This homework is due at 9pm on Monday, February 17. When you submit your homework, please make sure that your writing and scan are clear and legible.

Write complete solutions or proofs to the following problems. Numbered chapters and problems refer to Eccles’ book. Before you attempt a problem, make sure you have read your lecture notes and the corresponding parts of the chapter in the textbook.

Chapter 12 Exercises: 12.2, 12.5, 12.6
Chapter 14 Exercises: 14.1, 14.2, 14.3
Problems III: 18, 19, 20

Problem A (not from textbook): Suppose that \{A_n \mid n \in \mathbb{Z}^+\} is a denumerable set consisting of denumerable sets.

(a) For all \(n \in \mathbb{Z}^+\), let \(B_n = A_n - (A_1 \cup \cdots \cup A_{n-1})\). Show that the sets \(B_n\) are pairwise disjoint.

(b) Use part (a) and Exercise 14.3 to show that the union

\[
\bigcup_{n \in \mathbb{Z}^+} A_n = \{x \mid x \in A_n \text{ for some } n \in \mathbb{Z}^+\}
\]

is denumerable.

(c) Use part (b) to prove that the set \(\{A \in \mathcal{P}(\mathbb{Z}^+) \mid A \text{ is finite}\}\) of finite subsets of \(\mathbb{Z}^+\) is denumerable.

Comment: It follows from the above problem that a countable union of countable sets (pairwise disjoint or not) is countable.

If you feel like it, you can also think about the following problems, which will not be graded.

Bonus problems

Problems III: 28 (Hint: Problem A might help.)

Comment: The upshot of this problem is that, by Exercise 14.2 and the uncountability of \(\mathbb{R}\), almost all real numbers do not satisfy polynomial equations with rational coefficients (i.e. are not algebraic), so it is in a sense an existence result. However, it is much harder to prove that a particular number falls into this category (examples are \(e\) and \(\pi\)).

Problem C: A function \(f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+\) is called increasing if for all \(m, n \in \mathbb{Z}^+\) with \(m \leq n\) we have \(f(m) \leq f(n)\). It is called decreasing if for all \(m, n \in \mathbb{Z}^+\) with \(m \leq n\) we have \(f(m) \geq f(n)\).

(a) Is the set of all decreasing functions \(f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+\) countable?

(b) Is the set of all increasing functions \(f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+\) countable?

(Hint: You will need to use a diagonal argument for one of the two parts.)