

and similarly for g we have

$$\omega_g^\vee = \omega_f^\vee + \omega_f^\vee \circ B^\vee \circ (d_V \omega_f^\vee) \ \left(\operatorname{mod} I_U^2\right). \tag{4.14}$$

This proves the proposition. □

The following two lemmas show that Ω-equivalence is preserved by the operations of Kirwan blowup and taking slices of closed orbits.

Notation 4.2.3. *In what follows*

$$\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V),\ \bar{\Lambda}_V = (U, V, F_V, \bar{\omega}_V, D_V, \phi_V)$$

will denote data of a weak local model structure on V. Similarly on the Kirwan blowup we write

$$\Lambda_{\widehat{V}} = (\widehat{U}, \widehat{V}, F_{\widehat{V}}, \omega_{\widehat{V}}, D_{\widehat{V}}, \phi_{\widehat{V}}),\ \bar{\Lambda}_{\widehat{V}} = (\widehat{U}, \widehat{V}, F_{\widehat{V}}, \bar{\omega}_{\widehat{V}}, D_{\widehat{V}}, \phi_{\widehat{V}})$$

and on an étale slice thereof

$$\Lambda_S = (T, S, F_S, \omega_S, D_S, \phi_S),\ \bar{\Lambda}_S = (T, S, F_S, \bar{\omega}_S, D_S, \phi_S).$$

Lemma 4.2.4. *Let V be a smooth affine G-scheme and Λ_V, $\bar{\Lambda}_V$ as above, such that $\omega_V, \bar{\omega}_V$ are Ω-equivalent with A, B as in (4.9) and (4.10). Then the blowup sections $\omega_{\widehat{V}}, \bar{\omega}_{\widehat{V}}$ are Ω-equivalent (via induced equivariant morphisms $\widehat{A}, \widehat{B} \colon F_{\widehat{V}} \to T_{\widehat{V}}$).*

Proof. Let $i_E : E \to \widehat{V}$ be the exceptional divisor of the blowup $\pi \colon \widehat{V} \to V$. We have $N_{V^G/V} = T_V|^{mv}_{V^G}$ and therefore the relative Euler sequence

$$0 \longrightarrow \mathcal{O}_E(E) \longrightarrow \pi^* \left(T_V|^{mv}_{V^G}\right) \xrightarrow{\alpha} T_{E/V^G}(E) \longrightarrow 0. \tag{4.15}$$

We also have the tangent sequence

$$0 \longrightarrow T_{\widehat{V}} \xrightarrow{d\pi} \pi^* T_V \xrightarrow{\beta} i_{E*} T_{E/V^G}(E) \longrightarrow 0. \tag{4.16}$$

Using these and the definition of $F_{\widehat{V}}$ we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_{\widehat{V}} \arrow[r, "\gamma"] \arrow[d, dashed, "\widehat{A}"] & \pi^* F_V \arrow[r] \arrow[d, "\pi^* A"] & \pi^* F_V|_E \arrow[r] \arrow[d] & \pi^* \left(F_V|^{mv}_{V^G}\right) \arrow[d] \\
0 \arrow[r] & T_{\widehat{V}} \arrow[r, "d\pi"] & \pi^* T_V \arrow[r] \arrow[rrd, "\beta"'] & \pi^* T_V|_E \arrow[r] & \pi^* \left(T_V|^{mv}_{V^G}\right) \arrow[d, "\alpha"] \\
 & & & & T_{E/V^G}(E).
\end{tikzcd} \tag{4.17}$$

By (4.15), (4.16) and (4.17) it follows that the composition $\beta \circ \pi^* A \circ \gamma$ is zero and therefore we obtain an equivariant morphism $\widehat{A} \colon F_{\widehat{V}} \to T_{\widehat{V}}$ induced by $\pi^* A$.

It remains to check that $\widehat{A}$ satisfies (4.9) for the blowup sections $\omega_{\widehat{V}}$ and $\bar{\omega}_{\widehat{V}}$.

Let us denote

$$\eta = \omega_V^\vee - \bar{\omega}_V^\vee - \bar{\omega}_V \circ A \circ d_V \bar{\omega}_V^\vee \tag{4.18}$$

where we consider η as a $\mathbb{C}$-linear, equivariant morphism $F_V^\vee \to \mathcal{O}_V$.

It is easy to check that (4.9) is equivalent to $\eta(e_i^\vee) \in I_U^2$, where $\{e_1, ..., e_n\}$ is any local frame for F_V.

Let us choose $\{e_1, ..., e_n\}$ to be a local frame for F_V (possibly after shrinking V around $x \in V^G$) with a linear action of G. We can find such a frame by lifting equivariantly a basis of $F_V|_x$ to a neighbourhood of x, since G is reductive. We can also take a local frame

$\{dx_1, ..., dx_n\}$ for Ω_V on which G acts linearly by a similar argument.

Since G acts linearly on $e_1, ..., e_n$ we have a splitting $\mathbb{C}\langle e_1, ..., e_n\rangle = \mathbb{C}\langle w_i^{fix}\rangle \oplus \mathbb{C}\langle w_j^{mv}\rangle$ into invariant and moving subspaces. Then $\{\pi^* w_i^{fix}, \xi\pi^* w_j^{mv}\}$ is a local frame for $F_{\widehat{V}}$, where ξ is a local equation of the exceptional divisor of the blowup.

By equivariance, we see that

$$\eta((w_i^{fix})^\vee) \in (I_U^2)^{fix},\ \eta((w_j^{mv})^\vee) \in (I_U^2)^{mv} \tag{4.19}$$

We can pull back (4.9) to obtain

$$\pi^*\omega_V^\vee = \pi^*\bar{\omega}_V^\vee + \pi^*\bar{\omega}_V \circ \pi^* A \circ \pi^* \left(d_V \bar{\omega}_V^\vee\right) \left(\text{mod } \pi^* I_U^2\right). \tag{4.20}$$

The diagram of $\mathbb{C}$-linear morphisms

$$\begin{tikzcd}
\pi^* F_V^\vee \arrow[r, "\pi^*\left(d_V \bar{\omega}_V^\vee\right)"] \arrow[d, "\gamma^\vee"] & \pi^*\Omega_V \arrow[r, "\pi^* A^\vee"] \arrow[d] & \pi^* F_V^\vee \arrow[r, "\pi^*\bar{\omega}^\vee"] \arrow[d, "\gamma^\vee"] & \mathcal{O}_{\widehat{V}} \\
F_{\widehat{V}}^\vee \arrow[r, "d_{\widehat{V}}\bar{\omega}_{\widehat{V}}^\vee"'] & \Omega_{\widehat{V}} \arrow[r, "\widehat{A}^\vee"'] & F_{\widehat{V}}^\vee \arrow[ur, "\bar{\omega}_{\widehat{V}}^\vee"'] &
\end{tikzcd} \tag{4.21}$$

commutes when evaluated on the $\pi^* e_i^\vee$ and hence

$$\pi^*\bar{\omega}_V^\vee \circ \pi^* A^\vee \circ \left(d_V \bar{\omega}_V^\vee\right) = \bar{\omega}_{\widehat{V}}^\vee \circ \widehat{A}^\vee \circ \left(d_{\widehat{V}} \bar{\omega}_{\widehat{V}}^\vee\right) \circ \gamma^\vee. \tag{4.22}$$

Therefore we may re-write (4.20) as

$$\pi^*\eta = \left(\omega_{\widehat{V}}^\vee - \bar{\omega}_{\widehat{V}}^\vee - \bar{\omega}_{\widehat{V}}^\vee \circ \widehat{A}^\vee \circ \left(d_{\widehat{V}} \bar{\omega}_{\widehat{V}}^\vee\right)\right) \circ \gamma^\vee \left(\text{mod } \pi^* I_U^2\right). \tag{4.23}$$

For convenience, let us denote $\widehat{\eta} = \omega_{\widehat{V}}^\vee - \bar{\omega}_{\widehat{V}}^\vee - \bar{\omega}_{\widehat{V}}^\vee \circ \widehat{A}^\vee \circ \left(d_{\widehat{V}} \bar{\omega}_{\widehat{V}}^\vee\right)$.

By equivariance, we can un-twist by γ (which is multiplication by ξ, the equation of the exceptional divisor, on the moving part of $\pi^* F_V$) to obtain

$$\omega_{\widehat{V}}^\vee = \bar{\omega}_{\widehat{V}}^\vee - \bar{\omega}_{\widehat{V}}^\vee \circ \widehat{A}^\vee \circ \left(d_{\widehat{V}} \bar{\omega}_{\widehat{V}}^\vee\right) \left(\text{mod } I_{\widehat{U}}^2\right) \tag{4.24}$$

More precisely, we need to show that $\widehat{\eta}$ maps the dual to a local frame of $F_{\widehat{V}}$ into $I_{\widehat{U}}^2$.

(4.19) implies that

$$\widehat{\eta}\left(\pi^*(w_i^{fix})^\vee\right) \in \pi^*(I_U^2)^{fix} \subset I_{\widehat{U}}^2 \tag{4.25}$$

and $\widehat{\eta}\left(\pi^*(w_j^{mv})^\vee\right) \in \pi^*(I_U^2)^{mv}$. By the proposition following this lemma, we have $(I_U^2)^{mv} \subset I_U^{mv} I_U$ and therefore

$$\pi^*(I_U^2)^{mv} \subset \pi^* I_U^{mv} \pi^* I_U \subset \xi I_{\widehat{U}}^2.$$

In particular $\frac{1}{\xi}\widehat{\eta}\left(\pi^*(w_j^{mv})^\vee\right) \in I_{\widehat{U}}^2$.

By the definition of $\widehat{A}$ and equivariance, we have that $\widehat{A}^\vee(d_{\widehat{V}}\xi) \in \pi^*(F_V^{mv})^\vee \subset \xi F_{\widehat{V}}^\vee$ and thus $\widehat{A}^\vee(\frac{d_{\widehat{V}}\xi}{\xi}) \in F_{\widehat{V}}^\vee$. Since

$$\bar{\omega}_{\widehat{V}}^\vee \circ \widehat{A}^\vee \circ d_{\widehat{V}} \bar{\omega}_{\widehat{V}}^\vee \left(\frac{1}{\xi}\pi^*(w_j^{mv})^\vee\right) = \frac{1}{\xi}\bar{\omega}_{\widehat{V}}^\vee \circ \widehat{A}^\vee \circ d_{\widehat{V}} \bar{\omega}_{\widehat{V}}^\vee \left(\pi^*(w_j^{mv})^\vee\right) - \frac{1}{\xi}\bar{\omega}_{\widehat{V}}^\vee \left(\pi^*(w_j^{mv})^\vee\right) \bar{\omega}_{\widehat{V}}^\vee \circ \widehat{A}^\vee \circ \frac{d_{\widehat{V}}\xi}{\xi}$$

it follows that $\widehat{\eta}\left(\frac{1}{\xi}\pi^*(w_j^{mv})^\vee\right) = \frac{1}{\xi}\widehat{\eta}\left(\pi^*(w_j^{mv})^\vee\right) \left(\text{mod } I_{\widehat{U}}^2\right)$ and so

$$\widehat{\eta}\left(\frac{1}{\xi}\pi^*(w_j^{mv})^\vee\right) \in I_{\widehat{U}}^2. \tag{4.26}$$

Combining (4.25) and (4.26) concludes the proof and yields relation (4.9) for the blowup. (4.10) for the blowup follows by the same argument.

□

Proposition 4.2.5. *Suppose that V is a smooth affine G-scheme and U is an invariant closed subscheme with ideal $I_U \subset \mathcal{O}_V$. Then $(I_U^2)^{mv} \subset I_U^{mv} I_U$.*

Proof. Let $f \in (I_U^2)^{mv}$. Since the action of G on I_U^2 is rational, f is contained in an irreducible finite dimensional G-invariant subspace $V \subset (I_U^2)^{mv}$ on which G acts linearly. Let W be the $\mathbb{C}$-linear span of $\bigcup_T V_T^{mv}$ where T stands for any maximal torus in G and V_T^{mv} is the moving part of V with respect to the action of T. This is G-invariant, since $v \in V_T^{mv}$ implies that $gv \in V_{gTg^{-1}}^{mv}$, and hence by irreducibility $W = V$. In particular, there exists a finite collection $\{T_1, \dots, T_k\}$ of maximal tori such that V is the span of $\bigcup_{i=1}^k V_{T_i}^{mv}$. We may thus write $f = f_1 + ... + f_k$ with $f_i \in (I_U^2)_{T_i}^{mv} \subset (I_U^2)^{mv}$.

By working with f_i and T_i, we may assume that $G = T$ is a torus. By further splitting f into its summands, we can assume that it is an eigenvector for the action of T of weight λ. Since G is reductive, there are finite-dimensional G-invariant subspaces $W^{fix} \subset I_U^{fix}$ and

$W^{mv} \subset I_U^{mv}$ on which G acts linearly with diagonal bases $\{w_i^{fix}\}$ and $\{w_j^{mv}\}$ respectively such that $f \in W^2$ and $tw_j^{mv} = t^{\lambda_j} w_j^{mv}$ for $t \in G$. Since $W^{fix}W^{mv} \subset I_U^{mv} I_U$, we may further reduce to the case $f = \sum_{k,l} a_{kl} w_k^{fix} w_l^{fix} + \sum_{k,l} b_{kl} w_k^{mv} w_l^{mv}$ with $a_{kl}, b_{kl} \in \mathbb{C}$. By the G-action on f, we obtain

$$t^\lambda f = \sum_{k,l} a_{kl} w_k^{fix} w_l^{fix} + \sum_{k,l} b_{kl} t^{\lambda_k + \lambda_l} w_k^{mv} w_l^{mv}$$

and therefore

$$\sum_{k,l} a_{kl} w_k^{fix} w_l^{fix} + \sum_{\lambda_k + \lambda_l = 0} b_{kl} w_k^{mv} w_l^{mv} = 0$$

and

$$f = \sum_{\lambda_k + \lambda_l \neq 0} b_{kl} w_k^{mv} w_l^{mv} \in (I_U^{mv})^2 \subset I_U^{mv} I_U,$$

which concludes the proof. ☐

Lemma 4.2.6. *Let S be an étale slice of a closed point $x \in \widehat{V}$ with closed orbit and stabilizer H. Let $\Lambda_{\widehat{V}}$, $\bar{\Lambda}_{\widehat{V}}$ be data of a weak local model on $\widehat{V}$ such that $\omega_{\widehat{V}}, \bar{\omega}_{\widehat{V}}$ are Ω-equivalent. Let $\omega_S, \bar{\omega}_S$ be the two sections obtained as part of the weak local model induced on S using these two choices of data and Lemma 4.1.5. Then $\omega_S, \bar{\omega}_S$ are also Ω-equivalent.*

Proof. Let $\widehat{U}$ be the zero locus of $\bar{\omega}_{\widehat{V}}$ (or equivalently $\omega_{\widehat{V}}$) and T be the zero locus of $\bar{\omega}_S$ (equivalently ω_S). The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathfrak{g}/\mathfrak{h})^\vee \arrow[r] \arrow[d, equal] & \Omega_{\widehat{V}}|_S \arrow[r, "q_S"] \arrow[d, "\sigma_{\widehat{V}}^\vee"] & \Omega_S \arrow[r] \arrow[d, "\sigma_S^\vee"] & 0 \\
0 \arrow[r] & (\mathfrak{g}/\mathfrak{h})^\vee \arrow[r] & \mathfrak{g}^\vee \arrow[r] & \mathfrak{h}^\vee \arrow[r] & 0
\end{tikzcd}$$

induces an isomorphism of kernels $\Omega_{\widehat{V}}|_T^{\mathrm{red}} \xrightarrow{q_S|_T} \Omega_S|_T^{\mathrm{red}}$ of $\sigma_{\widehat{V}}^\vee|_T$ and $\sigma_S^\vee|_T$ respectively. Since both exact sequences are locally split we may shrink S around x and find a (H-equivariant) right inverse $r_S \colon \Omega_S \to \Omega_{\widehat{V}}|_S$ for q_S, which is then also an inverse for $q_S|_T$ when restricted to $\Omega_S|_T^{\mathrm{red}}$.

By the same argument for the commutative diagram (cf. the definition of F_S in the proof of Lemma 4.1.5)

$$\begin{tikzcd}
0 \arrow[r] & (\mathfrak{g}/\mathfrak{h})^{\vee}(-D_S) \arrow[r] \arrow[d, equal] & F_{\widehat{V}}|_S \arrow[r, "\gamma_S"] \arrow[d, "\phi_{\widehat{V}}"] & F_S \arrow[r] \arrow[d, "\phi_S"] & 0 \\
0 \arrow[r] & (\mathfrak{g}/\mathfrak{h})^{\vee}(-D_S) \arrow[r] & \mathfrak{g}^{\vee}(-D_S) \arrow[r] & \mathfrak{h}^{\vee}(-D_S) \arrow[r] & 0
\end{tikzcd}$$

we may find a right inverse $\delta_S \colon F_S \to F_{\widehat{V}}|_S$, which is an inverse for γ_S when restricted to $F_S|_T^{\mathrm{red}}$, the kernel of $\phi_S|_T$.

Consider the (non-commutative) diagram

$$\begin{tikzcd}
F_{\widehat{V}}^{\vee}|_T \arrow[r, "d_{\widehat{V}}\bar{\omega}_{\widehat{V}}^{\vee}"] & \Omega_{\widehat{V}}|_T \arrow[r, "\widehat{A}^{\vee}"] \arrow[d, "q_S"] & F_{\widehat{V}}^{\vee}|_T \arrow[r, "\bar{\omega}_{\widehat{V}}^{\vee}"] & I_{\widehat{U}}/I_{\widehat{U}}^2|_T \arrow[d, "q_S"] \\
F_S^{\vee}|_T \arrow[u, "\gamma_S^{\vee}"] \arrow[r, "d_S\bar{\omega}_S^{\vee}"'] & \Omega_S|_T \arrow[r, dashed, "\widehat{A}_S^{\vee}"'] & F_S^{\vee}|_T \arrow[u, "\gamma_S^{\vee}"] \arrow[r, "\bar{\omega}_S^{\vee}"'] & I_T/I_T^2,
\end{tikzcd}$$

where we define $\widehat{A}_S^{\vee} := \delta_S^{\vee} \circ \widehat{A}^{\vee} \circ r_S$. The left- and rightmost squares are commutative.

We check now that the composition $q_S \circ \bar{\omega}_{\widehat{V}}^{\vee} \circ \widehat{A}^{\vee} \circ d_{\widehat{V}}\bar{\omega}_{\widehat{V}}^{\vee} \circ \gamma_S^{\vee}$ is equal to $\bar{\omega}_S^{\vee} \circ \widehat{A}_S^{\vee} \circ d_S\bar{\omega}_S^{\vee}$.

By Lemma 4.1.5 and Lemma 4.1.8, the above diagram factors through the sub-diagram

$$\begin{tikzcd}
\Omega_{\widehat{V}}|_T^{\mathrm{red}} \arrow[r] \arrow[d, "q_S"] & \Omega_{\widehat{V}}|_T \arrow[r, "\widehat{A}^{\vee}"] \arrow[d, "q_S"] & F_{\widehat{V}}^{\vee}|_T \arrow[r] & \left(F_{\widehat{V}}|_T^{\mathrm{red}}\right)^{\vee} \\
\Omega_S|_T^{\mathrm{red}} \arrow[r] & \Omega_S|_T \arrow[r, dashed, "\widehat{A}_S^{\vee}"'] & F_S^{\vee}|_T \arrow[u, "\gamma_S^{\vee}"] \arrow[r] & \left(F_S|_T^{\mathrm{red}}\right)^{\vee} . \arrow[u, "\gamma_S^{\vee}"]
\end{tikzcd}$$

Observe now that by the definition of $\widehat{A}_S^{\vee}$ the outer square in this diagram commutes. This immediately implies that indeed

$$q_S \circ \bar{\omega}_{\widehat{V}}^{\vee} \circ \widehat{A}^{\vee} \circ d_{\widehat{V}}\bar{\omega}_{\widehat{V}}^{\vee} \circ \gamma_S^{\vee} = \bar{\omega}_S^{\vee} \circ \widehat{A}_S^{\vee} \circ d_S\bar{\omega}_S^{\vee}. \tag{4.27}$$

Since $\omega_S^{\vee} = q_S \circ \omega_{\widehat{V}}^{\vee} \circ \gamma_S^{\vee}$ and $\bar{\omega}_S^{\vee} = q_S \circ \bar{\omega}_{\widehat{V}}^{\vee} \circ \gamma_S^{\vee}$, applying $q_S \circ (\bullet) \circ \gamma_S^{\vee}$ to the restriction of (4.9) to the slice S and using (4.27) we obtain

$$\omega_S^{\vee} = \bar{\omega}_S^{\vee} + \bar{\omega}_S^{\vee} \circ \widehat{A}_S^{\vee} \circ d_S\bar{\omega}_S^{\vee} \left(\mathrm{mod}\ I_T^2\right).$$

The exact same argument can be used to show the existence of $\widehat{B}_S$. □

The next lemma states that for the purposes of comparing obstruction sheaves and assignments we may replace a section by any Ω-equivalent section without any effect.

Given two Ω-equivalent $\omega_V, \bar{\omega}_V \in H^0(F_V)$, denoting $U = (\omega_V = 0) = (\bar{\omega}_V = 0)$, we obtain

$$\left(d\omega_V^\vee\right)^\vee|_U,\ \left(d\bar{\omega}_V^\vee\right)^\vee|_U \colon T_V|_U \longrightarrow F_V|_U.$$

Lemma 4.2.7. *Let V be a smooth affine G-scheme and Λ_V, $\bar{\Lambda}_V$ data of a weak local model on V, such that $\omega_V, \bar{\omega}_V$ are Ω-equivalent. Then*

$$\operatorname{coker}\left(d\omega_V^\vee\right)^\vee|_U = \operatorname{coker}\left(d\bar{\omega}_V^\vee\right)^\vee|_U. \tag{4.28}$$

Moreover, the two obstruction theories on U induced by ω_V and $\bar{\omega}_V$ give the same obstruction assignments via the morphism (4.3.7).

Proof. We check that $\operatorname{im}(d\omega_V^\vee)^\vee|_U = \operatorname{im}(d\bar{\omega}_V^\vee)^\vee|_U$. Since $\omega_V, \bar{\omega}_V$ are Ω-equivalent, there exist equivariant morphisms $A, B \colon F \to T_V$ such that

$$\begin{aligned} \omega_V^\vee &= \bar{\omega}_V^\vee + \bar{\omega}_V^\vee \circ A^\vee \circ (d\bar{\omega}_V^\vee) \ \left(\operatorname{mod} I_U^2\right), \\ \bar{\omega}_V^\vee &= \omega_V^\vee + \omega_V^\vee \circ B^\vee \circ (d\omega_V^\vee) \ \left(\operatorname{mod} I_U^2\right). \end{aligned} \tag{4.29}$$

Differentiating (4.29) and dualizing, we obtain

$$\left(d\omega_V^\vee\right)^\vee = \left(d\bar{\omega}_V^\vee\right)^\vee + \left(d\bar{\omega}_V^\vee\right)^\vee \circ A \circ \left(d\bar{\omega}_V^\vee\right)^\vee (\operatorname{mod} I_U). \tag{4.30}$$

This implies that $\operatorname{im}(d\omega_V^\vee)^\vee|_U \subseteq \operatorname{im}(d\bar{\omega}_V^\vee)^\vee|_U$. By the same argument, using the second equation in (4.29), the first claim follows.

For the obstruction assignments, consider an infinitesimal lifting problem of U at p. Let $\mathcal{O}b = \operatorname{coker}(d\omega_V^\vee)^\vee|_U$ and

$$\rho \colon I \otimes_{\mathbb{C}} F_V|_p \to I \otimes_{\mathbb{C}} H^1(E|_p) = I \otimes_{\mathbb{C}} H^1(\bar{E}|_p)$$

be the quotient morphism. Then by Lemma 3.1.5, we need to show that $\rho(\omega_V \circ g') = \rho(\bar{\omega}_V \circ g')$. But this holds, since dualizing (4.29) and composing with g' we have that

$\omega_V \circ g' - \bar{\omega}_V \circ g' \in I \otimes_{\mathbb{C}} \operatorname{im}(d\omega_V^\vee)^\vee|_p$. ☐

4.3 Ω-compatibility

We begin with the following lemma, describing how normal bundles behave with respect to Kirwan blowups. It will be used repeatedly in the rest of this section.

Lemma 4.3.1. *Let $\Phi\colon V \hookrightarrow W$ be a G-equivariant embedding of smooth affine schemes. Let $N_{V/W}$ be the normal bundle of V in W, and let $\operatorname{bl}(N_{V/W})$ as in Definition 4.1.3. Then there is a natural isomorphism $\operatorname{bl}(N_{V/W}) \cong N_{\widehat{V}/\widehat{W}}$.*

Proof. Let $x \in V^G$. Up to shrinking, we have a commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\Phi"] & T_xV = \mathbb{A}^n \arrow[d, "d\Phi"] \\ W \arrow[r] & T_xW = \mathbb{A}^{n+m}, \end{tikzcd}$$

where the maps $V \to T_xV$, $W \to T_xW$ are equivariant étale and the G-action on the tangent spaces is linear. Since G is reductive, we may pick coordinates $x_1, \dots, x_n$ on $\mathbb{A}^n$ on which G acts linearly and extra coordinates $x_{n+1}, \dots, x_{n+m}$ with a linear G-action on $\mathbb{A}^{n+m}$ such that the embedding $\mathbb{A}^n \to \mathbb{A}^{n+m}$ takes the canonical form $(x_1, \dots, x_n) \to (x_1, \dots, x_n, 0, \dots, 0)$. In particular, we get étale coordinates $x_1|_W, \dots, x_{n+m}|_W$ on W, $x_1|_V, \dots, x_n|_V$ on V and may also arrange that $I_V = (x_{n+1}|_W, \dots, x_{n+m}|_W)$.

In what follows, we often write just x_i in place of $x_i|_W$ or $x_i|_V$ by abuse of notation.

Let us assume that $x_1, \dots, x_p$ and $x_{n+1}, \dots, x_{n+q}$ are moving and $x_{p+1}, \dots x_n$ and $x_{n+q+1}, \dots, x_{n+m}$ are fixed by G.

Since the question is local, we may localize at x and assume that V and $W = \operatorname{Spec} A$ are local and the maximal ideal of A is

$$\mathfrak{m} = (x_1, \dots, x_p, x_{n+1}, \dots, x_{n+q}).$$

Now $\widehat{W}$ is covered by open affines of the form (for $k = 1, \dots, p+q$)

$$R_k^n = \operatorname{Spec} A[T_{k,1}, \dots, T_{k,p+q}][\xi_k]/(T_{k,k} - 1), \tag{4.31}$$

where $x_i = \xi_k T_{k,i}$ if $i \leq p$, $x_{n-p+i} = \xi_k T_{k,i}$ if $i > p$ and $\xi_k = 0$ is the exceptional divisor in R^n_k. It is easy to see that $\widehat{V}$ is covered by such affines for $k \leq p$ and in each such we have that

$$\begin{aligned} I_{\widehat{V}}|_{R^n_k} &= \left(\frac{1}{\xi_k} x_{n+1}, \ldots, \frac{1}{\xi_k} x_{n+q}, x_{n+q+1}, \ldots, x_{n+m}\right) \\ &= \left(T_{k,p+1}, \ldots, T_{k,p+q}, x_{n+q+1}, \ldots, x_{n+m}\right). \end{aligned}$$

In particular, $N_{\widehat{V}/\widehat{W}}|_{R^n_k}$ has a basis of sections given by $\frac{\partial}{\partial T_{k,p+i}}, \frac{\partial}{\partial x_{n+q+j}}$.

Since $N_{V/W}$ has a basis by $\frac{\partial}{\partial x_{n+i}}$, we see that $\mathrm{bl}(N_{V/W})$ has a basis by $\xi_k \frac{\partial}{\partial x_{n+i}}$ for $1 \leq i \leq q$ and $\frac{\partial}{\partial x_{n+j}}$ for $q+1 \leq j \leq m$. But $x_{n+i} = \xi_k T_{k,p+i}$ implies that $\frac{1}{\xi_k} d_{\widehat{W}} x_{n+i} = d_{\widehat{W}} T_{k,p+i} \pmod{I_{\widehat{V}}}$ and thus by dualizing $\xi_k \frac{\partial}{\partial x_{n+i}} = \frac{\partial}{\partial T_{k,p+i}} \pmod{I_{\widehat{V}}}$. We see that the frames of the two bundles $N_{\widehat{V}/\widehat{W}}$ and $\mathrm{bl}(N_{V/W})$ match. This concludes the proof. □

For the purposes of comparing obstruction theories obtained by embeddings of d-critical charts and their Kirwan blowups and slices thereof, we introduce the following definition.

Definition 4.3.2. *Let $U \subset V \xrightarrow{\Phi} W$ be a sequence of G-equivariant embeddings of affine G-schemes such that U is in standard form for data $\Lambda_V = (U, V, F_V, \omega_V, D_V, \phi_V)$ and $\Lambda_W = (W, F_W, \omega_W, D_W, \phi_W)$. We say that ω_V and ω_W are Ω-compatible via Φ if the following hold:*

1. *D_W pulls back to D_V under Φ.*

2. *The embedding $\Phi \colon V \to W$ induces a surjective equivariant morphism $\eta_\Phi \colon F_W|_V \to F_V$, compatible with $\phi_W|_V$ and ϕ_V, such that $\eta_\Phi(\omega_W|_V)$ is Ω-equivalent to ω_V.*

3. *Let $\mathcal{O}b_W := \operatorname{coker}\left(d_W \omega_W^\vee\right)^\vee|_U$, $\mathcal{O}b_V := \operatorname{coker}\left(d_V \omega_V^\vee\right)^\vee|_U$. Then η_Φ induces an isomorphism*

$$\Phi^{\mathrm{ck}} \colon \mathcal{O}b_W \longrightarrow \mathcal{O}b_V.$$

Remark 4.3.3. *It makes sense to talk about an induced surjection η_Φ, since the local data we are considering arise either as the data of an embedding of d-critical charts or are obtained by such an embedding by performing Kirwan blowups and taking slices. In the former case η_Φ is just pullback of differential forms and in the latter it is canonically induced starting from pulling back differential forms and then blowing up or taking slices.*

The motivation behind the above definition is the following lemma.

Lemma 4.3.4. *Let* $(U,V,f,i) \xrightarrow{\Phi} (R,W,g,j)$ *be a* G*-equivariant embedding of invariant d-critical charts (see Definition 5.1.5), where* $V \subset W$ *as before is a pair of smooth* G*-schemes. Then* $\omega_g = dg$ *and* $\omega_f = df$ *are* Ω*-compatible.*

Proof. η_Φ is pullback of differential forms and $\eta_\Phi(\omega_g|_V) = \omega_f$. Moreover, we have exact sequences

$$\begin{tikzcd}
T_W|_U \arrow[r, "(d_W\omega_g^\vee)^\vee"] & \Omega_W|_U \arrow[r] \arrow[d, "\eta_\Phi"] & \Omega_U \arrow[r] \arrow[d, equal] & 0 \\
T_V|_U \arrow[r, "(d_V\omega_f^\vee)^\vee"] & \Omega_V|_U \arrow[r] & \Omega_U \arrow[r] & 0
\end{tikzcd}$$

from which we deduce that $\operatorname{coker}\left(d_W\omega_g^\vee\right)^\vee|_U = \operatorname{coker}(d_V\omega_f^\vee)^\vee|_U = \Omega_U$. We obtain an isomorphism on cokernels induced naturally by η_Φ. $\square$

In the rest of this subsection, we consider the following situation:

Notation 4.3.5. *Let* $U \subset V \xrightarrow{\Phi} W$ *be a sequence of* G*-equivariant embeddings of affine* G*-schemes such that* U *is in standard form for data* $\Lambda_V = (U,V,F_V,\omega_V,D_V,\phi_V)$ *and* $\Lambda_W = (U,W,F_W,\omega_W,D_W,\phi_W)$ *and* ω_V *and* ω_W *are* Ω*-compatible.*

We now check that one may compare obstruction theories given by different data of a local model with Ω-compatible sections.

Lemma 4.3.6. *Let* Λ_V*,* Λ_W *be as above. Consider the two complexes* E_W, E_V

$$(L_U^{\geq -1})^\vee \longrightarrow E_W = [T_W|_U \xrightarrow{(d_W\omega_W^\vee)^\vee} F_W|_U],$$
$$(L_U^{\geq -1})^\vee \longrightarrow E_V = [T_V|_U \xrightarrow{(d_V\omega_V^\vee)^\vee} F_V|_U],$$

on U*. The obstruction theories induced by the dual complexes* $E_W^\vee, E_V^\vee$ *give the same obstruction assignments via* Φ^{ck}*.*

Proof. Suppose we have an infinitesimal lifting problem at a closed point $p \in U$. Let

$$\rho_W \colon I \otimes F_W|_p \to I \otimes \mathcal{O}b_W|_p \quad \text{and} \quad \rho_V \colon I \otimes F_V|_p \to I \otimes \mathcal{O}b_V|_p$$

be the induced quotient morphisms. By Lemma 3.1.5, we need to show

$$\Phi^{\mathrm{ck}}\left(\rho_W(\omega_W \circ \Phi \circ g')\right) = \rho_V(\omega_V \circ g') \tag{4.32}$$

Since $\eta_\Phi(\omega_W|_V)$ and ω_V are Ω-equivalent, by Lemma 4.2.7, we may assume that $\eta_\Phi(\omega_W|_V) = \omega_V$.

Because of $\eta_\Phi \circ \omega_W \circ \Phi = \omega_V$, $\Phi^{\mathrm{ck}} \circ \rho_W = \rho_V \circ \eta_\Phi$, and the commutative diagram

$$\begin{tikzcd}
\bar{\Delta} \arrow[r, "g'"] & V \arrow[r, "\omega_V"] \arrow[d, "\Phi"] & F_V \arrow[r, "\rho_V"] & \mathcal{O}b_V \\
\Delta \arrow[u] \arrow[ur, "g"'] & W \arrow[r, "\omega_W"'] & F_W \arrow[r, "\rho_W"'] \arrow[u, "\eta_\Phi"'] & \mathcal{O}b_W, \arrow[u, "\Phi^{\mathrm{ck}}"']
\end{tikzcd}$$

(4.32) follows. ☐

We now show that Ω-compatibility is preserved under taking Kirwan blowups and slices thereof. We begin with a preparatory lemma.

Lemma 4.3.7. *Let Λ_V, Λ_W as above. Then the induced embedding of Kirwan blowups $\widehat{\Phi}\colon \widehat{V} \to \widehat{W}$ induces a morphism $\widehat{\Phi}^{\mathrm{ck}}\colon \mathcal{O}b_{\widehat{W}} \to \mathcal{O}b_{\widehat{V}}$. The same is true for an étale slice of a closed point of $\widehat{U}$ with closed orbit.*

Proof. We need to show that $\eta_{\widehat{\Phi}}$ maps $\mathrm{im}(d_{\widehat{W}}\omega_{\widehat{W}}^\vee)^\vee|_{\widehat{U}}$ to $\mathrm{im}(d_{\widehat{V}}\omega_{\widehat{V}}^\vee)^\vee|_{\widehat{U}}$. Since $\eta_{\widehat{\Phi}}$ maps $T_{\widehat{V}}|_{\widehat{U}}$ into $\mathrm{im}(d_{\widehat{V}}\omega_{\widehat{V}}^\vee)^\vee|_{\widehat{U}}$, it suffices to show that the same is true for $N_{\widehat{V}/\widehat{W}}|_{\widehat{U}}$, for any local splitting $T_{\widehat{W}}|_{\widehat{V}} = T_{\widehat{V}} \oplus N_{\widehat{V}/\widehat{W}}$.

η_Φ satisfies the same requirement, so by the same reasoning we may find a morphism $\alpha\colon N_{V/W}|_U \to T_V|_U$ such that the following diagram commutes (for a splitting $T_W|_V = T_V \oplus N_{V/W}$)

$$\begin{tikzcd}
N_{V/W}|_U \arrow[r] \arrow[dr, "\alpha"'] & T_W|_U \arrow[r, "(d_W\omega_W^\vee)^\vee"] & F_W|_U \arrow[d, "\eta_\Phi"] \\
 & T_V|_U \arrow[r, "(d_V\omega_V^\vee)^\vee"'] & F_V|_U.
\end{tikzcd}$$

By Lemma 4.2.4 and Lemma 4.3.1, we have that $\mathrm{bl}(T_V) \subset T_{\widehat{V}}$, $\mathrm{bl}(T_W|_V) \subset T_{\widehat{W}}|_{\widehat{V}}$ and

$\mathrm{bl}(N_{V/W}) = N_{\widehat{V}/\widehat{W}}$. We therefore obtain a commutative diagram

$$\begin{tikzcd}
N_{\widehat{V}/\widehat{W}}|_{\widehat{U}} \arrow[r] \arrow[dr, "\widehat{\alpha}"'] & \mathrm{bl}(T_W|_U) \arrow[r] & T_{\widehat{W}}|_{\widehat{U}} \arrow[r, "(d_{\widehat{W}}\omega^{\vee}_{\widehat{W}})^{\vee}"] & F_{\widehat{W}}|_{\widehat{U}} \arrow[d, "\eta_{\widehat{\Phi}}"] \\
 & \mathrm{bl}(T_V|_U) \arrow[r] & T_{\widehat{V}}|_{\widehat{U}} \arrow[r, "(d_{\widehat{V}}\omega^{\vee}_{\widehat{V}})^{\vee}"'] & F_{\widehat{V}}|_{\widehat{U}},
\end{tikzcd}$$

where the composition $N_{\widehat{V}/\widehat{W}}|_{\widehat{U}} \to \mathrm{bl}(T_W|_U) \to T_{\widehat{W}}|_{\widehat{U}}$ comes from the induced splitting $T_{\widehat{W}}|_{\widehat{V}} = T_{\widehat{V}} \oplus N_{\widehat{V}/\widehat{W}}$ and the desired conclusion follows.

Let now S be an étale slice for $\widehat{W}$ of a closed point $x \in \widehat{V} \xrightarrow{\Phi} \widehat{W}$ with stabilizer H and $T = \widehat{V} \cap S$, $R = \widehat{U} \cap T = \widehat{U} \cap S$. Then we have induced data Λ_T, Λ_S of a local model on T and S (cf. Lemma 4.1.5) respectively such that R is in standard form.

By the definition of ω_S, ω_T we see that the two horizontal compositions in the diagram

$$\begin{tikzcd}
T_S|_R \arrow[r] & T_{\widehat{W}}|_R \arrow[r] & F_{\widehat{W}}|_R \arrow[r] \arrow[d, "\eta_\Phi"] & F_S|_R \arrow[r, "\rho_S"] \arrow[d, "\eta_\Psi"] & \mathcal{O}b_S \\
T_T|_R \arrow[r] \arrow[u] & T_{\widehat{V}}|_R \arrow[r] \arrow[u] & F_{\widehat{V}}|_R \arrow[r] & F_T|_R \arrow[r, "\rho_T"] & \mathcal{O}b_T
\end{tikzcd}$$

equal $(d_S\omega_S^{\vee})^{\vee}$ and $(d_T\omega_T^{\vee})^{\vee}$. By Lemma 4.1.8 and an identical argument to Lemma 4.2.6, in order to show that η_Ψ maps the image $\mathrm{im}(d_S\omega_S^{\vee})^{\vee}|_R$ into $\mathrm{im}(d_T\omega_T^{\vee})^{\vee}|_R$ we may replace all bundles by their reduced analogues and get a commutative diagram of sheaves

$$\begin{tikzcd}
T_S|_R^{\mathrm{red}} \arrow[r] & T_{\widehat{W}}|_R^{\mathrm{red}} \arrow[r] & F_{\widehat{W}}|_R^{\mathrm{red}} \arrow[r] \arrow[d, "\eta_\Phi"] & F_S|_R^{\mathrm{red}} \arrow[d, "\eta_\Psi"] \\
T_T|_R^{\mathrm{red}} \arrow[r] \arrow[u] & T_{\widehat{V}}|_R^{\mathrm{red}} \arrow[r] \arrow[u] & F_{\widehat{V}}|_R^{\mathrm{red}} \arrow[r] & F_T|_R^{\mathrm{red}}
\end{tikzcd}$$

where all horizontal arrows except those in the middle are isomorphisms. It follows immediately that we obtain an induced morphism $\Psi^{\mathrm{ck}} \colon \mathcal{O}b_S \to \mathcal{O}b_T$. □

Lemma 4.3.8. *Let Λ_V, Λ_W be as above. Then the induced sections $\omega_{\widehat{V}}$ and $\omega_{\widehat{W}}$ (cf. Lemma 4.1.5) are Ω-compatible.*

Proof. By Lemma 4.2.4, it readily follows that conditions (1) and (2) in the definition of Ω-compatibility hold for the respective Kirwan blowups.

It remains to check conditions (3). By Lemma 4.3.7, we see that $\eta_{\widehat{\Phi}}$ induces a morphism $\widehat{\Phi}^{\mathrm{ck}}\colon \mathcal{O}b_{\widehat{W}} \to \mathcal{O}b_{\widehat{V}}$. Moreover, since $\eta_{\widehat{\Phi}}$ is surjective, $\widehat{\Phi}^{\mathrm{ck}}$ is surjective. We need to check that it is an isomorphism.

$$\begin{tikzcd}
T_{\widehat{W}}|_{\widehat{U}} \arrow[r, "(d_{\widehat{W}}\omega^\vee_{\widehat{W}})^\vee"] & F_{\widehat{W}}|_{\widehat{U}} \arrow[r, "\rho_W"] \arrow[d, "\eta_{\widehat{\Phi}}"] & \mathcal{O}b_{\widehat{W}} \arrow[r] \arrow[d, "\widehat{\Phi}^{\mathrm{ck}}"] & 0 \\
T_{\widehat{V}}|_{\widehat{U}} \arrow[r, "(d_{\widehat{V}}\omega^\vee_{\widehat{V}})^\vee"'] & F_{\widehat{V}}|_{\widehat{U}} \arrow[r, "\rho_V"] & \mathcal{O}b_{\widehat{V}} \arrow[r] & 0.
\end{tikzcd} \tag{4.33}$$

Consider a morphism $\Delta \to \widehat{U}$, where Δ is the spectrum of a local Artinian ring. We will show that $\ell(\mathcal{O}b_{\widehat{W}}|_\Delta) = \ell(\mathcal{O}b_{\widehat{V}}|_\Delta)$. Since length is additive in exact sequences, we have

$$\ell(\mathcal{O}b_{\widehat{W}}|_\Delta) = \ell(F_{\widehat{W}}|_\Delta) - \ell(\ker \rho_W|_\Delta) \tag{4.34}$$

We have an exact sequence

$$T_{\widehat{W}}|_\Delta \xrightarrow{(d_{\widehat{W}}\omega^\vee_{\widehat{W}})^\vee|_\Delta} F_{\widehat{W}}|_\Delta \xrightarrow{\rho_W|_\Delta} Ob_{\widehat{W}}|_\Delta \longrightarrow 0.$$

Thus $\ker \rho_W|_\Delta = \mathrm{im}(d_{\widehat{W}}\omega^\vee_{\widehat{W}})^\vee|_\Delta$ and it follows that

$$\ell(\ker \rho_W|_\Delta) = \ell(\mathrm{im}(d_{\widehat{W}}\omega^\vee_{\widehat{W}})^\vee|_\Delta) = \ell(T_{\widehat{W}}|_\Delta) - \ell(\ker(d_{\widehat{W}}\omega^\vee_{\widehat{W}})^\vee|_\Delta) \tag{4.35}$$

Notice now that

$$\ker(d_{\widehat{W}}\omega^\vee_{\widehat{W}})^\vee|_\Delta = \mathrm{coker}\left(d_{\widehat{W}}\omega^\vee_{\widehat{W}}|_\Delta\right)^\vee = \Omega_{\widehat{U}}|^\vee_\Delta \tag{4.36}$$

Therefore combining (4.34), (4.35) and (4.36) we get, since $F_{\widehat{W}}$ and $T_{\widehat{W}}$ have the same rank, $\ell(\mathcal{O}b_{\widehat{W}}|_\Delta) = \ell(\Omega_{\widehat{U}}|^\vee_\Delta)$.

An identical argument shows that $\ell(\mathcal{O}b_{\widehat{V}}|_\Delta) = \ell(\Omega_{\widehat{U}}|^\vee_\Delta)$.

Since $\ell(\mathcal{O}b_{\widehat{W}}|_\Delta) = \ell(\mathcal{O}b_{\widehat{V}}|_\Delta)$ for all such $\Delta \to \widehat{U}$ and $\widehat{\Phi}^{\mathrm{ck}}$ is surjective, we conclude that $\widehat{\Phi}^{\mathrm{ck}}$ is an isomorphism. □

Remark 4.3.9. *It might seem counterintuitive that there is an induced map $\widehat{\Phi}^{\mathrm{ck}}$, given that in the diagram (4.33) the derivative arrow $T_{\widehat{V}}|_{\widehat{U}} \to T_{\widehat{W}}|_{\widehat{U}}$ goes "the wrong way". However, the fact that we begin with d-critical charts and then perform Kirwan blowups allows us to* *show that this is possible. The corresponding situation for taking étale slices also owes to the special properties of Kirwan blowups.*

Moreover, the isomorphism property of $\widehat{\Phi}^{\mathrm{ck}}$ *depends crucially on the fact that the 2-term complexes in question induce perfect obstruction theories on* $\widehat{U}$*. This enables us to show that the kernels of* $(d_{\widehat{W}}\omega_{\widehat{W}}^{\vee})^{\vee}|_{\Delta}$ *and* $(d_{\widehat{V}}\omega_{\widehat{V}}^{\vee})^{\vee}|_{\Delta}$ *are the same and hence obtain an equality of lengths.*

Lemma 4.3.10. *Let* Λ_V*,* Λ_W *be as above. Let* S *be an étale slice for* $\widehat{W}$ *of a closed point* $x \in \widehat{V} \xrightarrow{\Phi} \widehat{W}$ *with stabilizer* H *and* $T = \widehat{V} \cap S$*,* $R = \widehat{U} \cap T = \widehat{U} \cap S$*. Then we have induced data* Λ_T*,* Λ_S *of a local model on* T *and* S *(cf. Lemma 4.1.5) respectively such that* R *is in standard form and* ω_T *and* ω_S *are* Ω*-compatible.*

Proof. By Lemma 4.2.6, $\eta_\Psi(\omega_S|_T)$ and ω_T are Ω-equivalent, since $\eta_\Phi(\omega_{\widehat{W}}|_{\widehat{V}})$ and $\omega_{\widehat{V}}$ are. Thus, by Lemma 4.3.7, conditions (1) and (2) of the definition of Ω-compatibility hold. Condition (3) follows by an identical argument involving lengths as in the previous lemma. ☐

The induced map Φ^{ck} is independent of the particular choice of embedding Φ.

Lemma 4.3.11. *Let* $U \subset V \xrightarrow{\Phi,\Psi} W$ *be* G*-equivariant embeddings of affine* G*-schemes such that* U *is in standard form for data* Λ_V *and* Λ_W*,* ω_V *and* $\eta_\Phi(\omega_W|_V)$ *are* Ω*-equivalent,* ω_V *and* $\eta_\Psi(\omega_W|_V)$ *are* Ω*-equivalent and the diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & V \arrow[d, "\Phi{,}\Psi"] \\ & W \end{tikzcd}$$

commutes. If $\Phi^{\mathrm{ck}} = \Psi^{\mathrm{ck}}$*, then the induced morphisms on the Kirwan blowups satisfy* $\widehat{\Phi}^{\mathrm{ck}} = \widehat{\Psi}^{\mathrm{ck}}$*. The same holds if we take slices of these blowups.*

Furthermore, if Λ_V *and* Λ_W *are data of d-critical charts for* U*, then we indeed have* $\Phi^{\mathrm{ck}} = \Psi^{\mathrm{ck}}$ *for any two choices of embedding* $V \to W$*.*

Proof. It suffices to show that if $\eta_\Phi - \eta_\Psi$ maps $F_W|_U$ to $\mathrm{im}(d_V\omega_V^{\vee})^{\vee}|_U$ then the same is true for $\widehat{\Phi} - \widehat{\Psi}$. Let us denote $\alpha = \eta_\Phi - \eta_\Psi$ for brevity.

Since $\alpha|_U$ maps $F_W|_U$ to $\mathrm{im}(d_V\omega_V^{\vee})^{\vee}|_U$, it follows that α factors as

$$F_W|_U \xrightarrow{\beta} T_V|_U \xrightarrow{(d_V\omega_V^{\vee})^{\vee}} F_V|_U.$$

Similarly to Lemma 4.3.7, we obtain a commutative diagram

$$\begin{tikzcd}
F_{\widehat{W}}|_{\widehat{U}} \arrow[r, "\widehat{\alpha}"] \arrow[dr, "\widehat{\beta}"'] & F_{\widehat{V}}|_{\widehat{U}} \arrow[r, "\gamma"] & \pi^*F_V|_{\widehat{U}} \\
 & T_{\widehat{V}}|_{\widehat{U}} \arrow[r, "d\pi"'] & \pi^*T_V|_{\widehat{U}}. \arrow[u, "\pi^*(d_V\omega_V^\vee)^\vee"']
\end{tikzcd}$$

Hence $\gamma \circ \widehat{\alpha} = \pi^*(d_V\omega_V^\vee)^\vee \circ d\pi \circ \widehat{\beta} = \gamma \circ (d_{\widehat{V}}\omega_{\widehat{V}}^\vee)^\vee \circ \widehat{\beta}$ on $\widehat{U}$, from which we deduce that $\gamma \circ \left(\widehat{\alpha} - (d_{\widehat{V}}\omega_{\widehat{V}}^\vee)^\vee \circ \widehat{\beta}\right) \in \pi^*I_U \cdot \pi^*F_V$, which in turn yields that $\widehat{\alpha} - (d_{\widehat{V}}\omega_{\widehat{V}}^\vee)^\vee \circ \widehat{\beta} \in I_{\widehat{U}} \cdot F_{\widehat{V}}$ and therefore $\widehat{\alpha}|_{\widehat{U}} = (d_{\widehat{V}}\omega_{\widehat{V}}^\vee)^\vee|_{\widehat{U}} \circ \widehat{\beta}|_{\widehat{U}}$, which implies what we want.

The proof for slices proceeds along the same lines of Lemma 4.2.6 and Lemma 4.3.7 and we omit it.

Finally if the two sets of data give d-critical charts of U, we have $\Phi^* - \Psi^* \colon \mathcal{O}_W \to \operatorname{im} \omega_V^\vee \subset \mathcal{O}_V$ for the map on coordinate rings and therefore by symmetry $\eta_\Phi - \eta_\Psi \colon \Omega_W|_U \to \operatorname{im}(d_V\omega_V^\vee)|_U = \operatorname{im}(d_V\omega_V^\vee)^\vee|_U \subset \Omega_V|_U$, which implies the desired equality $\Phi^{\mathrm{ck}} = \Psi^{\mathrm{ck}}$. $\square$

The following lemma asserts that taking a slice gives compatible reduced obstruction sheaves and assignments and concludes this section.

Lemma 4.3.12. *Let $(U, V, F_V, \omega_V, D_V, \phi_V)$ be the data of a local model on V. Let $\Phi \colon S \to V$ be an étale slice of a closed point of V with closed G-orbit and stabilizer H and $(T, S, F_S, \omega_S, D_S, \phi_S)$ be the induced data on S.*

Consider the diagram

$$\begin{tikzcd}
K_V|_T \arrow[r, equal] & {[\mathfrak{g}} \arrow[r] & T_V|_T \arrow[r, "(d_V\omega_V^\vee)^\vee"] & F_V|_T \arrow[r] \arrow[d] & \mathfrak{g}^\vee(-D_S)] \arrow[d] \\
K_S \arrow[r, equal] & {[\mathfrak{h}} \arrow[r] \arrow[u] & T_S|_T \arrow[r, "(d_S\omega_S^\vee)^\vee"'] \arrow[u] & F_S|_T \arrow[r] & \mathfrak{h}^\vee(-D_S)]
\end{tikzcd} \tag{4.37}$$

The surjection $\eta_\Phi \colon F_V|_S \to F_S$ induces an isomorphism $\Phi^b \colon H^1(K_V|_T) \to H^1(K_S)$ which restricts to an isomorphism $\mathcal{O}b_V^{\mathrm{red}}|_{T^s} \to \mathcal{O}b_S^{\mathrm{red}}$, via which the dual complexes $\left(K_V^{\mathrm{red}}|_T\right)^\vee, \left(K_S^{\mathrm{red}}\right)^\vee$ give the same obstruction assignments on T.

Proof. We have a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{h} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d] & \mathfrak{g}/\mathfrak{h} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_S|_T \arrow[r] & T_V|_T \arrow[r] & \mathfrak{g}/\mathfrak{h} \arrow[r] & 0
\end{tikzcd}$$

inducing an isomorphism $T_S|_T^{\mathrm{red}} \to T_V|_T^{\mathrm{red}}$.

By the definition of F_S, we have a diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & (\mathfrak{g}/\mathfrak{h})^\vee(-D_T) \arrow[r] \arrow[d, equal] & F_V|_T \arrow[r] \arrow[d] & F_S|_T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathfrak{g}/\mathfrak{h})^\vee(-D_T) \arrow[r] & \mathfrak{g}^\vee(-D_T) \arrow[r] & \mathfrak{h}^\vee(-D_T) \arrow[r] & 0
\end{tikzcd}$$

and similarly we deduce that $F_V|_T^{\mathrm{red}} \to F_S|_T^{\mathrm{red}}$ is an isomorphism. Then the central square of (4.37) factors through the commutative diagram

$$\begin{tikzcd}
T_V|_T^{\mathrm{red}} \arrow[r] & F_V|_T^{\mathrm{red}} \arrow[d] \\
T_S|_T^{\mathrm{red}} \arrow[r] \arrow[u] & F_S|_T^{\mathrm{red}},
\end{tikzcd}$$

where the vertical arrows are isomorphisms. We obtain an induced isomorphism $H^1(K_V|_T) \to H^1(K_S)$ restricting to an isomorphism $\mathcal{O}b_V^{\mathrm{red}}|_{T^s} \to \mathcal{O}b_S^{\mathrm{red}}$ by (4.8) and the obstruction assignments of the reduced complexes must match by standard arguments as in the proof of Proposition 4.3.6. □

Remark 4.3.13. *All of the above results on Ω-compatibility are true if one replaces locally closed embeddings $\Phi\colon V \to W$ by unramified morphisms and Zariski open embeddings by étale maps. This is because for our purposes it suffices to work étale locally and then Zariski open maps correspond to étale maps and locally closed embeddings to unramified morphisms. We will tacitly use this observation in the following chapters and sections of the thesis.*

One may alternatively choose to work in the complex analytic topology where everything works verbatim.

Chapter 5

(-1)-shifted Symplectic Stacks and their Truncations

5.1 D-critical loci

In this section, we recall Joyce's theory of d-critical loci, as developed in [Joy15], and establish some notation.

We comment that there is also a parallel theory of critical virtual manifolds, developed in [KL12], which is equivalent to the theory of d-critical loci for the cases considered here and could alternatively be used as well.

5.1.1 D-critical schemes

We begin by defining the notion of a d-critical chart.

Definition 5.1.1. (d-critical chart) *A d-critical chart for M is the data of (U, V, f, i) such that: $U \subseteq M$ is Zariski open, V is a smooth scheme, $f \colon V \to \mathbb{A}^1$ is a regular function on V and $U \overset{i}{\to} V$ is an embedding so that $U = (d_V f = 0) = \mathrm{Crit}(f) \subseteq V$.*

If $x \in U$, then we say that the d-critical chart (U, V, f, i) is centered at x.

Joyce defines a canonical sheaf $\mathcal{S}_M$ of $\mathbb{C}$-vector spaces with the property that for any Zariski open $U \subseteq M$ and an embedding $U \hookrightarrow V$ into a smooth scheme V with ideal I, $\mathcal{S}_M$ fits into an exact sequence

$$0 \longrightarrow \mathcal{S}_M|_U \longrightarrow \mathcal{O}_V/I^2 \xrightarrow{d_V} \Omega^1_V / I \cdot \Omega^1_V. \tag{5.1}$$

Example 5.1.2. *For a d-critical chart* (U, V, f, i) *of* M*, the element* $f + I^2 \in \Gamma(V, \mathcal{O}_V/I^2)$ *gives a section of* $\mathcal{S}_M|_U$*.*

Definition 5.1.3. (d-critical scheme) *A d-critical structure on a scheme* M *is a section* $s \in \Gamma(M, \mathcal{S}_M)$ *such that* M *admits a cover by d-critical charts* (U, V, f, i) *and* $s|_U$ *is given by* $f + I^2$ *as above on each such chart. We refer to the pair* (M, s) *as a d-critical scheme.*

For a d-critical scheme M and a Zariski open $U \subseteq M$, any embedding $U \hookrightarrow V$ into a smooth scheme can be locally made into a d-critical chart.

Proposition 5.1.4. [Joy15, Proposition 2.7] *Let* M *be a d-critical scheme,* $U \subseteq M$ *Zariski open and* $i\colon U \hookrightarrow V$ *a closed embedding into a smooth scheme* V*. Then for any* $x \in U$*, there exist Zariski open* $x \in U' \subseteq U$*,* $i(U') \subseteq V' \subseteq V$ *and a regular function* $f'\colon V' \to \mathbb{A}^1$ *such that* $(U', V', f', i|_{U'})$ *is a d-critical chart centered at* x*.*

In order to compare different d-critical charts, we need the notion of an embedding.

Definition 5.1.5. *Let* (U, V, f, i) *and* (R, W, g, j) *be two d-critical charts for a d-critical scheme* (M, s) *with* $U \subseteq R$ *Zariski open. We call a locally closed embedding* $\Phi : V \to W$ *an embedding between the two charts if* $f = g \circ \Phi\colon V \to \mathbb{A}^1$ *and the following diagram commutes*

$$\begin{tikzcd}
U \arrow[r, "i"] \arrow[d] & V \arrow[d, "\Phi"] \\
R \arrow[r, "j"] & W
\end{tikzcd}$$

By abuse of notation, we use $\Phi : (U, V, f, i) \to (R, W, g, j)$ *to denote this data.*

We then have the following way to compare different overlapping d-critical charts.

Proposition 5.1.6. [Joy15, Theorem 2.20] *Let* (U, V, f, i) *and* (S, W, g, j) *be two d-critical charts centered at* x *for a d-critical scheme* (M, s)*. Then, after possibly (Zariski) shrinking* V *and* W *around* x*, there exists a d-critical chart* (T, Z, h, k) *centered at* x *and embeddings* $\Phi : (U, V, f, i) \to (T, Z, h, k)$*,* $\Psi : (R, W, g, j) \to (T, Z, h, k)$*.*

5.1.2 Equivariant d-critical loci

For our purposes, we need equivariant analogues of the results of Section 5.1.1. The theory works in parallel as before (cf. [Joy15, Section 2.6]).

Definition 5.1.7. (Good action) *Let G be an algebraic group acting on a scheme M. We say that the action is good if M has a cover $\{U_\alpha\}_{\alpha \in A}$ where every $U_\alpha \subseteq M$ is an invariant open affine subscheme of M.*

Remark 5.1.8. *If M is obtained by GIT so that it is the semistable locus of a projective scheme with a linearized G-action, then the action of G on M is good.*

It is straightforward to extend Definitions 2.1, 2.2 and 2.3 and Proposition 5.1.4 in the equivariant setting (cf. [Joy15, Definition 2.40]).

Proposition 5.1.9. [Joy15, Remark 2.47] *Let G be a complex reductive group with a good action on a scheme M. Suppose that (M, s) is an invariant d-critical scheme. Then the following hold:*

1. *For any $x \in M$ fixed by G, there exists an invariant d-critical chart (U, V, f, i) centered at x, i.e. an invariant open affine $U \ni x$, a smooth scheme V with a G-action, an invariant regular function $f \colon V \to \mathbb{A}^1$ and an equivariant embedding $i \colon U \to V$ so that $U = \operatorname{Crit}(f) \subseteq V$.*
2. *Let (U, V, f, i) and (S, W, g, j) be two invariant d-critical charts centered at the fixed point $x \in M$. Then, after possibly shrinking V and W around x, there exists an invariant d-critical chart (T, Z, h, k) centered at x and equivariant embeddings $\Phi : (U, V, f, i) \to (T, Z, h, k)$, $\Psi : (S, W, g, j) \to (T, Z, h, k)$.*

Remark 5.1.10. *If G is a torus $(\mathbb{C}^*)^k$, then Proposition 5.1.9 is true without the assumption that x is a fixed point of G.*

Remark 5.1.11. *There is a notion of d-critical locus for Artin stacks $\mathcal{M}$ (cf. [Joy15, Section 2.8]). Then (cf. [Joy15, Example 2.55]) d-critical structures on quotient stacks $[M/G]$ are in bijective correspondence with invariant d-critical structures on M.*

Moreover, one may pull back d-critical structures along smooth morphisms between stacks.

5.2 (-1)-shifted symplectic stacks

In what follows, we assume that $C = \operatorname{Spec} S$ is a smooth affine curve over $\mathbb{C}$. Whenever we refer to a reductive group, we assume that it acts trivially on C. All cotangent complexes and Kähler differentials will be relative to C, unless noted otherwise.

5.2.1 Derived algebraic geometry: introduction and local model

In this subsection, we give a very brief introduction on derived algebraic geometry, as developed by Toën-Vezzosi and Lurie. Since we will be interested mostly in local calculations, we focus on the local picture. Results and properties that are quoted here can be found in the exposition of [BBJ13, Sections 2,3].

In classical algebraic geometry, the fundamental building blocks are affine schemes $\operatorname{Spec} R$, where R is a commutative S-algebra. In derived algebraic geometry, we consider commutative differential graded algebras $(A^\bullet, \delta)$ (cdga's) over S, which are negatively graded. A cdga $(A^\bullet, \delta)$ gives rise to the affine derived scheme $\mathbf{Spec}\, A^\bullet$ with underlying classical truncation the affine scheme $t_0(\mathbf{Spec}\, A^\bullet) = \operatorname{Spec} A^\bullet = \operatorname{Spec} H^0(A^\bullet)$. $\mathbf{Spec}\, A^\bullet$ and $\operatorname{Spec} A^\bullet$ have the same underlying topological space, but different rings of functions.

This analogy may be continued to define derived schemes and stacks. Derived Artin stacks are ∞-functors

$$\boldsymbol{\mathcal{M}} \colon \{\text{commutative differential graded } S\text{-algebras}\} \longrightarrow \{\text{simplicial sets}\}$$

satisying certain conditions. $\boldsymbol{\mathcal{M}}$ is an affine derived scheme if it is equivalent to $\mathbf{Spec}\, A^\bullet$ and a derived scheme if it has a Zariski open cover by affine derived schemes. As in the case of affine derived schemes, $\boldsymbol{\mathcal{M}}$ has a classical truncation $t_0(\boldsymbol{\mathcal{M}}) = \mathcal{M}$, which is a classical Artin stack. The categories of classical schemes and Artin stacks embed fully faithfully into the category of derived Artin stacks.

Notions from classical algebraic geometry naturally translate into derived algebraic geometry. In particular, there is a notion of derived cotangent complex, sharing similar properties with the classical cotangent complex, which is moreover preserved under base change.

We wish to work with the following local model.

Definition 5.2.1. (Standard form cdga) *We say that a cdga $(A^\bullet, \delta)$ is of standard form if A^0 is a smooth S-algebra and, as a graded algebra, it is freely generated over A^0 by finitely many generators in each negative degree (i.e. it is quasifree).*

If $A^\bullet$ is of standard form, then its derived cotangent complex $\mathbb{L}_{A^\bullet}$ is represented by the Kähler differentials together with the internal differential $(\Omega_{A^\bullet}, \delta)$. We also have the usual de Rham differential d on $\Omega_{A^\bullet}$, so that we obtain a mixed complex. Moreover, $\mathbb{L}_{A^\bullet} \otimes H^0(A^\bullet)$ is a complex of free $H^0(A^\bullet)$-modules.

Definition 5.2.2. (Minimality) *Let* $(A^\bullet, \delta)$ *be of standard form and* $x \in \boldsymbol{Spec}\, A^\bullet$*. We say that* $A^\bullet$ *is minimal at* x *if all the differentials in* $\mathbb{L}_{A^\bullet}|_x$ *are zero.*

If G is a reductive group acting on $(A^\bullet, \delta)$ (and by our convention trivially on C), then we have analogous equivariant statements. We will consider minimality at x only when x is fixed by G.

The next theorem shows that every derived scheme can be locally modelled by a minimal cdga of standard form.

Theorem 5.2.3. [BBJ13, Theorem 4.1] *Let* $\boldsymbol{X}$ *be a derived locally finitely presented* C*-scheme and* $x \in \boldsymbol{X}$*. Then there exists a Zariski open inclusion* $\boldsymbol{Spec}\, A^\bullet \to \boldsymbol{X}$*, mapping* $p \in \boldsymbol{Spec}\, A^\bullet$ *to* x*, where* $A^\bullet$ *is a cdga of standard form which is minimal at* p*.*

5.2.2 (−1)-shifted symplectic structures

We proceed to give a brief account of (−1)-shifted symplectic forms on an affine derived scheme $\mathbf{Spec}\, A^\bullet$, introduced in [PTVV13].

Definition 5.2.4. ((−1)-shifted symplectic form) *We say that* $\boldsymbol{\omega} = (\omega_0, \omega_1, ...)$ *is a* (-1)*-shifted symplectic form on* $\boldsymbol{Spec}\, A^\bullet$ *if* $\omega_i \in \left(\wedge^{2+i}\Omega_{A^\bullet}\right)^{-1-i}$ *such that*

1. ω_0 *gives a quasi-isomorphism* $\mathbb{L}_{A^\bullet} \to \mathbb{T}_{A^\bullet}[1]$,
2. $\delta\omega_0 = 0$ *and* $d\omega_i + \delta\omega_{i+1} = 0$ *for* $i \geq 0$.

We refer to condition (1) as the non-degeneracy property and condition (2) as the closedness property.

Two forms $\boldsymbol{\omega}, \boldsymbol{\omega'}$ *are equivalent if there exist* $\alpha_i \in \left(\wedge^{2+i}\Omega_{A^\bullet}\right)^{-2-i}$ *such that* $\omega_0 - \omega_0' = \delta\alpha_0$ *and for all* $i \geq 0$, $\omega_{i+1} - \omega_{i+1}' = d\alpha_i + \delta\alpha_{i+1}$.

A (−1)-shifted symplectic form can be defined on a derived Artin stack by smooth descent. Suppose now that the derived quotient stack $[\mathbf{Spec}\, A^\bullet/G]$ has a (−1)-shifted symplectic form $\boldsymbol{\omega}$, where $A^\bullet$ is in standard form and minimal at a fixed point x. Then ω_0 induces a quasi-isomorphism

$$\mathbb{L}_{[\mathbf{Spec}\, A^\bullet/G]}|_{H^0(A^\bullet)} \to \mathbb{T}_{[\mathbf{Spec}\, A^\bullet/G]}|_{H^0(A^\bullet)}[1].$$

This implies that $\mathbb{L}_{A^\bullet}$ must have Tor-amplitude $[-2,0]$ and therefore $A^\bullet$ is freely generated over A^0 in degrees -1 and -2 by generators $y_j \in A^{-1}$ and $w_k \in A^{-2}$ respectively. We may write the above quasi-isomorphism as the following equivariant morphism of complexes

$$\begin{tikzcd}
V^{-2} \arrow[r] \arrow[d] & V^{-1} \arrow[r] \arrow[d] & \Omega_{A^0} \arrow[r] \arrow[d] & \mathfrak{g}^\vee \arrow[d] \\
\mathfrak{g} \arrow[r] & T_{A^0} \arrow[r] & (V^{-1})^\vee \arrow[r] & (V^{-2})^\vee .
\end{tikzcd} \tag{5.2}$$

Since $A^\bullet$ is minimal at the fixed point x, we may localize G-invariantly around x and assume that the vertical arrows are isomorphisms. In particular, we obtain an isomorphism $V^{-1} \simeq T_{A^0}$. Let x_i be a set of étale coordinates for A^0. We may now choose generators $y_i \in A^{-1}$ such that $dy_i \in V^{-1}$ are the dual basis to $dx_i \in \Omega_{A^0}$ under this isomorphism. We may thus identify A^{-1} and T_{A^0} as A^0-modules. Therefore the differential $\delta\colon T_{A^0} \to A^0$ induces an invariant section ω of Ω_{A^0} whose zero locus is precisely $\operatorname{Spec} H^0(A^\bullet)$.

Definition 5.2.5. *(Special cdga) We say that a cdga $(A^\bullet, \delta)$ with a G-action, minimal at a fixed point x of G, is special if it is freely generated over A^0 in degrees -1 and -2 by generators y_i and w_j respectively, together with an identification $A^{-1} = V^{-1}$ (where V^{-1} is as in* (5.2)*) mapping y_i to dy_i.*

We denote $U = \operatorname{Spec} H^0(A^\bullet)$ and $V = \operatorname{Spec} A^0$.

By Theorem 5.2.3, every finitely presented affine derived scheme is (up to Zariski shrinking) equivalent to $\mathbf{Spec}\, A^\bullet$ with $A^\bullet$ a standard form cdga and this is also true in the equivariant setting around a fixed point x of G. For more details, we refer to the proof of [BBJ13, Theorem 4.1], which is valid in the equivariant setting as well. We deduce the following proposition.

Proposition 5.2.6. *Let U be an affine G-scheme over C, which is the classical scheme associated to a derived affine G-scheme $\boldsymbol{U}$ such that the stack $[\boldsymbol{U}/G]$ is (-1)-shifted symplectic with form $\boldsymbol{\omega}$. Moreover, let $x \in U$ be a fixed point of G. Then, up to equivariant Zariski shrinking, $\boldsymbol{U}$ is equivalent to $\boldsymbol{Spec}\, A^\bullet$, where $(A^\bullet, \delta)$ is a special cdga, minimal at x.*

In particular, there exists a smooth affine G-scheme $V \to C$, a G-equivariant embedding $U \to V$ over C minimal at x and an induced invariant 1-form $\omega \in H^0(\Omega_{V/C})$ such that $U = Z(\omega) \subset V$ is the zero locus of ω.

We can use the above to understand the local structure of quotient stacks that arise as the truncation of (−1)-shifted symplectic derived stacks. The following proposition can be deduced by work of Halpern-Leistner[1] [HL].

Proposition 5.2.7. *Let $\boldsymbol{\mathcal{M}} \to C$ be a (-1)-shifted symplectic derived stack whose truncation $\mathcal{M} = [X/G] \to C$ is a quotient stack such that the action of G on X is good (and trivial on C). Let $x \in \mathcal{M}$ be a closed point with reductive stabilizer R. Then there exists an étale morphism $\boldsymbol{f} \colon [\boldsymbol{T}/R] \to \boldsymbol{\mathcal{M}}$, a point $t \in [\boldsymbol{T}/R]$ fixed by R, where $\boldsymbol{T}$ is equivalent to $\boldsymbol{Spec}\, A^\bullet$ with $(A^\bullet, \delta)$ a special cdga, minimal at t, mapping t to x and inducing the inclusion $R \subset G$ on stabilizer groups.*

At the classical level, we get an étale morphism $f \colon [T/R] \to \mathcal{M}$. There exists a smooth affine R-scheme $S \to C$, a G-equivariant embedding $T \to S$ over C minimal at t and an induced invariant 1-form $\omega \in H^0(\Omega_{S/C})$ such that $T = Z(\omega) \subset S$ is the zero locus of ω.

Proof. Let $x \in U$ be a G-invariant affine open in X (such exists since the G-action on X is good). Since U is affine and G is reductive, by [HL, Lemma 2.4], there exists an affine derived G-scheme $\boldsymbol{U} = \mathbf{Spec}\, B^\bullet$ such that we have a fiber diagram

$$\begin{tikzcd}
{[U/G]} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\
{[\boldsymbol{U}/G]} \arrow[r] & \boldsymbol{\mathcal{M}}.
\end{tikzcd}$$

Let $V = \operatorname{Spec} B^0$. By Luna's étale slice theorem, we may pick an affine slice $x \in S$ in V and obtain an étale map $[T/R] \to [U/G]$, where $T = U \cap S$, induced from the étale map $[S/R] \to [V/G]$. Using $S \to V$, there exists a derived affine scheme $\boldsymbol{T} = \mathbf{Spec}\, C^\bullet$ with an R-action and $C^0 = B^0$, whose classical truncation is T, fitting in a fiber diagram

$$\begin{tikzcd}
{[T/R]} \arrow[r] \arrow[d] & {[U/G]} \arrow[d] \\
{[\boldsymbol{T}/R]} \arrow[r] & {[\boldsymbol{U}/G]}.
\end{tikzcd}$$

where the lower horizontal arrow is étale. Therefore $[\boldsymbol{T}/R]$ is (-1)-shifted symplectic and we may apply Proposition 5.2.6 to deduce that $\boldsymbol{T}$ is equivalent to $\mathbf{Spec}\, A^\bullet$, where $(A^\bullet, \delta)$ is special and minimal at t. ☐

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[1] At the time of writing of this thesis, it is available online at `http://www.math.columbia.edu/~danhl/derived_equivalences_2016_09_18.pdf`.

Remark 5.2.8. *One of the main results of [BBJ13] is that, in the absolute case, if $A^\bullet$ is a standard form $\mathbb{C}$-cdga with a (-1)-shifted symplectic form $\boldsymbol{\omega}$, then up to equivalence and possible shrinking we have $\boldsymbol{\omega} = (\omega_0, 0, 0, \dots)$, where $\omega_0 = d\phi$ and $\delta\phi = d\Phi$ for $\phi \in (\Omega^\bullet_A)^{-1}$ and $\Phi \in A^0$. In particular, this implies that $\operatorname{Spec} H^0(A^\bullet)$ is the critical locus of Φ inside $\operatorname{Spec} A^0$. This also works equivariantly as in the above.*

In [BBBBJ15], it is shown that the classical truncation of a (-1)-shifted symplectic Artin stack inherits a d-critical structure. Thus one gets a truncation functor from (-1)-shifted symplectic derived Artin stacks to d-critical Artin stacks.

5.2.3 Comparison of local presentations

We first examine how the 1-form ω changes if one moves $\boldsymbol{\omega}$ within its equivalence class.

Proposition 5.2.9. *Let $(A^\bullet, \delta)$ be a special cdga, minimal at x. Suppose that $\boldsymbol{\omega}, \boldsymbol{\eta}$ are equivalent (-1)-shifted symplectic forms on $[\boldsymbol{Spec}\, A^\bullet/G]$. Then, up to equivariant shrinking of V around x, these induce 1-forms $\omega, \eta \in H^0(\Omega_{V/C})$ which are Ω-equivalent, meaning that:*

1. *We have an equality of ideals in A^0, $(\omega) = (\eta) = I_U$.*
2. *There exist equivariant morphisms $B, C \colon \Omega_{V/C} \to T_{V/C}$ such that*

$$\omega^\vee - \eta^\vee = \eta^\vee B^\vee d\eta^\vee \ \left(mod\ I_U^2\right)$$

and

$$\eta^\vee - \omega^\vee = \omega^\vee C^\vee d\omega^\vee \ \left(mod\ I_U^2\right).$$

Proof. Let x_i be an étale basis for V over C. As in the discussion preceding Definition 5.2.5, we may write $\omega_0 = \sum_i dy_i^\omega dx_i$ and $\eta_0 = \sum_i dy_i^\eta dx_i$, where y_i^ω and y_i^η are bases for A^{-1}, for the 0-part of the pullbacks of $\boldsymbol{\omega}$ and $\boldsymbol{\eta}$ to $\mathbf{Spec}\, A^\bullet$. Thus we have

$$y_i^\omega = \sum_k J_{ik}^\omega y_k,\ \ y_i^\eta = \sum_k J_{ik}^\eta y_k,\ \ J_{ik}^\omega, J_{ik}^\eta \in A^0$$

The induced 1-forms are then given by

$$\begin{aligned}\omega &= \sum_i \delta y_i^\omega \, dx_i = \sum_{i,k} J_{ik}^\omega s_k dx_i, \\ \eta &= \sum_i \delta y_i^\eta \, dx_i = \sum_{i,k} J_{ik}^\eta s_k dx_i,\end{aligned} \tag{5.3}$$

where we write $s_i = \delta y_i$ for convenience. Since $y_i, y_i^\omega, y_i^\eta$ are all bases for A^{-1} it is clear that $I_U = (\delta y_i) = (\delta y_i^\omega) = (\delta y_i^\eta)$, which proves (1).

Let $\delta w_j = \sum_i W_{ji} y_i$, where $W_{ji} \in A^0$. Since $\delta^2(w_j) = 0$ we have

$$\sum_{i,j} W_{ji} s_i = 0 \tag{5.4}$$

Now $\boldsymbol{\omega}, \boldsymbol{\eta}$ are equivalent as symplectic forms so in particular we have $\omega_0 - \eta_0 = \delta\alpha_0$ for some $\alpha_0 \in \left(\wedge^2 \Omega_{A^\bullet}\right)^{-2}$. By degree considerations, we may write

$$\alpha_0 = \sum_{ij} E_{ij} dy_i dy_j + \sum_{ik} F_{ik} dw_k dx_i + \alpha_0', \;\; E_{ij}, F_{ik} \in A^0,$$

where α_0' is a 2-form, whose every term is divisible by some of the y_i, and we may assume without loss of generality that E_{ij} is symmetric. Thus $\delta\alpha_0' \in I_U \cdot \left(\wedge^2 \Omega^1_{A^\bullet}\right)^{-1}$ and we have

$$\begin{aligned}\delta\alpha_0 &= -2\sum_{i,j} E_{ij} ds_i dy_j - \sum_{i,k} F_{ik} d(\delta w_k) dx_i \pmod{I_U} \\ &= -2\sum_{i,j} E_{ij} ds_i dy_j - \sum_{i,j,k} F_{ik} W_{kj} dx_i dy_j \pmod{I_U}.\end{aligned}$$

We have also

$$\begin{aligned}\omega_0 - \eta_0 &= \sum_i dy_i^\omega dx_i - \sum_i dy_i^\eta dx_i \\ &= \sum_{i,k} dJ_{ik}^\omega y_k dx_i - \sum_{i,k} J_{ik}^\omega dy_k dx_i - \sum_{i,k} dJ_{ik}^\eta y_k dx_i + \sum_{i,k} J_{ik}^\eta dy_k dx_i.\end{aligned}$$

By comparing the coefficients of each dy_j, we obtain a relation

$$\sum_i J_{ij}^\omega dx_i - \sum_i J_{ij}^\eta dx_i = 2\sum_i E_{ij} ds_i + \sum_{i,k} F_{ik} W_{kj} dx_i \pmod{I_U}$$

for each index j. Then, using (5.3),

$$\begin{aligned}\omega - \eta &= \sum_{i,k} (J^{\omega}_{ik} dx_i - J^{\eta}_{ik} dx_i) s_k \\ &= 2 \sum_{i,k} E_{ik} ds_i \; s_k + \sum_{i,k,j} F_{ij} W_{jk} s_k dx_i \;\; \left(\operatorname{mod} I_U^2\right) \\ &= 2 \sum_{i,k} E_{ik} ds_i \; s_k \;\; \left(\operatorname{mod} I_U^2\right),\end{aligned} \tag{5.5}$$

where in the second expression the second term is zero using (5.4).

Note that in order to derive (5.5), we only used the fact that y_i is a basis for A^{-1}. So we could repeat the exact same analysis and obtain a similar equation with s_i replaced by $s_i^{\omega} = \delta y_i^{\omega}$ or $s_i^{\eta} = \delta y_i^{\eta}$ and E_{ik} by E^{ω}_{ik} or E^{η}_{ik} respectively. It is then easy to see that these exactly imply (2), with the coefficients of B, C being determined by E^{ω}_{ik}, E^{η}_{ik} after averaging over G to make the morphism equivariant ($\omega - \eta$ is already invariant). □

Suppose now that we have étale morphisms $\boldsymbol{f}_\alpha \colon [\boldsymbol{T}_\alpha / R_\alpha] \to \boldsymbol{\mathcal{M}}$ and $\boldsymbol{f}_\beta \colon [\boldsymbol{T}_\beta / R_\beta] \to \boldsymbol{\mathcal{M}}$ as in Proposition 5.2.7, where $\boldsymbol{T}_\alpha$ is equivalent to $\mathbf{Spec}\, A^\bullet$ and $\boldsymbol{T}_\beta$ is equivalent to $\mathbf{Spec}\, B^\bullet$, with $A^\bullet, B^\bullet$ special equivariant cdgas. Let $z \in [\boldsymbol{T}_\alpha / R_\alpha] \times_{\boldsymbol{\mathcal{M}}} [\boldsymbol{T}_\beta / R_\beta]$ be a closed point with stabilizer H.

Note that $\boldsymbol{f}_\alpha, \boldsymbol{f}_\beta$ are also affine. Then, since $T_\alpha, T_\beta \to \mathcal{M}$ are affine, the diagonal of $\mathcal{M}$ is affine, and a derived scheme is affine if and only if its truncation is affine, we obtain that $\boldsymbol{T}_\alpha \times_{\boldsymbol{\mathcal{M}}} \boldsymbol{T}_\beta$ is an affine derived scheme with an action of $R_\alpha \times R_\beta$ such that $[\boldsymbol{T}_\alpha \times_{\boldsymbol{\mathcal{M}}} \boldsymbol{T}_\beta / R_\alpha \times R_\beta]$ is (-1)-shifted symplectic. Then, there exists a special cdga $C^\bullet$ with an H-action, minimal at $t_{\alpha\beta}$, such that we have H-equivariant morphisms $\alpha \colon A^\bullet \to C^\bullet$, $\beta \colon B^\bullet \to C^\bullet$ and a commutative diagram of étale arrows

$$\begin{tikzcd}
{[\mathbf{Spec}\, C^\bullet / H]} \arrow[r, "\boldsymbol{\theta_\beta}"] \arrow[d, "\boldsymbol{\theta_\alpha}"'] & {[\mathbf{Spec}\, B^\bullet / R_\beta]} \arrow[d, "\boldsymbol{f}_\beta"] \\
{[\mathbf{Spec}\, A^\bullet / R_\alpha]} \arrow[r, "\boldsymbol{f}_\alpha"'] & \boldsymbol{\mathcal{M}}.
\end{tikzcd} \tag{5.6}$$

Moreover, the morphism $[\mathbf{Spec}\, C^\bullet / H] \to [\boldsymbol{T}_\alpha \times_{\boldsymbol{\mathcal{M}}} \boldsymbol{T}_\beta / R_\alpha \times R_\beta]$ is étale and maps $t_{\alpha\beta}$ to z.

We can recast the above data at the level of classical stacks and schemes. We have an étale map $f_\alpha \colon [T_\alpha / R_\alpha] \to \mathcal{M}$, where $T_\alpha \subset S_\alpha$ is the zero locus of an invariant section ω_α of $\Omega_{S_\alpha / C}$ and $S_\alpha \to C$ is smooth, R_α-equivariant. Similar data is obtained for the étale map

$[T_\beta/R_\beta] \to \mathcal{M}$. The above diagram shows that we have the following comparison data:

1. We have an affine, smooth H-scheme $S_{\alpha\beta} \to C$ with an invariant section $\omega_{\alpha\beta}$ of $\Omega_{S_{\alpha\beta}/C}$ with zero locus $T_{\alpha\beta}$, minimal at a point $t_{\alpha\beta}$ fixed by H. Here $T_{\alpha\beta}$ is the truncation of the derived scheme $\mathbf{Spec}\, C^\bullet$.

2. There exist H-equivariant unramified morphisms $\theta_\alpha \colon S_{\alpha\beta} \to S_\alpha$ and $\theta_\beta \colon S_{\alpha\beta} \to S_\beta$, inducing unramified morphisms $T_{\alpha\beta} \to T_\alpha$ and $T_{\alpha\beta} \to T_\beta$.

3. $\eta_{\theta_\alpha}(\omega_\alpha),\ \eta_{\theta_\beta}(\omega_\beta)$ are Ω-equivalent to $\omega_{\alpha\beta}$.

4. We have a commutative diagram with étale arrows

$$\begin{tikzcd}
{[T_{\alpha\beta}/H]} \arrow[r, "\theta_\beta"] \arrow[d, "\theta_\alpha"'] & {[T_\beta/R_\beta]} \arrow[d, "f_\beta"] \\
{[T_\alpha/R_\alpha]} \arrow[r, "f_\alpha"'] & \mathcal{M}.
\end{tikzcd} \tag{5.7}$$

Definition 5.2.10. (Common roof) *If the above four conditions hold, coming from a diagram* (5.6)*, we say that the quasi-critical chart* $\Lambda_{S_{\alpha\beta}}$ *is a common roof for the quasi-critical charts* Λ_{S_α} *and* Λ_{S_β}*. More generally, the same definition applies to any two relative local models which are not necessarily quasi-critical charts.*

Remark 5.2.11. *Suppose that we have a common roof coming from a commutative diagram* (5.6) *where* $\boldsymbol{\mathcal{M}} = [\boldsymbol{Spec}\, D^\bullet/G]$ *and* $R_\alpha = R_\beta = H$*. Moreover, assume that we have two compositions* $g_\alpha \colon D^\bullet \to A^\bullet \to C^\bullet$ *and* $g_\beta \colon D^\bullet \to B^\bullet \to C^\bullet$ *such that* $\boldsymbol{Spec}\, g_\alpha$*,* $\boldsymbol{Spec}\, g_\beta$ *become equivalent when composed with the quotient morphism* $\boldsymbol{Spec}\, D^\bullet \to [\boldsymbol{Spec}\, D^\bullet/G]$ *and induce the diagram* (5.6)*. If* $g_\alpha,\ g_\beta \colon D_0 \to C_0$ *are the induced morphisms and we denote* $V = \operatorname{Spec} D^0$*,*

$$g^b_\alpha = \theta^b_\alpha \circ f^b_\alpha,\ g^b_\beta = \theta^b_\beta \circ f^b_\beta \colon \mathcal{O}b_V^{\mathrm{red}}|_{T^s_{\alpha\beta}} \longrightarrow \mathcal{O}b^{\mathrm{red}}_{S_{\alpha\beta}}$$

are obtained by the corresponding maps from (5.6) *on cotangent complexes of quotient stacks, as in Lemma 4.3.12. It follows that for the (non-commutative) diagram of quotient stacks*

$$\begin{tikzcd}
{[\operatorname{Spec} C^0/H]} \arrow[r, "\theta_\beta"] \arrow[d, "\theta_\alpha"'] & {[\operatorname{Spec} B^0/H]} \arrow[d, "f_\beta"] \\
{[\operatorname{Spec} A^0/H]} \arrow[r, "f_\alpha"'] & {[\operatorname{Spec} D^0/G].}
\end{tikzcd}$$

which is compatible with (5.7)*, there is a natural equivalence between the compositions* $\theta^b_\alpha \circ f^b_\alpha$ *and* $\theta^b_\beta \circ f^b_\beta$*, which is also compatible with the commutativity of* (5.7)*.*

Another way to see this more concretely is when G *is a Zariski open subscheme of an affine space. This is the case in our application to Donaldson-Thomas invariants, since closed points correspond to polystable sheaves whose stabilizers are products of general linear groups. The fact that* g_α, g_β *composed with* $\boldsymbol{Spec}\, D^\bullet \to [\boldsymbol{Spec}\, D^\bullet/G]$ *are equivalent implies then that there exists a morphism* $\boldsymbol{h} = \boldsymbol{Spec}\, h\colon \boldsymbol{Spec}\, C^\bullet \to G$ *such that the composition*

$$\boldsymbol{Spec}\, g'_\beta\colon \boldsymbol{Spec}\, C^\bullet \xrightarrow{(id,\boldsymbol{h})} \boldsymbol{Spec}\, C^\bullet \times G \xrightarrow{(\boldsymbol{Spec}\, g_\beta, id)} \boldsymbol{Spec}\, D^\bullet \times G \longrightarrow \boldsymbol{Spec}\, D^\bullet,$$

where the last arrow is given by the group action, gives a map of cdgas $D^\bullet \to C^\bullet$ *which is homotopic to* g_β*. In particular, the induced morphisms* $g_\beta\colon D^0 \to C^0$ *and* $g'_\beta\colon D^0 \to C^0$ *satisfy* $g_\beta - g'_\beta\colon D^0 \to \operatorname{im} C^{-1} = \operatorname{im} \omega^\vee_{\alpha\beta}$ *and thus, as in Lemma 4.3.11, we get that*

$$g^b_\beta = (g'_\beta)^b\colon \mathcal{O}b^{\mathrm{red}}_V|_{T^s_{\alpha\beta}} \longrightarrow \mathcal{O}b^{\mathrm{red}}_{S_{\alpha\beta}}$$

and the induced morphism $h\colon \operatorname{Spec} C^0 \to G$ *gives the data for the equivalence mentioned above.*

Chapter 6

Generalized Donaldson-Thomas Invariants

This chapter leads up to the main result of this thesis, the construction of the generalized DT invariant via Kirwan blowups. We treat the cases of moduli stacks of semistable sheaves and semistable perfect complexes.

6.1 The case of sheaves

For background on Gieseker stability of sheaves, we refer the reader to [HL10].

6.1.1 Obstruction theory of Kirwan partial desingularization of equivariant d-critical loci

As in (2.1), let $\mathcal{M} = [X/G]$ be a quotient stack obtained by GIT. We have therefore equivariant embeddings $X \subset P \subset (\mathbb{P}^N)^{ss}$ with G acting on $\mathbb{P}^N$ via a homomorphism $G \to \mathrm{GL}(N+1)$.

Suppose also that $\mathcal{M}$ admits a d-critical structure $s \in \Gamma(\mathcal{M}, \mathcal{S}_{\mathcal{M}})$ so that $(\mathcal{M}, s)$ is a d-critical stack. By Remark 5.1.11 this is equivalent to a G-invariant d-critical structure on X.

In this section, we carefully follow the steps of Chapter 2 tailored to the case of a d-critical locus to show that the Kirwan partial desingularization $\widetilde{\mathcal{M}}$ admits a semi-perfect obstruction theory.

Let $\mathfrak{R}(X) = \{R_1, ..., R_m, \{1\}\}$ in order of decreasing dimension.

Let $x \in X$ be a closed point with closed G-orbit, fixed by R_1. Since the action of G is good, we have a G-invariant affine open $x \in U \subset X$. Therefore, we may apply the étale slice theorem [Dré04, Theorem 5.3] to get a locally closed affine $x \in T \subset U$ such that $[T/R_1] \to [U/G] \subset [X/G]$ is étale. We thus obtain an étale cover

$$\coprod [T^1_\alpha/R_1] \coprod [(X - GZ_{R_1})/G] \longrightarrow [X/G] = \mathcal{M},$$

where each T^1_α is an R_1-invariant d-critical locus. In particular, for each α there exist data of a local model $(T^1_\alpha, S^1_\alpha, \Omega_{S^1_\alpha}, df_\alpha, 0, \sigma_{S^1_\alpha})$ with $T^1_\alpha = (df_\alpha = 0) \subset S^1_\alpha$ in standard form and we may take S^1_α to be affine, locally closed in P.

Let $X_1 = \widehat{X}$ be the Kirwan blowup of X_1 associated with R_1 and set $X_1^\circ = X - GZ_{R_1}$. Then we obtain an induced étale cover

$$\coprod [\widehat{T}^1_\alpha/R_1] \coprod [X_1^o/G] \longrightarrow [X_1/G] = \mathcal{M}_1 \subset \mathcal{P}_1 := [P_1/G].$$

For each α, we have induced data $(\widehat{T}^1_\alpha, \widehat{S}^1_\alpha, F_{\widehat{S}^1_\alpha}, \omega_{\widehat{S}^1_\alpha}, D_{\widehat{S}^1_\alpha}, \phi_{\widehat{S}^1_\alpha})$ with $\widehat{T}^1_\alpha = (\omega_{\widehat{S}^1_\alpha} = 0) \subset \widehat{S}^1_\alpha \subset \widehat{P} =: P_1$. Note also that $[X_1^\circ/G] \longrightarrow \mathcal{M}_1$ factors through $\mathcal{M}$ as an open immersion.

Now $R_2 \in \mathfrak{R}(X_1)$ is of maximal dimension. Let $x \in X_1$ be a closed point with closed G-orbit, fixed by R_2. Then its orbit will lie in X_1^o or it will be contained in the image of some $[\widehat{T}^1_\alpha/R_1]$. Applying the same reasoning, we get an induced étale cover

$$\coprod [\widehat{T}^2_\beta/R_2] \coprod [(\widehat{T}^1_\alpha)^\circ/R_1] \coprod [X_2^\circ/G] \longrightarrow [X_2/G] = \mathcal{M}_2 \subset \mathcal{P}_2 := [P_2/G],$$

where $(\widehat{T}^1_\alpha)^\circ = \widehat{T}^1_\alpha - \widehat{T}^1_\alpha \cap GZ_{R_2}$, $X_2^\circ = X_1^\circ - GZ_{R_2}$ and the various $T^2_\beta \subset S^2_\beta$ are étale slices of $G \times_{R_1} \widehat{T}^1_\alpha$ or X_1° in standard form, where we may take S^2_β to be slices in $P_2 = \widehat{P_1}$.

Continuing inductively, we have for any n an étale cover

$$\coprod [\widehat{T}^n_\gamma/R_n] \coprod \cdots \coprod [(\widehat{T}^1_\alpha)^\circ/R_1] \coprod [X_n^\circ/G] \to \mathcal{M}_n \subset \mathcal{P}_n := [P_n/G],$$

where by abuse of notation we write

$$(\widehat{T}^i_\alpha)^\circ = \widehat{T}^i_\alpha - GZ_{R_{i+1}} \cap \widehat{T}^i_\alpha - ... - GZ_{R_n} \cap \widehat{T}^i_\alpha$$

and so on, and the T^n_α are appropriate slices of the elements of the étale cover for $\mathcal{M}_{n-1}$.

Note also that for each i, $[(\widehat{T}^i_\alpha)^\circ/R_i]$ factors through an étale morphism to $\mathcal{M}_i$.

We see that $\widetilde{\mathcal{M}} = \mathcal{M}_m = [X_m/G] \subset [P_m/G]$, a DM stack, is the Kirwan partial desingularization of $\mathcal{M}$. We formalize the above procedure in the following proposition, where we also keep track of obstruction sheaves and their comparison data.

Lemma 6.1.1. *For each $n \geq 0$, let $\mathcal{E}^X_n, \mathcal{E}^P_n$ be the union of all exceptional divisors in $\mathcal{M}_n$ and $\mathcal{P}_n$ respectively. There exist collections of étale morphisms $[T_\alpha/R_\alpha] \to \mathcal{M}_n$ and $[S_\alpha/R_\alpha] \to \mathcal{P}_n$ such that:*

1. *For each α, $R_\alpha \in \{R_1, ..., R_n\}$.*
2. *Each T_α is in standard form for data $(T_\alpha, S_\alpha, F_{S_\alpha}, \omega_{S_\alpha}, D_{S_\alpha}, \phi_{S_\alpha})$ of a local model on a smooth affine R_α-scheme $S_\alpha \subset P_n$.*
3. *The collections cover $\mathcal{E}^X_n$ and $\mathcal{E}^P_n$ respectively.*
4. *The identity components of stabilizers that occur in $\mathcal{M}_n$ lie, up to conjugacy, in the set $\{R_{n+1}, ..., R_m\}$.*
5. *The data $(T_\alpha, S_\alpha, F_{S_\alpha}, \omega_{S_\alpha}, D_{S_\alpha}, \phi_{S_\alpha})$ restricted to the complement of $\mathcal{E}^P_n \subset \mathcal{P}_n$ are the same as those of a d-critical chart on T_α.*
6. *For α, β and $q \in [T_\alpha/R_\alpha] \times_{\mathcal{M}_n} [T_\beta/R_\beta]$ whose stabilizer has identity component conjugate to R_q, there exist an affine $T_{\alpha\beta}$ in standard form for data $(T_{\alpha\beta}, S_{\alpha\beta}, F_{\alpha\beta}, \omega_{\alpha\beta}, D_{\alpha\beta}, \phi_{\alpha\beta})$ of a local model and an equivariant commutative diagram*

$$\begin{tikzcd}
& T_{\alpha\beta} \arrow[rr] \arrow[dl, "i_\alpha"'] & & S_{\alpha\beta} \arrow[dl, "\theta_\alpha"] \arrow[dr, "\theta_\beta"'] & & T_{\alpha\beta} \arrow[ll] \arrow[dr, "i_\beta"] & \\
T_\alpha \arrow[rr] \arrow[dr] & & S_\alpha \arrow[dr] & & S_\beta \arrow[dl] & & T_\beta \arrow[ll] \arrow[dl] \\
& \mathcal{M}_n \arrow[rr] & & \mathcal{P}_n & & \mathcal{M}_n, \arrow[ll] &
\end{tikzcd} \tag{6.1}$$

inducing a commutative diagram consisting of étale maps on the corresponding quotient stacks for arrows pointing downwards. $T_{\alpha\beta}, S_{\alpha\beta}$ are étale slices for both T_α, T_β and S_α, S_β respectively. All horizontal arrows are embeddings and $\theta_\alpha, \theta_\beta$ are unramified and R_q-equivariant. $\eta_{\theta_\alpha}(\omega_{S_\alpha})$ and $\eta_{\theta_\beta}(\omega_{S_\beta})$ are Ω-equivalent to $\omega_{\alpha\beta}$.

7. *For each index* α*, consider the 4-term complex*

$$K_\alpha = [\mathfrak{r}_\alpha \longrightarrow T_{S_\alpha}|_{T_\alpha} \xrightarrow{(d\omega_{S_\alpha}^\vee)^\vee} F_{S_\alpha}|_{T_\alpha} \xrightarrow{\phi_{S_\alpha}} \mathfrak{r}_\alpha^\vee(-D_{S_\alpha})],$$

where by convention we place $F_{S_\alpha}|_{T_\alpha}$ *in degree* 1*.* θ_α *induces an isomorphism* $\theta_\alpha^b \colon \mathcal{O}b_{S_\alpha}^{\mathrm{red}}|_{T_{\alpha\beta}^s} \to \mathcal{O}b_{S_{\alpha\beta}}^{\mathrm{red}}$*. This does not change if we replace* ω_{S_α} *by any* Ω*-equivalent section. Analogous statements are true for the index* β*.*

8. *We obtain comparison isomorphisms*

$$\theta_{\alpha\beta}^b := (\theta_\beta^b)^{-1}\theta_\alpha^b \colon \mathcal{O}b_{S_\alpha}^{\mathrm{red}}|_{T_{\alpha\beta}^s} \to \mathcal{O}b_{S_\beta}^{\mathrm{red}}|_{T_{\alpha\beta}^s}.$$

These give the same obstruction assignments for the complexes K_α^{red}*,* K_β^{red} *on* $T_{\alpha\beta}^s$*.*

9. *Let now* q *be a point in the triple intersection*

$$q \in [T_\alpha/R_\alpha] \times_{\mathcal{M}_n} [T_\beta/R_\beta] \times_{\mathcal{M}_n} [T_\gamma/R_\gamma]$$

with stabilizer in class R_q*. Then we have* $S_{\alpha\beta}$ *as in (6) for the indices* α, β*,* $S_{\beta\gamma}$ *as in (6) for the indices* β, γ *and* $S_{\gamma\alpha}$ *for the indices* γ, α *with a common* R_q*-invariant étale neighbourhood* $S_{\alpha\beta\gamma}$ *fitting on top of the diagrams of the form* (6.1)*. The descents of* $\theta_{\alpha\beta}^b, \theta_{\beta\gamma}^b, \theta_{\gamma\alpha}^b$ *to* $[T_{\alpha\beta\gamma}^s/R_q]$ *satisfy the cocycle condition.*

Proof. For $n = 0$ there is nothing to show, as we may take an empty set of étale morphisms. We proceed by induction. Suppose the claim is true for n.

Consider the locus of closed points $x \in \mathcal{M}_n$ whose stabilizer has identity component conjugate to R_{n+1}. Then, either x is in the image of some $[T_\alpha/R_\alpha]$ or not.

Let's examine the first case. Take $[T_\alpha/R_\alpha]$ and consider all such closed points $x \in [T_\alpha/R_\alpha]$. For each x, let $[T_\alpha^x/R_{n+1}] \to [T_\alpha/R_\alpha]$ be induced by an étale slice $[S_\alpha^x/R_{n+1}] \to [S_\alpha/R_\alpha]$. Then we consider the collection of étale maps $[\widehat{T_\alpha^x}/R_{n+1}] \to \mathcal{M}_{n+1}$ together with $[T_\alpha^\circ/R_\alpha] \to \mathcal{M}_{n+1}$ where T_α° is the complement in T_α of the locus of closed points with stabilizer whose identity component is conjugate to R_{n+1}. We may repeat this process for all $[T_\alpha/R_\alpha] \to \mathcal{M}_n$ and x. We say that these étale maps are of type I.

So, consider $x \in \mathcal{M}_n$ such that x does not lie in the image of any $[T_\alpha/R_\alpha]$. In particular, by (3) x does not lie in $\mathcal{E}_n^X$ and thus we may assume that it is a closed point of $[X/G]$. Then there exist slices $T \subset S \subset P$ such that T is in standard form for data $(T, S, \Omega_S, df_S, 0, \sigma_S)$

of a local model on S and we have étale morphisms $[T/R_{n+1}] \to \mathcal{M}_n,\ [S/R_{n+1}] \to \mathcal{P}_n$. Then, we may take étale maps $[\widehat{T}/R_{n+1}] \to \mathcal{M}_{n+1},\ [\widehat{S}/R_{n+1}] \to \mathcal{P}_{n+1}$. We may repeat this process for all such x. We say that these maps are of type II.

We have thus produced a collection of étale morphisms for $\mathcal{M}_{n+1}$ and $\mathcal{P}_{n+1}$. It is clear by our choice and the inductive hypothesis that (1)-(5) are automatically satisfied by Lemma 4.1.5 and the properties of Kirwan blowups.

To check (6), the fact that the maps are unramified follows from the slice property since the derivatives are injective around the point of interest. Furthermore, since by (5) the restriction of a map of type I to the complement of $\mathcal{E}_n^X$ clearly yields rise to a map of type II for $\mathcal{M}_{n+1}$ it suffices to produce comparison data for maps of the same type.

Now, for two maps of type I, we may assume that $S_{\alpha\beta}$ in (6.1) factors through S_α^x and S_β^y and therefore we may use $\widehat{\theta}_\alpha$ and $\widehat{\theta}_\beta$ to get comparison data, which satisfy the requirements, since Ω-equivalence is preserved by Kirwan blowups and taking slices by Lemma 4.2.4 and Lemma 4.2.6.

For two maps of type II, coming from two choices of $[T/R_{n+1}] \to \mathcal{M}$ and $[S/R_{n+1}] \to \mathcal{P}$, we may find (up to shrinking), as in the proof of Proposition 2.3.7 and using the properties of d-critical loci, a common étale refinement $T_{\alpha\beta} \to T_\alpha,\ T_{\alpha\beta} \to T_\beta$ with $T_{\alpha\beta} = (\omega_{\alpha\beta} = df_{\alpha\beta} = 0) \subset S_{\alpha\beta}$ and commutative comparison diagrams

$$\begin{tikzcd}
T_{\alpha\beta} \arrow[r] \arrow[d] & S_{\alpha\beta} \arrow[d, "\theta_\alpha"] \\
T_\alpha \arrow[r] & S_\alpha \arrow[d, "f_\alpha"] \\
 & \mathbb{A}^1
\end{tikzcd} \tag{6.2}$$

fitting in a diagram of the form (6.1) for $\mathcal{M}_0 := \mathcal{M}$ and $\mathcal{P}_0 := \mathcal{P}$, such that $\theta_\alpha, \theta_\beta$ are étale and $df_\alpha|_{S_{\alpha\beta}}$, $df_\beta|_{S_{\alpha\beta}}$ are Ω-equivalent to $df_{\alpha\beta}$. Since Ω-equivalence is preserved for Kirwan blowups and taking slices, we see that (6) is satisfied in this case as well and exceptional divisors pull back to exceptional divisors (cf. Lemma 4.1.5).

Finally (7) and (8) are an immediate consequence of the inductive hypothesis and Lemma 4.3.12 of the preceding section. The existence of $S_{\alpha\beta\gamma}$ in (9) can be seen, using the inductive hypothesis for maps of type I, and the d-critical structure for maps of type II. The cocycle condition follows from the commutativity of the diagrams of quotient stacks induced by (6.1) and the fact that by construction θ_α^b is induced by pullback, is functorial with respect to compositions and both the reduced obstruction sheaves $\mathcal{O}b_{S_\alpha}^{\mathrm{red}}$ and the morphisms θ_α descend to the level of quotient stacks. We note here that all the results of Section 6 hold true if one replaces embeddings by unramified morphisms, so they apply to the present situation as well (cf. Remark 4.3.13). ☐

We are now in position to show that $\widetilde{\mathcal{M}} = \mathcal{M}_m$ is equipped with a semi-perfect obstruction theory of dimension 0. Let $[T_\alpha/R_\alpha] \to \widetilde{\mathcal{M}}$ be the cover granted by the previous proposition. Each $[T_\alpha/R_\alpha]$ is a DM stack and together they cover the strictly semistable locus of $\mathcal{M}$. Then for any $x \in \mathcal{M}^s$, as in the proof of the preceding proposition, we have an étale map $T_x \to \mathcal{M}$, where T_x is a d-critical locus with $T_x \subset S_x \to P$. The stable locus $\mathcal{M}^s$ is not affected during the partial desingularization procedure and so $\mathcal{M}^s$ is an open substack of $\widetilde{\mathcal{M}}$ and we get étale maps $T_x \to \widetilde{\mathcal{M}}$. We obtain an étale surjective cover

$$\coprod [T_\alpha/R_\alpha] \coprod T_x \longrightarrow \widetilde{\mathcal{M}}.$$

It is easy to check that the obstruction sheaves and assignments of the T_x are compatible with those of the cover $[T_\alpha/R_\alpha] \to \widetilde{\mathcal{M}}$ by an identical argument as in the above proof. We thus see that the reduced obstruction theories indeed give a semi-perfect obstruction theory of dimension 0 on $\widetilde{\mathcal{M}}$.

We have proved our main theorem.

Theorem 6.1.2. *Let $\mathcal{M} = [X/G]$, where X is a quasi-projective scheme, which is the semistable part of a projective scheme $X^\dagger$ with a linearized G-action. Suppose also that $\mathcal{M}$ admits a d-critical structure $s \in \Gamma(\mathcal{M}, \mathcal{S}_{\mathcal{M}})$ so that $(\mathcal{M}, s)$ is a d-critical stack. Then the Kirwan partial desingularization $\widetilde{\mathcal{M}}$ is a proper DM stack endowed with a canonical semi-perfect obstruction theory of dimension zero induced by s, giving rise to a 0-dimensional virtual fundamental cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}}$.*

Remark 6.1.3. *Since at any point of our construction, we are working with stabilizers of closed points and slices thereof, $\widetilde{\mathcal{M}}$ and its obstruction theory are independent of the particular presentation of $\mathcal{M}$ as a quotient stack $[X/G]$.*

Remark 6.1.4. *If we replace the d-critical structure s by $s' = c \cdot s$, where $c \in \mathbb{C}^\times$, the virtual cycle stays the same. This is because rescaling the d-critical structure is equivalent* *to replacing every d-critical chart (U, V, f, i) of X by (U, V, cf, i) and this does not affect the intrinsic normal cone of the obstruction theory of $\widetilde{\mathcal{M}}$.*

6.1.2 Definition of the DTK invariant

Let $\mathcal{M} = \mathcal{M}^{ss}(\gamma)$ be the moduli stack of Gieseker semistable sheaves with fixed Chern character γ on a Calabi-Yau threefold W. Let $\kappa \colon \mathcal{O}_W \to K_W$ be a trivialization of K_W.

The stack $\mathcal{M}$ is the truncation of a derived Artin stack $\boldsymbol{\mathcal{M}}$. Moreover, by [PTVV13] κ induces a (-1)-shifted symplectic structure on $\boldsymbol{\mathcal{M}}$ and therefore by [BBBBJ15] a d-critical structure on $\mathcal{M}$. $\mathcal{M} = [X/G]$ is obtained by GIT (cf. [HL10, Section 4]), where $G = \mathrm{GL}(n, \mathbb{C})$, and we may rigidify $\mathbb{C}^\times$-scaling automorphisms by taking $G = \mathrm{PGL}(n, \mathbb{C})$. The conditions of Theorem 6.1.2 hold and we have the Kirwan partial desingularization $\widetilde{\mathcal{M}}$ and its virtual fundamental cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}}$.

Since choosing a different trivialization amounts to rescaling κ and subsequently the d-critical structure, by Remarks 6.1.3 and 6.1.4, the virtual cycle does not depend on the choice of κ or presentation as a quotient stack.

Theorem-Definition 6.1.5. (Generalized DT invariant) *We define the generalized DT invariant via Kirwan blowups of Chern character γ to be*

$$\mathrm{DTK}(\mathcal{M}) := \deg [\widetilde{\mathcal{M}}]^{\mathrm{vir}}.$$

6.2 Deformation invariance

In this section, we use the relative versions of the results of this thesis, obtained via derived symplectic geometry, to conclude that the generalized Donaldson-Thomas invariant via Kirwan blowup is invariant under deformations of the complex structure of the Calabi-Yau threefold W.

6.2.1 Obstruction theory in the relative case

As before, suppose that $\boldsymbol{\mathcal{M}} \to C$ is a (-1)-shifted symplectic derived stack, whose truncation $\mathcal{M} = [X/G]$ is a quotient stack obtained by GIT, where X is a C-scheme and G acts trivially on C.

The Kirwan partial desingularization procedure goes through in exactly the same way as in the absolute case. We have the following modified version of Proposition 6.1.1 in this relative situation.

The main difference stems from the fact that we need to use local models, arising from special cdga's (cf. Definition 8.8), which satisfy a minimality property (cf. Definition 8.6). Here, for the sake of full generality, we no longer use slices S_α coming from a global embedding $\mathcal{M} \to \mathcal{P}$ into a smooth stack and we will need to use the notion of Ω-compatibility to address the issue of extra coordinates. In particular, in the analogue of diagram (6.1), we will be missing the arrows $S_\alpha \to \mathcal{P}_n,\ S_\beta \to \mathcal{P}_n$ and S_α, S_β can have extra coordinates.

Proposition 6.2.1. *For each $n \geq 0$, let $\mathcal{E}_n^X$ be the union of all exceptional divisors in $\mathcal{M}_n$. There exist collections of étale morphisms $[T_\alpha/R_\alpha] \to \mathcal{M}_n$ such that:*

1. *For each α, $R_\alpha \in \{R_1, ..., R_n\}$.*
2. *Each T_α is in relative standard form for data*

$$\Lambda_{S_\alpha} = (T_\alpha, S_\alpha, F_\alpha, \omega_\alpha, D_\alpha, \phi_\alpha)$$

   of a relative local model on a smooth affine R_α-scheme S_α.

3. *The collections cover $\mathcal{E}_n^X$.*
4. *The identity components of stabilizers that occur in $\mathcal{M}_n$ lie, up to conjugacy, in the set $\{R_{n+1}, ..., R_m\}$.*
5. *Λ_{S_α} restricted to the complement of the union of exceptional divisors $\mathcal{E}_{S_\alpha} \subset S_\alpha$ are the same as those of a quasi-critical chart on T_α.*
6. *For indices α, β and $q \in [T_\alpha/R_\alpha] \times_{\mathcal{M}_n} [T_\beta/R_\beta]$ whose stabilizer has identity component conjugate to R_q, there exist an affine $T_{\alpha\beta}$ in relative standard form for data*

$$\Lambda_{S_{\alpha\beta}} = (T_{\alpha\beta}, S_{\alpha\beta}, F_{\alpha\beta}, \omega_{\alpha\beta}, D_{\alpha\beta}, \phi_{\alpha\beta})$$

of a relative local model and an equivariant commutative diagram

$$\begin{tikzcd}
 & S_{\alpha\beta} \arrow[dl, "\theta_\alpha"'] & T_{\alpha\beta} \arrow[l] \arrow[r] \arrow[dl, "i_\alpha"] \arrow[dr, "i_\beta"'] & S_{\alpha\beta} \arrow[dr, "\theta_\beta"] & \\
S_\alpha & T_\alpha \arrow[l] \arrow[dr] & & T_\beta \arrow[r] \arrow[dl] & S_\beta \\
 & & \mathcal{M}_n, & &
\end{tikzcd} \tag{6.3}$$

such that $\Lambda_{S_{\alpha\beta}}$ is a common roof for Λ_{S_α} and Λ_{S_β} coming from a diagram of the form (5.6)*. Moreover, θ_α factors as $S_{\alpha\beta} \to S'_\alpha \to S_\alpha$ where S'_α is an étale slice for S_α at the image of q, $T_{\alpha\beta} \to T'_\alpha$ is étale and $S_{\alpha\beta} \to S'_\alpha$ is unramified. Analogous conditions hold for the index β.*

7. *For each index α, consider the 4-term complex*

$$K_\alpha = [\mathfrak{r}_\alpha \longrightarrow T_{S_\alpha}|_{T_\alpha} \xrightarrow{(d\omega_{S_\alpha}^\vee)^\vee} F_\alpha|_{T_\alpha} \xrightarrow{\phi_{S_\alpha}} \mathfrak{r}_\alpha^\vee(-D_\alpha)],$$

where by convention we place $F_{S_\alpha}|_{T_\alpha}$ in degree 1. θ_α induces an isomorphism $\theta_\alpha^b \colon \mathcal{O}b_{S_\alpha}^{\mathrm{red}}|_{T_{\alpha\beta}^s} \to \mathcal{O}b_{S_{\alpha\beta}}^{\mathrm{red}}$. This also does not change if we replace ω_{S_α} by any Ω-equivalent section. Analogous statements are true for the index β.

8. *We obtain comparison isomorphisms*

$$\theta_{\alpha\beta}^b := (\theta_\beta^b)^{-1}\theta_\alpha^b \colon \mathcal{O}b_{S_\alpha}^{\mathrm{red}}|_{T_{\alpha\beta}^s} \to \mathcal{O}b_{S_\beta}^{\mathrm{red}}|_{T_{\alpha\beta}^s}.$$

These give the same obstruction assignments for the complexes K_α^{red}, K_β^{red} on $T_{\alpha\beta}^s$.

9. *Let now q be a point in the triple intersection*

$$q \in [T_\alpha/R_\alpha] \times_{\mathcal{M}_n} [T_\beta/R_\beta] \times_{\mathcal{M}_n} [T_\gamma/R_\gamma]$$

with stabilizer in class R_q. We then have a commutative R_q-equivariant diagram of *common roofs*

$$\begin{tikzcd}
 & & & \Lambda_{\alpha\beta\gamma} \arrow[dl] \arrow[dr] & & & \\
 & & \Lambda_{\alpha\beta,\beta\gamma} \arrow[dl] \arrow[dr] & & \Lambda_{\beta\gamma,\gamma\alpha} \arrow[dl] \arrow[dr] & & \\
 & \Lambda_{\alpha\beta} \arrow[dl] \arrow[dr] & & \Lambda_{\beta\gamma} \arrow[dl] \arrow[dr] & & \Lambda_{\gamma\alpha} \arrow[dl] \arrow[dr] & \\
\Lambda_{\alpha} & & \Lambda_{\beta} & & \Lambda_{\gamma} & & \Lambda_{\alpha}
\end{tikzcd}$$

such that all the morphisms between local models $\Lambda_\lambda \to \Lambda_\mu$, where λ, μ are multi-indices, satisfy the properties of a roof, the morphisms $T_\lambda \to T_\mu$ are étale, and we have induced isomorphisms $\mathcal{O}b_\mu^{\mathrm{red}}|_{T_\lambda^s} \to \mathcal{O}b_\lambda^{\mathrm{red}}$. The descents of $\theta_{\alpha\beta}^b, \theta_{\beta\gamma}^b, \theta_{\gamma\alpha}^b$ to $[T_{\alpha\beta\gamma}^s/R_q]$ satisfy the cocycle condition.

Proof. In the same way as in the absolute case, all conditions (1)-(9) are preserved at each inductive step. So we only need to check that they hold in the beginning and also address the differences in (6), (7) and (9) from the absolute case.

By the previous subsection, roofs exist and they satisfy the conditions of (6) by construction, since we may first replace $\boldsymbol{T_\alpha}/R_\alpha$ by an étale slice $\boldsymbol{T_\alpha}'/H$ (and similarly for the index β) and then take a common roof. In particular, by Lemma 4.3.8 and Lemma 4.3.12 we immediately deduce that θ_α^b indeed induce isomorphisms on the reduced obstruction sheaves. It is at this point where we have to use the notion of Ω-compatibility to deal with possible extra coordinates.

Finally, the cocycle condition holds by applying Remark 5.2.11 and using Lemma 4.3.8, 4.3.10, 4.3.11 and 4.3.12, which are valid in the relative setting, and going through the appropriate diagrams of roofs granted by (9). We leave the details to the reader. $\square$

We have shown that we can follow the same steps as in the absolute case to obtain the following theorem.

Theorem 6.2.2. *Let $\mathcal{M} = [X/G] \to C$, where $X \to C$ is a quasi-projective scheme, which is the semistable part of a projective scheme $X^\dagger \to C$ with a linearized G-action and C is a smooth, quasi-projective curve with a trivial G-action. Suppose also that $\mathcal{M}$ is the truncation of a relative (-1)-shifted symplectic derived stack $\boldsymbol{\mathcal{M}}$ over C. Then the Kirwan partial desingularization $\widetilde{\mathcal{M}} \to C$ is a proper DM stack over C endowed with a canonical* *relative semi-perfect obstruction theory induced by the relative (-1)-shifted symplectic form, giving rise to a virtual fundamental cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}}$.*

Furthermore, the restriction of the obstruction theory to the fiber $\widetilde{\mathcal{M}}_c$ over a point $c \in C$ is the semi-perfect obstruction theory of Theorem 6.1.2, constructed in the absolute case.

6.2.2 Deformation invariance of DTK invariants

Let $W \to C$ be a family of Calabi-Yau threefolds over a smooth affine curve. The relative canonical bundle $K_{W/C}$ is then trivial.

Let $\mathcal{M}$ be the moduli stack of relatively semistable sheaves over C of Chern character γ on W. As in the absolute case, a trivialization of $K_{W/C}$ induces a relative (-1)-shifted symplectic structure on the derived stack $\boldsymbol{\mathcal{M}} \to C$. Then by Theorem 6.2.2, it follows that applying the same construction we obtain a Kirwan partial desingularization $\widetilde{\mathcal{M}} \to C$ with a relative semi-perfect obstruction theory, which induces the absolute semi-perfect obstruction theory on each fiber $\widetilde{\mathcal{M}}_c$ we constructed earlier.

Moreover, its fiber over $c \in C$ is the Kirwan partial desingularization of the moduli stack of semistable sheaves on W_c by the results of Section 2.4. Therefore, by "conservation of number" for semi-perfect obstruction theories [CL11, Proposition 3.8], the generalized Donaldson-Thomas invariant stays the same along the family, i.e. $\mathrm{DTK}(\mathcal{M}_c)$ is constant for $c \in C$.

Theorem 6.2.3. *The generalized DT invariant defined in Theorem-Definition 6.1.5 is invariant under deformation of the complex structure of the Calabi-Yau threefold.*

Remark 6.2.4. *Choosing a different trivialization of $K_{W/C}$ amounts to rescaling the induced trivializations of K_{W_c} on each fiber, so by Remark 6.1.4 our discussion does not depend on the specific choice of trivialization.*

6.3 The case of perfect complexes

In this section, as usual W will denote a smooth, projective Calabi-Yau threefold over $\mathbb{C}$. We first describe the stability conditions σ on $D^b(\mathrm{Coh}\, W)$ that we will be interested in. We then quote relevant results from the theory of Θ-reductivity, which are necessary in order to apply a criterion for the existence of a good moduli space. We proceed to show that moduli stacks of σ-semistable complexes are stacks of DT type and explain how to rigidify the $\mathbb{C}^\times$-automorphisms of stable objects. Finally, we define generalized DT invariants via Kirwan blowups and state a sufficient condition for their deformation invariance.

We will be interested in stacks which are nicely behaved in the sense of the following definition.

Definition 6.3.1. (Stack of DT type) *Let $\mathcal{M}$ be an Artin stack. We say that $\mathcal{M}$ is of DT type if the following are true:*

1. *$\mathcal{M}$ is quasi-separated and finite type over $\mathbb{C}$.*
2. *There exists a good moduli space $\pi\colon \mathcal{M} \to M$, where π is of finite type and has affine diagonal.*
3. *$\mathcal{M}$ is the truncation of a (-1)-shifted symplectic derived Artin stack.*

Remark 6.3.2. *A stack $\mathcal{M}$ parametrizing Gieseker semistable sheaves on W of a fixed Chern character is of DT type.*

6.3.1 Stability conditions

By [Lie06], there is an Artin stack $\mathrm{Perf}(W)$ of (universally gluable) perfect complexes on W, which is locally of finite type. We will be interested in the following setup.

Setup 6.3.3. *Consider the data:*

1. *A heart $\mathcal{A} \subset D^b(\mathrm{Coh}\, W)$ of a t-structure.*
2. *A vector $\gamma \in H^*(W, \mathbb{Q})$.*
3. *A stability condition σ on $D^b(\mathrm{Coh}\, W)$.*

These are required to satisfy the conditions:

1. *The stack $\mathcal{M} := \mathcal{M}^{\sigma-ss}(\gamma)$ of σ-semistable objects in $\mathcal{A}$ of Chern character γ is an Artin stack of finite type.*
2. *$\mathcal{M}$ is an open substack of $\mathrm{Perf}(W)$.*
3. *$\mathcal{M}$ satisfies the existence part of the valuative criterion of properness. We then say that $\mathcal{M}$ is quasi-proper or universally closed.*

4. *Any modification of a map* $\operatorname{Spec} R \to \mathcal{M}$*, where* R *is a discrete valuation ring, can be factored as a sequence of elementary modifications.*

Let us explain the last condition in more detail. Let R be a discrete valuation ring with fraction field K and closed point $s \in \operatorname{Spec} R$. Consider a morphism $h\colon \operatorname{Spec} R \to \mathcal{M}$, corresponding to a complex $F \in D^b(\operatorname{Coh} W_R)$.

Definition 6.3.4. *A morphism* $h'\colon \operatorname{Spec} R \to \mathcal{M}$ *is called a modification of* h *if* $h'_K \simeq h_K$*.*

Let us write F_s for the (derived) restriction of F to the special fiber $W_s \xrightarrow{i} W$. Consider an exact sequence

$$0 \longrightarrow E \longrightarrow F_s \longrightarrow Q \longrightarrow 0$$

in $\mathcal{A}$. By adjunction, this induces a map $F \to i_*Q$. Let G be its kernel.

Definition 6.3.5. G *is called the elementary modification of* F *at* Q*.*

These definitions explain the meaning of condition (4) above.

Remark 6.3.6. *If* G *is an elementary modification of* F*, then we obtain another exact sequence*

$$0 \longrightarrow Q \longrightarrow G_s \longrightarrow E \longrightarrow 0.$$

Hence F_s *and* G_s *are S-equivalent. In particular, if* F_K *is the complex corresponding to a morphism* $\operatorname{Spec} K \to \mathcal{M}$*, any extension* F_R *to a family over* $\operatorname{Spec} R$ *is unique up to S-equivalence.*

Definition 6.3.7. (Stability condition) *A stability condition* σ *on* W *will be one of the following:*

1. *A Bridgeland stability condition in the sense of Toda-Piyaratne [PT15].*
2. *A polynomial stability condition in the sense of Lo [Lo11, Lo13].*
3. *A weak stability condition in the sense of Joyce-Song [JS12, Definition 3.5], where we take* $\mathcal{A} = \operatorname{Coh} W$ *and* $K(A) = N(W)$*, such as Gieseker stability and slope stability.*

By the results of the mentioned authors, these satisfy all the required properties given in Setup 6.3.3.

Remark 6.3.8. *More generally, one may consider any set of data that make the conditions of Setup 6.3.3 true.*

6.3.2 Θ-reductive stacks

Θ-reductivity is a notion that will be helpful in the following subsections. We state the definition and properties we will use.

Definition 6.3.9. [Hal14, Definition 2.27] *Let* $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$. *We say that* $\mathcal{M}$ *is* Θ-*reductive if for any discrete valuation ring* R *with fraction field* K *and any morphism* $\Theta_K \cup_{\operatorname{Spec} K} \operatorname{Spec} R \to \mathcal{M}$ *there exists a unique extension* $\Theta_R \to \mathcal{M}$.

Lemma 6.3.10. [Hal14, Lemma 4.22] *Let* $\mathcal{A} \subset D^b(W)$ *be the heart of a t-structure. The moduli stack* $\mathcal{M}(\gamma)$ *of objects in* $\mathcal{A}$ *of Chern character* γ *is* Θ-*reductive.*

Proposition 6.3.11. *If* σ *is a stability condition on* W, *then the stack* $\mathcal{M}^{\sigma-ss}(\gamma)$ *of* σ-*semistable objects of Chern character* γ *is* Θ-*reductive.*

Proof. This is a formal consequence of the previous lemma. It suffices to notice that the proof of Lemma 6.3.10 goes through if one replaces the heart $\mathcal{A} \subset D^b(W)$ with an abelian subcategory $\mathcal{C} \subset \mathcal{A}$ and that σ-semistable objects of fixed slope ϕ (determined by γ) form an abelian category. ☐

6.3.3 Moduli stacks of semistable complexes are of DT type

We will make use of the following criterion.

Theorem 6.3.12. [AHLH] *Let* $\mathcal{M}$ *be an Artin stack of finite type over* $\mathbb{C}$ *with affine diagonal. Suppose that the following are true:*

1. *Closed points of* $\mathcal{M}$ *have reductive stabilizers.*
2. $\mathcal{M}$ *is* Θ-*reductive.*
3. *For every discrete valuation ring* R *with fraction field* K *and any morphism* $h\colon \operatorname{Spec} R \to \mathcal{M}$ *with an automorphism* $\phi_K \in \operatorname{Aut}(h_K)$ *of finite order, there exists* $h'\colon \operatorname{Spec} R \to \mathcal{M}$ *such that* $h'_K \simeq h_K$ *and* ϕ_K *extends to an element of* $\operatorname{Aut}(h')$.

Then $\mathcal{M}$ *admits a good moduli space* $\pi\colon \mathcal{M} \to M$.

Remark 6.3.13. *Property (3) is a special case of what is called "unpunctured inertia". It is also true that if* $\mathcal{M}$ *has connected stabilizers then* $\mathcal{M}$ *has unpunctured inertia.*

From now on, we write $\mathcal{M} = \mathcal{M}^{\sigma-ss}(\gamma)$ for convenience, where σ is one of the stability conditions of Definition 6.3.7. We immediately obtain the following corollary.

Theorem 6.3.14. [AHLH] *$\mathcal{M}$ admits a good moduli space $\pi\colon \mathcal{M} \to M$.*

Proof. We will apply the previous theorem.

Closed points of $\mathcal{M}$ correspond to polystable objects. Their stabilizers are products of general linear groups and hence reductive. This shows (1).

By Proposition 6.3.11, we know that $\mathcal{M}$ is Θ-reductive. This shows (2).

Finally, if ϕ_K is an automorphism of $h_K\colon \operatorname{Spec} K \to \mathcal{M}$, i.e. a semistable perfect complex $E \otimes K$ of $\mathcal{O}_W \otimes K$-modules, then since the objects of $\mathcal{M}$ lie in an abelian category $\mathcal{A}$, we have an injection $E \to E \otimes K$ by the arguments of [Hal14, Lemma 4.22]. h_K is of finite order and so we may decompose $E \otimes K = \bigoplus_\lambda E_K^\lambda$ as a direct sum of eigenspaces for different roots of unity λ. Then, we clearly obtain $E = \bigoplus_\lambda E \cap E_K^\lambda$ and we may extend h_K by acting by multiplication by λ on each summand $E \cap E_K^\lambda$. This shows (3) and concludes the proof. Alternatively, $\mathcal{M}$ has connected stabilizers, so we can also use the preceding remark to conclude. □

We also have the following proposition.

Proposition 6.3.15. *M is separated and of finite type. $\pi\colon \mathcal{M} \to M$ is of finite type with affine diagonal.*

Proof. Since $\mathcal{M}$ is of finite type over $\mathbb{C}$, the same holds for M by Proposition 2.5.2(7). By Proposition 2.5.2(3), M parametrizes S-equivalence classes of complexes in $\mathcal{M}$. It follows by condition (4) of Setup 6.3.3 and Remark 6.3.6 that M must be separated.

We have a diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\Delta_\pi"] \arrow[dr, "\Delta_{\mathcal{M}}"'] & \mathcal{M} \times_M \mathcal{M} \arrow[r] \arrow[d, "\alpha"] & M \arrow[d] \\
& \mathcal{M} \times \mathcal{M} \arrow[r] & M \times M
\end{tikzcd}$$

where the right square is cartesian. Since M is separated, α is a closed immersion and hence by the usual cancellation property, since the diagonal of an immersion is an isomorphism, and $\Delta_{\mathcal{M}}$ is affine, we deduce that Δ_π is affine. □

By condition (2) of Setup 6.3.3, $\mathcal{M}$ is an open substack of $\operatorname{Perf}(W)$. Since W is a

Calabi-Yau threefold, by the results of [PTVV13], $\mathrm{Perf}(W)$ is the truncation of a (-1)-shifted symplectic derived Artin stack. In particular, $\mathcal{M}$ is the truncation of a (-1)-shifted symplectic derived Artin stack.

We have therefore verified the assertion of the following theorem.

Theorem 6.3.16. *Let σ be a stability condition as in Definition 6.3.7 and W a Calabi-Yau threefold. Then the moduli stack $\mathcal{M}^{\sigma-ss}(\gamma)$ of σ-semistable objects in $\mathcal{A}$ of Chern character γ is an Artin stack of DT type with separated good moduli space M.*

6.3.4 Rigidification of $\mathbb{C}^\times$-automorphisms of objects

Let $\mathcal{M}$ be as in Theorem 6.3.16. We denote $T = \mathbb{C}^\times$.

To obtain a meaningful DT invariant, it will be necessary to rigidify the $\mathbb{C}^\times$-automorphisms of objects in $\mathcal{M}$.

For each family of complexes $E_S \in \mathcal{M}(S)$ there exists an embedding

$$\mathbb{G}_m(S) \to \mathrm{Aut}(E_S)$$

which is compatible with pullbacks and moreover $\mathbb{G}_m(S)$ is central. In the terminology used in [AGV08], we say that $\mathcal{M}$ has a $\mathbb{G}_m$-2-structure.

Using the results of [AOV08] or [AGV08], we may take the $\mathbb{G}_m$-rigification $\mathcal{M} /\!\!/ \mathbb{G}_m$ of $\mathcal{M}$. From the properties of rigification, for any point $x \in \mathcal{M}$, one has $\mathrm{Aut}_{\mathcal{M}/\!\!/\mathbb{G}_m}(x) = \mathrm{Aut}_{\mathcal{M}}(x)/T$. In particular, if $x \in \mathcal{M}^s$ is stable, then $\mathrm{Aut}_{\mathcal{M}/\!\!/\mathbb{G}_m}(x) = \{\mathrm{id}\}$. $\mathcal{M} /\!\!/ \mathbb{G}_m$ has the same good moduli space M and, as in Section 4, an étale cover by cartesian diagrams of the form

$$\begin{tikzcd}
{[U_\alpha/(G_\alpha/T)]} \arrow[r] \arrow[d] & \mathcal{M} /\!\!/ \mathbb{G}_m \arrow[d] \\
U_\alpha /\!\!/ (G_\alpha/T) = U_\alpha /\!\!/ G_\alpha \arrow[r] & M.
\end{tikzcd}$$

We may now replace $\mathcal{M}$ by $\mathcal{M} /\!\!/ \mathbb{G}_m$ and carry out the Kirwan partial desingularization procedure to get a DM stack $\widetilde{\mathcal{M}}$. One may check that the results of Chapter 2 go through by identical arguments.

Lemma 6.3.17. *$\widetilde{\mathcal{M}}$ is a proper Deligne-Mumford stack.*

Proof. Since $\mathcal{M}$ is quasi-proper by condition (3) of Setup 6.3.3, $\mathcal{M}/\!\!/\mathbb{G}_m$ is also quasi-proper. Since $\widetilde{\mathcal{M}}$ is quasi-proper over $\mathcal{M}/\!\!/\mathbb{G}_m$, $\widetilde{\mathcal{M}}$ is also quasi-proper.

It thus remains to show that $\widetilde{\mathcal{M}}$ is separated. We first check that the map $\widetilde{\pi}\colon \widetilde{\mathcal{M}} \to \widetilde{M}$ is separated. By the above discussion we may assume that this map is of the form $[U/G] \to U/\!\!/G$ where U is affine and G is finite. But this is separated by a standard fact from GIT (cf. [MFK94, Proposition 0.8]). In particular, we deduce that

$$\widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}} \times_{\widetilde{M}} \widetilde{\mathcal{M}}$$

is proper.

Since $\widetilde{M}$ is also separated (it is proper over M), the cartesian diagram

$$\begin{tikzcd}
\widetilde{\mathcal{M}} \times_{\widetilde{M}} \widetilde{\mathcal{M}} \arrow[r] \arrow[d] & \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}} \arrow[d] \\
\widetilde{M} \arrow[r] & \widetilde{M} \times \widetilde{M}
\end{tikzcd} \tag{6.4}$$

shows that $\widetilde{\mathcal{M}} \times_{\widetilde{M}} \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}}$ is a closed embedding. It immediately follows that the diagonal $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}}$ is proper. □

Remark 6.3.18. *In the case of semistable sheaves, rigidification is much simpler, since the moduli stack is a global GIT quotient* $\mathcal{M} = [X/G]$ *where* $G = \mathrm{GL}(N, \mathbb{C})$*, and then one may work directly with* $[X/\mathrm{PGL}(N, \mathbb{C})]$ *which is the* $\mathbb{G}_m$*-rigidication.*

We can also first take the Kirwan partial desingularization and then rigidify: We may follow the same steps as in Chapter 2 to obtain an Artin stack $\widetilde{\mathcal{M}}'$ whose stabilizers are all of dimension one, by resolving all higher dimensional stabilizers in order of decreasing dimension. Moreover, for any object $E_S \in \widetilde{\mathcal{M}}'(S)$ we get an induced embedding $\mathbb{G}_m(S) \to \mathrm{Aut}(E_S)$, compatible with pullbacks, such that $\mathbb{G}_m(S)$ is central.

By construction, $\widetilde{\mathcal{M}}'$ has a good moduli space $\widetilde{M}$ and is covered by étale morphisms

$$[U_\alpha/G_\alpha] \to \widetilde{\mathcal{M}}'$$

where each $[U_\alpha/G_\alpha]$ is in standard form for data $\Lambda_\alpha = (U_\alpha, V_\alpha, F_\alpha, \omega_\alpha, \phi_\alpha)$. It has a $\mathbb{G}_m$-2-structure, compatible with the one of $\widetilde{\mathcal{M}}'$, so that $T \subset G_\alpha$ acts trivially on U_α.

Then we have the $\mathbb{G}_m$-rigification $\widetilde{\mathcal{M}}' /\!\!/ \mathbb{G}_m$ of $\widetilde{\mathcal{M}}'$. From the properties of rigidification, for any point $x \in \widetilde{\mathcal{M}}$, one has $\mathrm{Aut}_{\widetilde{\mathcal{M}}'/\!\!/\mathbb{G}_m}(x) = \mathrm{Aut}_{\widetilde{\mathcal{M}}'}(x)/T$, and this is now zero-dimensional, hence finite. In particular, G_α/T is finite and $\widetilde{\mathcal{M}}' /\!\!/ \mathbb{G}_m$ is then a DM stack, which has the same good moduli space $\widetilde{M}$. It has an étale cover of the form

$$[U_\alpha/(G_\alpha/T)] \to \widetilde{\mathcal{M}}' /\!\!/ \mathbb{G}_m$$

Each $[U_\alpha/G_\alpha]$ comes with a 4-term complex

$$\mathfrak{g}_\alpha \to T_{V_\alpha}|_{U_\alpha} \xrightarrow{(d\omega_{V_\alpha}^\vee)^\vee} F_\alpha|_{U_\alpha} \xrightarrow{\phi_V} \mathfrak{g}_\alpha^\vee(-D_\alpha). \tag{6.5}$$

Setting $\mathfrak{t} = \mathrm{Lie}(T)$, we see that the compositions $\mathfrak{t} \to \mathfrak{g}_\alpha \to T_{V_\alpha}|_{U_\alpha}$ and $F_\alpha|_{U_\alpha} \to \mathfrak{g}_\alpha^\vee(-D_\alpha) \to \mathfrak{t}^\vee(-D_\alpha)$ are zero, since the T-action on V_α is trivial and by property (3) of Setup 4.1.1. It follows that we have an induced 4-term complex

$$\mathfrak{g}_\alpha/\mathfrak{t} \to T_{V_\alpha}|_{U_\alpha} \xrightarrow{(d\omega_{V_\alpha}^\vee)^\vee} F_\alpha|_{U_\alpha} \xrightarrow{\phi_V} (\mathfrak{g}_\alpha/\mathfrak{t})^\vee(-D_\alpha), \tag{6.6}$$

with injective first arrow and surjective last arrow, since $[U_\alpha/(G_\alpha/T)]$ is DM. As in Chapter 4, by taking the kernel and cokernel of these two arrows, this yields a reduced 2-term complex

$$E_\alpha = [(F_\alpha^{\mathrm{red}})^\vee|_{U_\alpha} \xrightarrow{d(\omega_\alpha^{\mathrm{red}})^\vee} \Omega_{V_\alpha}|_{U_\alpha}] \tag{6.7}$$

giving a reduced perfect obstruction theory on $[U_\alpha/(G_\alpha/T)]$. These glue to a semi-perfect obstruction theory on $\widetilde{\mathcal{M}}' /\!\!/ \mathbb{G}_m$.

It is easy to observe that the following proposition is true.

Proposition 6.3.19. *We have an isomorphism $\widetilde{\mathcal{M}} \simeq \widetilde{\mathcal{M}}'/\!\!/\mathbb{G}_m$ and a commutative diagram*

$$\begin{tikzcd}
\widetilde{\mathcal{M}}' \arrow[r] \arrow[d] & \widetilde{\mathcal{M}} \simeq \widetilde{\mathcal{M}}' /\!\!/ \mathbb{G}_m \arrow[r] \arrow[d] & \widetilde{M} \arrow[d] \\
\mathcal{M} \arrow[r] & \mathcal{M} /\!\!/ \mathbb{G}_m \arrow[r] & M,
\end{tikzcd}$$

where the vertical arrows are the Kirwan desingularization morphisms.

Combining the above, we get the following theorem.

Theorem-Definition 6.3.20. *$\widetilde{\mathcal{M}}$ is called the $\mathbb{C}^\times$-rigidified Kirwan partial desingularization of $\mathcal{M}$. It is a proper DM stack with a morphism $\widetilde{\mathcal{M}} \to \mathcal{M} /\!\!/ \mathbb{G}_m$ and a semi-perfect obstruction theory of virtual dimension zero.*

Remark 6.3.21. *By identical reasoning, all of the above hold in greater generality when $\mathcal{M}$ is an Artin stack of DT type with a $\mathbb{G}_m$-2-structure.*

An alternative way to rigidify when the objects of $\mathcal{M}^{\sigma-ss}(\gamma)$ have non-zero rank, would be to consider the moduli stack $\mathcal{M}_L^{\sigma-ss}(\gamma)$ of fixed determinant L.

6.3.5 Definition of the DTK invariant

Suppose as before that $\mathcal{M}$ is an Artin stack of DT type parametrizing σ-semistable objects in a heart $\mathcal{A}$ of a t-structure. Then there is an induced $\mathbb{C}^\times$-rigidified Kirwan partial desingularization $\widetilde{\mathcal{M}}$ with a good moduli space $\widetilde{M}$ and its induced semi-perfect obstruction theory and virtual cycle of dimension zero.

Combining all of the above, Theorem-Definition 6.3.20 yields the following:

Theorem-Definition 6.3.22. *Let W be a smooth, projective Calabi-Yau threefold, σ a stability condition on a heart $\mathcal{A} \subset D^b(X)$ of a t-structure, as in Definition 6.3.7, and $\gamma \in H^*(W)$. Then we may define the generalized Donaldson-Thomas invariant via Kirwan blowup as*

$$\mathrm{DTK}\left(\mathcal{M}^{\sigma-ss}(\gamma)\right) := \deg\, [\widetilde{\mathcal{M}}^{\sigma-ss}(\gamma)]^{\mathrm{vir}}.$$

6.3.6 The relative case

Let C be a smooth quasi-projective curve over $\mathbb{C}$ and $W \to C$ be a family of Calabi-Yau threefolds. We make the following assumption.

Assumption 6.3.23. *There is a family σ_t of stability conditions, where, for each $t \in C$, σ_t is as in Definition 6.3.7, stability is an open condition and we have a stack $\mathcal{M} \to C$ of relatively semistable objects in $D^b(\mathrm{Coh}\, W)$. This parametrizes perfect complexes E such that the (derived) restriction $E_t := E|_{W_t}$ is σ_t-semistable for all $t \in C$. Moreover, the morphism $\mathcal{M} \to C$ is universally closed.*

Remark 6.3.24. *When we have a GIT description of $\mathcal{M} \to C$, then Assumption 6.3.23 is satisfied. This is the case for Gieseker stability of coherent sheaves.*

Given this, all the above results extend to the relative case, since we have analogous results for good moduli spaces, the local structure of stacks with moduli spaces, Kirwan partial desingularizations and semi-perfect obstruction theories. In particular, a Luna étale slice theorem for stacks over a general base will appear in [AHR].

Theorem 6.3.25. *Let $f\colon W \to C$ be a family of Calabi-Yau threefolds over a smooth, quasi-projective curve C, $\{\sigma_t\}_{t\in C}$ a family of stability conditions on $D^b(\mathrm{Coh}\, W)$, $\gamma \in \Gamma(C, \mathbf{R}f_*\mathbb{Q})$ and $\mathcal{M} \to C$ the stack of fiberwise σ_t-semistable objects of Chern character γ_t in $D^b(\mathrm{Coh}\, W)$, satisfying Assumption 6.3.23.*

Then there exists an induced $\mathbb{C}^\times$-rigidified Kirwan partial desingularization $\widetilde{\mathcal{M}} \to C$, a proper DM stack over C, endowed with a semi-perfect obstruction theory of virtual dimension zero and a virtual fundamental cycle $[\widetilde{\mathcal{M}}]^{\mathrm{vir}}$. $\widetilde{\mathcal{M}}_t$ is the $\mathbb{C}^\times$-rigidified Kirwan partial desingularization of $\mathcal{M}_t$ and the obstruction theory restricts to the one constructed in the absolute case.

As an immediate corollary, we have the following theorem.

Theorem 6.3.26. *The generalized DT invariant via Kirwan blowups is invariant under deformations of the complex structure of the Calabi-Yau threefold.*

Remark 6.3.27. *Assumption 6.3.23 is not too much to ask for. Upcoming work announced in [BLM$^+$] will establish it in the cases of stability conditions that we consider in Definition 6.3.7 and therefore the above theorem holds unconditionally.*

Bibliography

[AGV08] Dan Abramovich, Tom Graber, and Angelo Vistoli. Gromov-Witten theory of Deligne-Mumford stacks. *Amer. J. Math.*, 130(5):1337–1398, 2008.

[AHLH] Jarod Alper, Daniel Halpern-Leistner, and Jochen Heinloth. To appear.

[AHR] Jarod Alper, Jack Hall, and David Rydh. The étale local structure of algebraic stacks.

[AHR15] Jarod Alper, Jack Hall, and David Rydh. A Luna étale slice theorem for algebraic stacks. *ArXiv e-prints*, April 2015.

[Alp13] Jarod Alper. Good moduli spaces for Artin stacks [bons espaces de modules pour les champs d'Artin]. *Annales de l'institut Fourier*, 63(6):2349–2402, 2013.

[AOV08] Dan Abramovich, Martin Olsson, and Angelo Vistoli. Tame stacks in positive characteristic. *Ann. Inst. Fourier (Grenoble)*, 58(4):1057–1091, 2008.

[Bay09] Arend Bayer. Polynomial bridgeland stability conditions and the large volume limit. *Geometry & Topology*, 13(4):2389–2425, 2009.

[BBBBJ15] Oren Ben-Bassat, Christopher Brav, Vittoria Bussi, and Dominic Joyce. A 'Darboux theorem' for shifted symplectic structures on derived Artin stacks, with applications. *Geom. Topol.*, 19(3):1287–1359, 2015.

[BBD$^+$12] Christopher Brav, Vittoria Bussi, Delphine Dupont, Dominic Joyce, and Balazs Szendroi. Symmetries and stabilization for sheaves of vanishing cycles. *arXiv preprint arXiv:1211.3259*, 2012.

[BBJ13] Christopher Brav, Vittoria Bussi, and Dominic Joyce. A 'Darboux theorem' for derived schemes with shifted symplectic structure. *ArXiv e-prints*, May 2013.

[Beh09] Kai Behrend. Donaldson-Thomas type invariants via microlocal geometry. *Ann. of Math. (2)*, 170(3):1307–1338, 2009.

[BF97] Kai Behrend and Barbara Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[BLM$^+$] Arend Bayer, Martí Lahoz, Emanuele Macrì, Howard Nuer, Alexander Perry, and Paolo Stellari. Stability conditions in families and families of hyperkähler varieties.

[BR16a] Kai Behrend and Pooyah Ronagh. The Eigenvalue Spectrum of the Inertia Operator. *ArXiv e-prints*, December 2016.

[BR16b] Kai Behrend and Pooyah Ronagh. The inertia operator on the motivic Hall algebra. *ArXiv e-prints*, December 2016.

[CL11] Huai-Liang Chang and Jun Li. Semi-perfect obstruction theory and Donaldson-Thomas invariants of derived objects. *Comm. Anal. Geom.*, 19(4):807–830, 2011.

[DM16] Ben Davison and Sven Meinhardt. Cohomological Donaldson-Thomas theory of a quiver with potential and quantum enveloping algebras. *ArXiv e-prints*, January 2016.

[Dré04] Jean-Marc Drézet. Luna's slice theorem and applications. In Jaroslaw A. Wisniewski, editor, *Algebraic group actions and quotients*, pages 39–90. Hindawi Publishing Corporation, 2004.

[ER17] Dan Edidin and David Rydh. Canonical reduction of stabilizers for Artin stacks with good moduli spaces. *ArXiv e-prints*, October 2017.

[Hal14] Daniel Halpern-Leistner. On the structure of instability in moduli theory. *ArXiv e-prints*, November 2014.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HL] Daniel Halpern-Leistner. The D-Equivalence Conjecture for Moduli Spaces of Sheaves.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[HT10] Daniel Huybrechts and Richard P. Thomas. Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes. *Math. Ann.*, 346(3):545–569, 2010.

[Ill71] Luc Illusie. *Complexe Cotangent et Deformations I.* Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1971.

[Joy15] Dominic Joyce. A classical model for derived critical loci. *J. Differential Geom.*, 101(2):289–367, 2015.

[JS12] Dominic Joyce and Yinan Song. A theory of generalized Donaldson-Thomas invariants. *Mem. Amer. Math. Soc.*, 217(1020):iv+199, 2012.

[Kir85] Frances Clare Kirwan. Partial desingularisations of quotients of nonsingular varieties and their Betti numbers. *Ann. of Math. (2)*, 122(1):41–85, 1985.

[KL12] Young-Hoon Kiem and Jun Li. Categorification of Donaldson-Thomas invariants via Perverse Sheaves. *ArXiv e-prints*, December 2012.

[KL13a] Young-Hoon Kiem and Jun Li. Localizing virtual cycles by cosections. *J. Amer. Math. Soc.*, 26(4):1025–1050, 2013.

[KL13b] Young-Hoon Kiem and Jun Li. A wall crossing formula of Donaldson-Thomas invariants without Chern-Simons functional. *Asian J. Math.*, 17(1):63–94, 2013.

[KLS17] Young-Hoon Kiem, Jun Li, and Michail Savvas. Generalized Donaldson-Thomas Invariants via Kirwan Blowups. *ArXiv e-prints*, December 2017.

[KS10] Maxim Kontsevich and Yan Soibelman. Motivic Donaldson-Thomas invariants: summary of results. In *Mirror symmetry and tropical geometry*, volume 527 of *Contemp. Math.*, pages 55–89. Amer. Math. Soc., Providence, RI, 2010.

[Lie06] Max Lieblich. Moduli of complexes on a proper morphism. *J. Algebraic Geom.*, 15(1):175–206, 2006.

[Lo11] Jason Lo. Moduli of PT-semistable objects I. *J. Algebra*, 339:203–222, 2011.

[Lo13] Jason Lo. Moduli of PT-semistable objects II. *Trans. Amer. Math. Soc.*, 365(9):4539–4573, 2013.

[LT98] Jun Li and Gang Tian. Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. *J. Amer. Math. Soc.*, 11(1):119–174, 1998.

[MFK94] David Mumford, James Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[MNOP06] Davesh Maulik, Nikita Nekrasov, Andrei Okounkov, and Rahul Pandharipande. Gromov-Witten theory and Donaldson-Thomas theory. I. *Compos. Math.*, 142(5):1263–1285, 2006.

[MT11] Davesh Maulik and David Treumann. Constructible functions and Lagrangian cycles on orbifolds. *arXiv preprint arXiv:1110.3866*, 2011.

[MT16] Davesh Maulik and Yukinobu Toda. Gopakumar-Vafa invariants via vanishing cycles. *ArXiv e-prints*, October 2016.

[PT09] Rahul Pandharipande and Richard P. Thomas. Curve counting via stable pairs in the derived category. *Invent. Math.*, 178(2):407–447, 2009.

[PT14] Rahul Pandharipande and Richard P. Thomas. Almost closed 1-forms. *Glasgow Mathematical Journal*, 56(1):169182, 2014.

[PT15] Dulip Piyaratne and Yukinobu Toda. Moduli of Bridgeland semistable objects on 3-folds and Donaldson-Thomas invariants. *ArXiv e-prints*, April 2015.

[PTVV13] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted symplectic structures. *Publications mathématiques de l'IHÉS*, 117(1):271–328, 2013.

[Rei89] Zinovy Reichstein. Stability and equivariant maps. *Inventiones mathematicae*, 96(2):349–383, Jun 1989.

[Sav] Michail Savvas. *To appear.*

[Tho00] Richard P. Thomas. A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations. *J. Differential Geom.*, 54(2):367–438, 2000.