1. (10 pt) (a) (7 pts) Let $n$ be an integer. Show that there does not exist an integer $q$ satisfying $n^2 = 5q + 2$.

**Solution:** By Euclidean division, we know that every integer $n$ is of the form $n = 5q + r$ where $0 \leq r \leq 4$. We have

\[
\begin{align*}
n = 5q &\Rightarrow n^2 = 5(5q^2) \\
n = 5q + 1 &\Rightarrow n^2 = 5(5q^2 + 2q) + 1 \\
n = 5q + 2 &\Rightarrow n^2 = 5(5q^2 + 4q) + 4 \\
n = 5q + 3 &\Rightarrow n^2 = 5(5q^2 + 6q + 1) + 4 \\
n = 5q + 4 &\Rightarrow n^2 = 5(5q^2 + 8q + 3) + 1
\end{align*}
\]

Thus we see that the possible remainders for the Euclidean division of $n^2$ by 5 are 0, 1, 4, which implies that there exists no integer $q$ with $n^2 = 5q + 2$.

(b) (3 pts) Show that 1002 is not a square.

**Solution:** $1002 = 5 \times 200 + 2$ and thus by (a) it cannot be a square as its remainder for Euclidean division by 5 is 2.
2. (10 pt) Let $f: X \to Y$ and $g: Y \to X$ be functions such that $g \circ f = I_X$.

(a) (4 pts) Show that $f$ is injective.

**Solution:** Suppose that $f(x_1) = f(x_2)$. Then

$$x_1 = g(f(x_1)) = g(f(x_2)) = x_2$$

and thus $f$ is injective.

(b) (4 pts) Show that $g$ is surjective.

**Solution:** Let $x \in X$. For $y = f(x) \in Y$ we have $g(y) = g(f(x)) = x$ and therefore $g$ is surjective.

(c) (2 pts) Give an explicit example of two sets $X, Y$ and two functions $f: X \to Y$ and $g: Y \to X$ such that $g \circ f = I_X$ and $f$ and $g$ are not bijective.

**Solution:** Let $X = \{1\}$, $Y = \{1, 2\}$, $f: X \to Y$ defined by $f(1) = 1$ and $g: Y \to X$ defined by $g(1) = g(2) = 1$. Then we have that $f$ is not surjective and $g$ is not injective, but $g \circ f = I_X$. 
3. (a) (4 pts) Let $n$ be a positive integer. Prove that

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

**Solution:** For $n > 0$, using the Binomial Theorem with $a = -1$, $b = 1$ we have

$$0 = (-1 + 1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i 1^{n-i} \Rightarrow \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

(b) (6 pts) Let $A$ be a set consisting of 10 distinct numbers from the set \{1, \ldots, 20\}. Show that there exist two disjoint subsets $X, Y$ of $A$ with $|X| = |Y| = 2$ such that the sum of the elements of $X$ is the same as the sum of the elements of $Y$.

**Solution:** The maximum possible sum of the elements of $X \subseteq A$ with $|X| = 2$ is $20 + 19 = 39$. Consider the function $f: \mathcal{P}_2(A) \to \mathbb{N}_{39}$ that maps $X \subseteq A$ with $|X| = 2$ to the sum of its elements. Then

$$|\mathcal{P}_2(A)| = \binom{10}{2} = \frac{10 \times 9}{2} = 45 > 39 = |\mathbb{N}_{39}|$$

so by the Pigeonhole Principle there exist $X \neq Y \subseteq A$ with $|X| = |Y| = 2$ with the same sum of elements $s = f(X) = f(Y)$.

If $x \in X \cap Y$ then $s - x$ also belongs to $X$ and $Y$ (since for both the sum of their two elements is $s$) and must be different than $x$ (if $s = 2x$ then $X, Y$ would only contain the element $x$, which is not possible), thus $X = Y = \{x, s - x\}$ which contradicts $X \neq Y$. Thus $X \cap Y = \emptyset$ and $X$ and $Y$ must be disjoint.
4. Let $A, B$ be disjoint infinite sets. Prove that their union $A \cup B$ is denumerable if and only if both $A$ and $B$ are denumerable.

**Solution:** Suppose that $A \cup B$ are denumerable. $A$ and $B$ are subsets of $A \cup B$ and are infinite, so they are denumerable.

Conversely, suppose that $A$ and $B$ are denumerable. Then we have bijections $f : \mathbb{Z}^+ \to A$ and $g : \mathbb{Z}^+ \to B$. We construct a function $h : \mathbb{Z}^+ \to A \cup B$ by the formula

$$h(n) = \begin{cases} f(k) & \text{if } n = 2k - 1, \ k \geq 1 \\ g(k) & \text{if } n = 2k, \ k \geq 1 \end{cases}$$

$h$ is surjective, since for any $a \in A$ we have $f(k) = a$ for some $k \geq 1$, as $f$ is surjective, and then $h(2k - 1) = f(k) = a$ and for any $b \in B$ we have $g(k) = b$ for some $k \geq 1$, as $g$ is surjective, and then $h(2k) = g(k) = b$.

Similarly, $h$ is injective. Suppose $h(n_1) = h(n_2) = c \in A \cup B$. If $c \in A$ then $c \notin B$ ($A, B$ are disjoint) and $n_1, n_2$ must both be odd, so that $n_1 = 2k_1 - 1$ and $n_2 = 2k_2 - 1$. Then $f(k_1) = h(n_1) = h(n_2) = f(k_2)$ and since $f$ is injective we have $k_1 = k_2 \Rightarrow n_1 = n_2$. If $c \in B$ then $c \notin A$ and $n_1, n_2$ must both be even, so that $n_1 = 2k_1$ and $n_2 = 2k_2$. Then $g(k_1) = h(n_1) = h(n_2) = g(k_2)$ and since $g$ is injective we have $k_1 = k_2 \Rightarrow n_1 = n_2$.

We conclude that $h$ is a bijection and $A \cup B$ is denumerable.