MATH 109 Winter 2020 – Final Exam Study Guide

The following provides a list of concepts that you should be familiar with for the final exam. It consists of essential points we have covered as well as some examples that appeared in lecture or in the homework together with occasional comments. The exam is cumulative and will cover all the material we have covered up to Chapter 22. You should also refer to the textbook, lecture notes and homework for an idea of what you may be expected to know. Reviewing examples done in lecture and assigned as homework will provide you with a solid understanding of concepts which may be tested. However, you may be asked to apply understanding of these concepts in new ways on the exam, so it is important that you master the underlying concepts and fully understand the motivation of each step of the solution in addition to knowing how to solve the exercises you review.

**Chapter 1 - The language of mathematics**

- Propositions, predicates, statements.
- The connectives ‘or’, ‘and’ and ‘not’.
- Truth tables.

**Chapter 2 - Implications**

- Truth table of $P \Rightarrow Q$.
- Language associated to implications: converse, necessary and sufficient conditions, if and only if.
- Equivalence of two statements.
- Divisibility.
- Even and odd integers.

**Chapter 3 - Proofs**

- Direct proofs.
- Proofs by cases.
- Constructing proofs backwards.
- Examples with algebra of real numbers.

**Chapter 4 - Proofs by contradiction**

- Proofs by contradiction.
- Proofs by contrapositive.
- Examples with arithmetic and algebra.
Chapter 5 - Induction

- The principle of induction.
- Examples with inequalities and divisibility.
- Definitions by induction.
- Examples with sum notation and factorial.

Chapter 6 - Sets

- Sets, elements of sets, subsets, equality of sets, empty set.
- Union, intersection, difference of two sets.
- The power set.
- Properties of union, intersection and complement.
- Using truth tables to prove statements about sets.
- Venn diagrams.
- Constructing proofs for statements about sets.

Chapter 7 - Quantifiers

- Universal statements (∀) and existential statements (∃).
- Proving and disproving universal and existential statements and combinations thereof.
- The Cartesian product of two sets.

Chapter 8 - Functions

- Functions, domain, codomain, image, graph.
- Equality of functions.
- Composition of two functions.

Chapter 9 - Injections, surjections, bijections

- Injections, surjections, bijections.
- Invertible functions and inverse functions.
- A function is bijective if and only if it is invertible.
- Investigating these properties in abstract and explicit examples.
Chapter 10 - Counting

- Definition of finite sets.
- Addition principle for the union of two disjoint finite sets.
- Multiplication principle for the Cartesian product of two finite sets.
- Inclusion-exclusion principle for the union of two or three finite sets.
- Applying these principles in counting problems.

Chapter 11 - Pigeonhole principle

- The cardinality of a finite set is well-defined.
- Proof that the existence of an injection \( \mathbb{N}_n \rightarrow \mathbb{N}_m \) implies that \( n \leq m \).
- The pigeonhole principle.
- Using the pigeonhole principle in existence problems.

Chapter 12 - Counting subsets

- Cardinality of the power set of a finite set.
- \( r \)-subsets and binomial coefficients \( \binom{n}{r} \).
- Identities of binomial coefficients and explicit formula and their proofs.
- Binomial theorem and its proof.
- Proving identities between binomial coefficients using counting and algebraic arguments.

Chapter 13 - \( \sqrt{2} \) is irrational

- Proof that \( \sqrt{2} \) is not rational.

Chapter 14 - Counting infinite sets

- Two equivalent definitions of denumerable and countable sets.
- Examples of denumerable sets.
- A subset of a countable set is countable.
- The Cartesian product of two denumerable sets is denumerable (via “diagonal counting”).
- \( \mathbb{Q} \) is denumerable.
- Cantor’s diagonal argument: \( \mathbb{R} \) is uncountable, for any set \( X \) there exists no bijection \( X \rightarrow \mathcal{P}(X) \).
Chapter 15 - Euclidean division

• The division theorem and its proof.
• $b$ divides $a$ if and only if the remainder of the division of $a$ by $b$ is zero.
• Using the division theorem to prove statements about squares of integers by examining possible remainders.

Chapter 16 - Euclidean algorithm

• Greatest common divisor of two integers.
• If $a = bq + r$, then $\gcd(a, b) = \gcd(b, r)$.
• Euclid’s algorithm for computing the greatest common divisor.

Chapter 17 - Consequences of the Euclidean algorithm

• Integral linear combinations of integers.
• The greatest common divisor of two integers is an integral linear combination of them.
• Using Euclid’s algorithm to write $\gcd(a, b)$ as an integral linear combination of $a, b$.
• Coprime integers and their properties.

Chapter 18 - Linear diophantine equations

• Diophantine equations.
• Solving homogeneous linear diophantine equations $am + bn = 0$.
• Determining when a linear diophantine equation $am + bn = c$ has solutions.
• Solving linear diophantine equations $am + bn = c$.

Chapter 19 - Modular arithmetic

• Congruence of two integers modulo $m$.
• Reflexivity, symmetry and transitivity of congruence.
• Addition, subtraction and multiplication modulo $m$.
• Set of remainders $R_m$ and remainder map $r_m$ modulo $m$.
• Using modular arithmetic to prove divisibility criteria, compute squares modulo $m$, investigate whether diophantine equations have solutions, find decimal digits of large powers etc.
Chapter 21 - Congruence classes

- Congruence classes modulo $m$.
- Properties of congruence classes corresponding to congruence modulo $m$.
- The set of congruence classes $\mathbb{Z}_m$ and its cardinality $|\mathbb{Z}_m| = m$.
- Addition, subtraction and multiplication on $\mathbb{Z}_m$ and proof they are well-defined.
- Addition, subtraction and multiplication on $\mathbb{R}_m$ and compatibility with the same operations on $\mathbb{Z}_m$.

Chapter 22 - Partitions and equivalence classes

- Partition $\Pi$ of a set $X$.
- Relations $\sim$ on a set $X$.
- Reflexivity, symmetry and transitivity of a relation. Equivalence relations.
- Equivalence classes for $\sim$ and set of equivalence classes $X/\sim$.
- Any partition $\Pi$ induces an equivalence relation $\sim_\Pi$ on $X$.
- For an equivalence relation $\sim$, $\Pi = X/\sim$ is a partition of $X$ and the induced equivalence relation $\sim_\Pi$ is the same as $\sim$.
- Examples of the above. Definition of $\mathbb{Q}$ and its algebraic operations as a set of equivalence classes.