Topological dynamics beyond Polish groups

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The universal minimal flow of a topological group

Given a topological group $G$, a $G$-flow is a compact space $X$ together with a continuous action of $G$ on $X$. A $G$-flow is minimal if each of its orbits is dense, or equivalently, if it contains no proper subflows. If $X, Y$ are two $G$-flows a continuous map $f : X \to Y$ is a $G$-map if it commutes with the action. If $Y$ is minimal then $f$ has to be onto.

Fact (Ellis ’60)

For each topological group $G$ there exists a universal minimal flow $M(G)$ which is unique up to isomorphism.
First facts on the UMF

If $G$ is compact, then $M(G)$ is $G$ itself with the natural action by translation. Indeed, $G$ is minimal and if $X$ is a minimal $G$-flow, the map $\rho_x(g) = g \cdot x$, with $x \in X$, is a $G$-map.

If $G$ is infinite discrete then it acts freely on $M(G)$, which is a non-metrizable subset of the space $\beta G$ of ultrafilters on $G$.

By a Theorem of Veech, also for $G$ locally compact the action on $M(G)$ is free. If $G$ is not compact then $M(G)$ is again a non-metrizable space.
Extreme amenability

There are large groups $G$ such that $|M(G)| = 1$. For example the unitary group $U(\ell^2)$ with the strong operator topology (Gromov-Milman). Such groups are called *extremely amenable*.

**Recall:** A topological group is *amenable* if every $G$-flow admits an invariant probability measure.

![Diagram](https://via.placeholder.com/150)

- extremely amenable
- $M(G)$ metrizable
- locally cpt non cpt

**GOOD DYNAMICS**

**BAD DYNAMICS**
Polish groups and non-archimedean groups

A topological group is *Polish* if it is separable and completely metrizable.

**Examples:** second countable locally compact groups, \( \text{Homeo}(X) \) of a compact metrizable space \( X \), the group \( \text{Sym}(\mathbb{N}) \) of all permutations of a countable set.

**Definition:** A topological group is *non-archimedean* if the identity has a basis consisting of open subgroups.

**Fact:** The Polish non-archimedean groups are exactly the closed subgroups of \( \text{Sym}(\mathbb{N}) \). They are also exactly the automorphism groups of countable \( (\omega\text{-homogeneous}) \) structures.

**Examples:** \( \text{Aut}(\mathbb{Q},<), \text{Aut}(\mathbb{R}) \) of the Rado graph.

**Fact:** \( \text{Aut}(\mathbb{Q},<) \) is extremely amenable (Pestov ’98). This is equivalent to the classic Ramsey theorem.
The case of $\text{Aut}(K)$

For a relational structure $K$, let $\text{Age}(K)$ be the class of finite substructures of $K$. Then $K$ is $\omega$-homogeneous if any isomorphism of finite substructures of $K$ extends to an automorphism of $K$. A countable $\omega$-homogeneous structure is a Fraïssé structure.

**Fact (Kechris-Pestov-Todorcevic)**

Let $K$ be a Fraïssé structure. Then $\text{Aut}(K)$ is extremely amenable if and only if $\text{Age}(K)$ has the Ramsey property.

**Fact (Zucker)**

Let $K$ be a Fraïssé structure. Then $M(\text{Aut}(K))$ is metrizable if and only if each $A \in \text{Age}(K)$ has finite Ramsey degrees if and only if $\text{Age}(K)$ admits an appropriate expansion class with the Ramsey property. In such a case, $M(\text{Aut}(K))$ has a concrete representation as a space of expansions of $K$. 
Metrizability of the UMF of Polish groups

Fact (Ben Yaacov-Melleray-Tsankov; Bartošová-Zucker; Jahel-Zucker)

Let $G$ be a Polish group. TFAE:

1. $M(G)$ is metrizable.
2. The UEB metric on $M(G)$ is compatible.
3. $\beta\mathbb{N}$ does not embed in $M(G)$.
4. There is a closed extremely amenable subgroup $H \leq G$ such that the completion of $G/H$ is a minimal $G$-flow (equiv. is the UMF).
5. For any $G$-flow $X$, the set $AP(X)$ is closed, thus a subflow.

Definition: If $X$ is a $G$-flow, the set $AP(X) \subseteq X$ of almost periodic points is the union of the minimal subflows of $X$.

Does there exist a meaningful extension of this dividing line beyond Polish?
The first step outside Polish

Fact (Zucker)
Let $K$ be a Fraïssé structure. Then $M(\text{Aut}(K))$ is metrizable if and only if $\text{Age}(K)$ admits an appropriate expansion class with the Ramsey property. In such a case, $M(\text{Aut}(K))$ has a concrete representation as a space of expansions of $K$.

Fact (Bartošová)
Let $K$ be a $\omega$-homogeneous structure. If $\text{Age}(K)$ admits an appropriate expansion class with the Ramsey property then $M(\text{Aut}(K))$ has a concrete representation as a space of expansions of $K$.

Example: $M(\text{Sym}(\kappa)) = \text{LO}(\kappa)$ is the space of linear orders on $\kappa$.

Theorem (B.-Zucker) Under the above conditions, $\text{AP}(X)$ is closed for any $\text{Aut}(K)$-flow $X$. 
CAP groups

Definition (B.-Zucker)

A topological group $G$ is CAP if $AP(X)$ is closed for every $G$-flow $X$.

Recall:

- $AP(X)$ is the union of the minimal subflows of $X$.
- A subflow $Y \subseteq X$ is minimal if $Gy = Y$ for all $y \in Y$.

Definition: Let $x \sim_{AP(X)} y \iff \overline{Gx} = \overline{Gy}$ be the equivalence relation on $AP(X)$ whose equivalence classes are the minimal flows of which $AP(X)$ is composed.

Next goal: define a canonical uniformity on $M(G)$. 
Uniform spaces

A *uniform structure* $\mathcal{U}$ on a set $X$ is a filter of supersets of the diagonal $\Delta \subseteq X \times X$, called *entourages*, such that:

- for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ with $V^2 = \{(x, z) \mid \exists y (x, y), (y, z) \in V\} \subseteq U$,
- if $U \in \mathcal{U}$, then $U^{-1} = \{(y, x) \mid (x, y) \in U\} \in \mathcal{U}$,
- $\bigcap_{U \in \mathcal{U}} U = \Delta$.

Topological groups admit a canonical compatible uniform structure, the *right uniformity*, which is generated by

$$\left\{ (g, h) \in G \times G \mid gh^{-1} \in U \right\},$$

for $U$ an open neighborhood of the identity.

Compact spaces admit a *unique* compatible uniform structure: all neighborhoods of the diagonal.
The Samuel compactification

A function \( f : X \to Y \) is *uniformly continuous* if for each entourage \( V \) of \( Y \) there is an entourage \( U \) of \( X \) such that \((f(x), f(y)) \in V\) for all \((x, y) \in U\).

The *Samuel compactification* \( S(G) \) is a \( G \)-flow which densely embeds \( G \) and has the following universal property: if \( X \) is a uniform space, each uniformly continuous \( f : G \to X \) uniquely extends to a continuous \( \hat{f} : S(G) \to X \).

\[
\begin{array}{ccc}
S(G) & \xrightarrow{\hat{f}} & X \\
\uparrow & & \downarrow \\
G & \xrightarrow{f} & X
\end{array}
\]

Suppose \( X \) is a minimal \( G \) flow and \( f = \rho_x : g \mapsto g \cdot x \) for some \( x \in X \). Then \( \hat{\rho_x|_M} \) is a \( G \)-map for any minimal subflow \( M \subseteq S(G) \).

**Fact:** Each minimal subflow of \( S(G) \) is isomorphic to \( M(G) \).
The UEB uniformity

A set \( H \) of functions \( G \to [0, 1] \) is uniformly equicontinuous if for every \( \varepsilon > 0 \) there is \( U \ni 1_G \) so that for any \( g, h \in G \) with \( gh^{-1} \in U \), we have \( |f(g) - f(h)| < \varepsilon \) for each \( f \in H \).

**Definition:** The *UEB uniformity* on \( S(G) \) is given by the basic entourages, for \( H \subseteq C(G, [0, 1]) \) uniformly equicontinuous and \( \varepsilon > 0 \):

\[
[H, \varepsilon] = \left\{ (p, q) \in S(G) \times S(G) : |\hat{f}(p) - \hat{f}(q)| < \varepsilon \text{ for all } f \in H \right\}.
\]

The restriction of this uniformity to \( M(G) \subseteq S(G) \) does not depend on the choice of minimal subflow.

When \( G \) is Polish, the UEB uniformity is actually a metric which is lower semi-continuous on \( M(G) \). We can define it directly as:

\[
d(p, q) = \sup \left\{ |\hat{f}(p) - \hat{f}(q)| : f \in Lip(G) \right\}
\]
In general the UEB uniformity on $M(G)$ is not compatible with the compact topology.

**Theorem (B.-Zucker)**

The space $(M(G), \tau)$ together with the UEB uniformity form a topo-uniform space, that is:

- each $(\tau \times \tau)$-open neighborhood of the diagonal is an entourage,
- the uniformity has a basis of $(\tau \times \tau)$-closed entourages.
Characterization theorem

Theorem (B.-Zucker)

Let $G$ be a topological group. TFAE:

1. $G$ is CAP.
2. $G$ is CAP and $x \sim_{AP(X)} y \iff \overline{Gx} = \overline{Gy}$ is closed for each $G$-flow $X$.
3. The UEB uniformity on $M(G)$ is compatible with the compact topology.
4. $M(G \times G) \cong M(G) \times M(G)$.

Question: Are the above equivalent to “$AP(S(G))$ is closed”?

It would follow from a positive answer to the ambitability/unique amenability question (Pachl):

If $G$ admits a unique $G$-invariant probability measure on any flow with a dense orbit, is $G$ precompact?
Which groups are CAP?

Theorem (B.-Zucker)
1. Every precompact group is CAP.
2. Every group with metrizable UMF is CAP.
3. The class of CAP groups is closed under quotients, group extensions, inverse limits and products.
4. If $K$ is a $\omega$-homogeneous structure, then $\text{Aut}(K)$ is CAP if and only if $\text{Age}(K)$ has finite Ramsey degrees.
5. Locally compact not compact groups are not CAP.

Theorem (B.-Zucker)
If $G_i$ is CAP for all $i \in \mathcal{I}$, then

$$M\left(\prod_{i \in \mathcal{I}} G_i\right) = \prod_{i \in \mathcal{I}} M(G_i).$$
Scattered spaces

A topological space is *scattered* if it does not contain any nonempty perfect subspace.

Fact (Gheysens '20+)

If $X$ is scattered the topology of pointwise convergence agrees with the topology of discrete pointwise convergence on $\text{Homeo}(X)$.

Therefore $\text{Homeo}(X)$ embeds in $\text{Sym}(|X|)$.

Any ordinal with the order topology is scattered, in particular $\omega_1$. 
Homeo(ω₁) and its UMF

Fact (Gheysens '20+)
Homeo(ω₁) is amenable, Roelcke-precompact, not Baire, and admits no nontrivial homomorphism to any metrizable group.

Fact (Gheysens '20+)
The closure of Homeo(ω₁) in Sym(ω₁) is isomorphic to Sym(ω₁)ω₁.

Theorem (B.-Zucker)
Homeo(ω₁) is CAP and M(Homeo(ω₁)) = LO(ω₁)ω₁.
**Missing converses**

**Theorem (B.-Zucker)**

If $G$ is not CAP then $\beta\mathbb{N}$ embeds in $M(G)$. If $G$ is CAP, then there is a $\supseteq$-monotone and cofinal map from $\mathcal{N}_G$ to $\text{Nbhd}(\Delta_{M(G)})$.

**Question:** Is there a condition on the “size” of $M(G)$ which is equivalent to being CAP?

**Theorem (B.-Zucker)**

If $G$ admits a closed extremely amenable subgroup $H$ such that the completion of $G/H$ is a minimal $G$-flow, then $G$ is CAP and $M(G)$ is the completion of $G/H$.

**Question:** Does the converse hold for complete groups $G$?

**For instance:** if $K$ is an uncountable, $\omega$-homogeneous graph which embeds every finite graph, does there exist a linear order on $K$ so that $(K, <)$ is also $\omega$-homogeneous? Here: $G = \text{Aut}(K)$, $H = \text{Aut}(K, <)$.

**Thank you!**