**Math 109** **Final Exam** **Name:** ______________

**Jun 14, 2019** **(Total points: 50)** **PID:** ______________

**Instructions**
1. Write your Name and PID in the spaces provided above.
2. Make sure your Name is on every page.
3. Write your solutions clearly in the spaces provided. Work on scratch paper will not be graded.
4. Note that since this class is about proofs, every statement should be proved. The only exceptions are statements that were proven in the text-book or in class.

1. (10 points) Check all the correct statements (in this question only the answers will be graded).

☐ The sets $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{R}$ are equipotent.

☐ The sets $\{m \in \mathbb{N} : \exists n \in \mathbb{N}\, m \leq n\}$ and $\{m \in \mathbb{N} : \forall n \in \mathbb{N}\, m \leq n\}$ are equal.

☐ For any set $X$ we have $\left|2^X\right| = |X|$.

☐ $\binom{4}{2} = 6$.

☐ $\gcd(15, 42) = 3$.

2. (8 points) (a) Show that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

(b) Show that

$$\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}.$$

3. (8 points) Show that the the open interval $(0, 1)$ has the same cardinality as:
   (a) The open interval $(-1, 1)$.

   (b) The real line $\mathbb{R}$.

4. (8 points) Let $X$ and $Y$ be subsets of $[n] = \{1, 2, \ldots, n\}$, for some positive integer $n$. Assume $|X| + |Y| > n$.

   (a) Prove that $X \cap Y \neq \emptyset$ using the pigeonhole principle

   (b) Prove that $X \cap Y \neq \emptyset$ using the inclusion-exclusion principal.

5. (8 points) (a) Find all the integers solutions to

$$56m + 72n = 38.$$

(b) Solve

$$56x \equiv 40 \pmod{72}.$$

6. (8 points) Let $p \in \mathbb{N}$ be prime. Show that for any $x, y \in \mathbb{Z}$

$$(x+y)^p \equiv x^p + y^p \pmod{p}.$$

Hint: Use the binomial theorem.