5.4
Base case: \( n = 0 \). LHS = 1 and RHS = 1. So the base case is true.
Inductive step: Assume
\[
\sum_{i=0}^{k} x^i = \frac{1 - x^{k-1}}{1 - x}
\]
Then
\[
\sum_{i=0}^{k+1} x^i = \sum_{i=0}^{k} x^i + x^{k+1} = \frac{1 - x^{k-1}}{1 - x} + x^{k+1}
\]
Simplifying this expression, after some algebraic manipulations we get:
\[
\sum_{i=0}^{k+1} x^i = \frac{1 - x^{k-1}}{1 - x} + x^{k+1} = \frac{1 - x^k}{1 - x}
\]
Therefore, by induction we have
\[
\sum_{i=0}^{n} x^i = \frac{1 - x^{n-1}}{1 - x}
\]
for all \( n \geq 0 \).

6.1
(i)
0 is not less than 0, so 0 \( \not\in \) (0, 1)
0 \( \leq \) 0 \( \leq \) 1, so 0 \( \in \) [0, 1]
0 \( \leq \) 0 < 1, so 0 \( \in \) [0, 1)
0 is not less than 0, so 0 \( \not\in \) (0, 1]

(ii)
\{ a,b \}

(iii)
We prove the contrapositive
\((a, b) \neq \emptyset \leftrightarrow a < b \)
\( \Rightarrow \): Suppose \((a, b) \neq \emptyset \). Then we can choose \( x \in (a, b) \), and so we have \( a < x < b \), therefore \( a < b \).
\( \Leftarrow \): Suppose \( a < b \), then \( a < \frac{a+b}{2} < b \) so \( \frac{a+b}{2} \in (a, b) \).

(iv)
Suppose \([a, b] \subseteq (c, d) \). Then since \( a \in [a, b] \), we must have \( a \in (c, d) \), therefore \( c < a \). Similarly, we have \( b < d \).
Conversely, suppose \( c < a \) and \( b < d \). Then \( x \in [a, b] \) implies that \( a \leq x \leq b \). Thus \( c < a \leq x \leq b < d \), so \( c < x < d \). So \( x \in (c, d) \). Thus \([a, b] \subseteq (c, d) \).
6.2

(i) Suppose \( a \in \{ x \in \mathbb{R} \mid x^2 + x - 2 = 0 \} \). Then \( a^2 + a - 2 = 0 \), solving this equation gives us \( a = 1, -2 \). Therefore \( \{ x \in \mathbb{R} \mid x^2 + x - 2 = 0 \} \subseteq \{ 1, -2 \} \). The other direction \( \{ x \in \mathbb{R} \mid x^2 + x - 2 = 0 \} \supseteq \{ 1, -2 \} \) is easy to check, so the two sets are equal.

(ii) Suppose \( a \in \{ x \in \mathbb{R} \mid x^2 + x - 2 < 0 \} \). Then \( a^2 + a - 2 < 0 \), solving this equation gives us \( a \in (-2, 1) \). Therefore \( \{ x \in \mathbb{R} \mid x^2 + x - 2 < 0 \} \subseteq (-2, 1) \).

(iii) Same method as above.

6.4

<table>
<thead>
<tr>
<th>( x \in A )</th>
<th>( x \in B )</th>
<th>( x \in C )</th>
<th>( x \in B \cup C )</th>
<th>( x \in A \cap (B \cup C) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

One can similarly construct a truth table for \( x \in A \cap B \), \( x \in A \cap C \) and \( x \in (A \cap B) \cup (A \cap C) \), and see that the final column of the truth tables are the same.

The Venn diagram in the 'three' bubble Venn diagram.

6.5

(i) Suppose \( A \subseteq B \). We want to show \( A \cup B = B \). By definition we already know \( B \subseteq A \cup B \). Given \( x \in A \cup B \), then either (1) \( x \in A \), in which case \( x \in A \subseteq B \). The second case is \( x \in B \), in which case we also have \( x \in B \). Therefore \( B \supseteq A \cup B \). Consider the converse: Suppose \( A \cup B = B \), then \( A \subseteq A \cup B = B \), so we are done.

(ii) Suppose \( A \subseteq B \). Then by definition we have \( A \cap B \subseteq A \). Let \( x \in A \), then we know \( x \in A \subseteq B \), so \( x \in B \). Thus \( x \in A \cap B \), so \( A \cap B \supseteq A \). Thus \( A \cap B = A \). Consider the converse: Suppose \( A \cap B = A \). then \( A = A \cap B \subseteq B \), so we are done.

6.6

We know \( A \cap B \subseteq C \) and \( x \in B \). Suppose for contradiction \( x \in A \cap C \). Then \( x \in A \). Since \( x \in B \) also, we know \( x \in A \cap B \subseteq C \). So \( x \in A \cap C \). A contradiction.

6.7

\( A \subseteq B \) means:

\[ x \in A \Rightarrow x \in B \]

So the contrapositive of this is

\[ x \notin B \Rightarrow x \notin A \]

But this is the same as \( B^c \subseteq A^c \):

\[ x \in B^c \Rightarrow x \in A^c \]

The two statements are contrapositive of each other and so equivalent.

7.1

\( \mathbb{Z}^+, \{1\}, \mathbb{Z}^+, \emptyset \).
7.2

(i) False, by counterexample.
(ii) True, by $m = n = 1$ or any other valid choices for $m, n$.
(iii) True. See 7.1.(i)
(iv) True. See 7.1.(ii)
(v) True. See 7.1.(iii)
(vi) False. See 7.1.(iv)

I.13

Base case: $n = 4$, $4! > 2^4$ is true.
Inductive step: Suppose $k! > 2^k$ is true. Then multiplying by $(k + 1)$ on both side we have

$$(k + 1)! = k!(k + 1) > 2^k(k + 1)$$

Since $k \geq 4$, we know

$$2^k(k + 1) > 2^k2 = 2^{k+1}$$

So $(k + 1)! > 2^{k+1}$.
Therefore by induction we have, $(n)! > 2^n$ for $n \geq 4$.

I.16

Base case: $n = 1$. LHS = $1/2$ and RHS = $1/2$. So the base case is true.
Inductive step: Assume

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

Then

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

Simplifying this expression, after some algebraic manipulations we get:

$$\sum_{i=1}^{k+1} x^i = \frac{k + 1}{k + 2}$$

Therefore, by induction we have

$$\sum_{i=1}^{n} x^i = \frac{k}{k + 1}$$

for all $n \geq 1$.  

3