9.5

You can show that \( g \circ f \) and \((g \circ f)^{-1}\) are injective and surjective directly.

Or we can show that \( g \circ f \) is invertible: Since \( f \) and \( g \) are bijections, we know \( f^{-1} \) and \( g^{-1} \) are well-defined. Consider

\[
(g \circ f)^{-1} \circ (g \circ f) = f^{-1} \circ g^{-1} \circ g \circ f = f^{-1} \circ (g^{-1} \circ g) \circ f
\]

Since \( g^{-1} \circ g = \text{Id}_Y \)

\[
(g \circ f)^{-1} \circ (g \circ f) = f^{-1} \circ (\text{Id}_Y) \circ f = f^{-1} \circ f
\]

By a similar argument, we have

\[
(g \circ f)^{-1} \circ (g \circ f) = \text{Id}_X
\]

So \( g \circ f \) is invertible with inverse \( f^{-1} \circ g^{-1} \).

By Theorem 9.2.3, we know invertible functions are bijections.

10.1

Since \( X \) is finite of cardinality \( n \) there exists a bijection \( f : \mathbb{N}_n \to X \). Suppose that \( Y \) is also finite of cardinality \( n \). Then there is a bijection \( g : \mathbb{N}_n \to Y \). Since a bijection is invertible, we have a bijection \( g \circ f^{-1} : X \to Y \).

Conversely, given a bijection \( h : X \to Y \), we can deduce that \( |Y| \) is finite of cardinality \( n \) by writing down the bijection \( h \circ f : \mathbb{N}_n \to Y \).

10.2

Use induction, the base case is easy to verify.

For the inductive step: suppose the statement holds for \( k \). Since the \( X_i \)'s are pairwise disjoint, we know \( X_1 \cup X_2 \cup ... \cup X_k \) and \( X_{k+1} \) are also disjoint. Then by the addition theorem

\[
|X_1 \cup X_2 \cup ... \cup X_k \cup X_{k+1}| = |X_1 \cup X_2 \cup ... \cup X_k| + |X_{k+1}|
\]

which by the induction hypothesis equals \( \sum_{i=1}^{k+1} |X_i| \).

11.1

We do this by induction on \( n \).

Base case: For \( n = 1 \), if \( f : \mathbb{N}_1 \to X \) is a surjection then \( X = \{f(x)\} \) and so \( |X| = 1 \).

Induction step: suppose the result holds for some \( k \geq 1 \) and \( f : \mathbb{N}_{k+1} \to X \) is a bijection. Then we have two cases:

(i) \( f|_{\mathbb{N}_k} \) is a surjection. Then, by inductive hypothesis, \( X \) is finite and \( |X| \leq k \), so that certainly \( |X| \leq k+1 \).

(ii) Otherwise define \( f_1 : \mathbb{N}_k \to X - \{f(k+1)\} \) by \( f_1(i) = f(i) \). This is a surjection and by the inductive hypothesis,

\[
|X - \{f(k+1)\}| \leq k
\]

and so

\[
|X| \leq k + 1
\]
III.1
Use the principle of inclusion-exclusion. Let $A, B, C$ be the set of people who like reasoning, algebra and calculus respectively. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |C \cap B| + |A \cap B \cap C| = 181$$

So $200 - 181 = 19$ people like none of the subjects.

III.3
Let $A, B, C$ be the set of people who speak English, Spanish and Swahili respectively.

Let $a, e, g$ be the number of people who only speak English, Spanish and Swahili respectively. Let $b, d, f$ be the number of people who speak English + Spanish, English + Swahili, Swahili + Spanish respectively. Let $c$ be the number of people who speak all three languages.

Then

$$a + b + c + d + e + f + g = 100 \quad (1)$$
$$a + b + c + d = 75 \quad (2)$$
$$b + c + e + f = 60 \quad (3)$$
$$d + c + f + g = 45 \quad (4)$$

Then $2 \times eq(1) - eq(2) - eq(3) - eq(5)$ gives us:

$$a + e + g = 20 + c \quad (5)$$

For part (i), there are many ways to see this, one way is to observe that

$$a + e + g + c \leq 100$$

Plugging in $eq(5)$ gives

$$c \leq 40$$

For $c = 40$ we can have $e = 20, g = 5, a = 35, b = d = f = 0$. So this solution that maximises only English speakers.

For part (ii), observe that is only happens if we made sure that every Swahili speaker speaks Spanish too, no one speaks Swahili and English and not Spanish, and no one speaks Spanish and English and not Swahili. Then we just have 60 Spanish speakers and 75 English speakers. Since there are 100 people in total, 35 people speak all three languages. So 40 people only speak English, 35 people speaks all 3 language, 10 people speak Spanish + Swahili, 15 people only speak Spanish.

For part (iii), this is evident in $eq(5)$.

III.4
See https://faculty.math.illinois.edu/~nirobles/files453/iep_proof.pdf for detailed work through.

III.8
(Proof sketch): For the inductive step suppose $N_k \to N_n$ injective implies $k \leq n$. Let $f : N_{k+1} \to N_n$ be injective, by by a procedure similar to that of the original proof is Lemma 10.1.4 , we can first restrict $f$ to $N_k$ the post compose it the map that sends $N_n - f(k+1)$ to $N_{n-1}$ so that we have an injection $f' : N_k \to N_{n-1}$, so by the induction hypothesis $k \leq n - 1$, thus $k + 1 \leq n$.

III.12
Suppose by contradiction that $X$ is finite, then $|X| = n$ for some $n$. We have a bijection $g : N_n \to X$. Then $g^{-1} \circ f : \mathbb{Z}^+ \to N_n$ is an injection. Then consider $f|_{N_{n+1}}, N_{n+1} \to N_n$ is still an injection, so by cor.11.1.1, $N_{n+1} \leq N_n$, so $n + 1 \leq n$ which is a contradiction.
III.14

Let $O$ be the odd numbers in $\mathbb{N}_{2n}$, then $f$ defined in the hints is a function from $A$ to $O$. Since there are $n$ odd numbers in $\mathbb{N}_{2n}$, we know that $|O| = n$. Since $|A| = n + 1$, then by the pigeon hole principle, there exist $x, y \in A$ such that $f(x) = f(y) = z$. That means, there exist $i, j \in \mathbb{Z}_{\geq 0}$, such that

$$x = 2^i z, \quad y = 2^j z,$$

Without loss of generality, assume $i \leq j$, then $x$ divides $y$, so we are done.