Math 18 | **Final Exam 1 v.1** | **Name:** _____
December 11, 2018 | **(Total points: 50)** | **PID:** _____

Instructions
1. Write your Name and PID in the spaces provided above.
2. Make sure your Name is on every page.
3. No calculators, tablets, phones, or other electronic devices are allowed during this exam.
4. You may use one handwritten page of notes, but no books or other assistance during this exam.
5. Read each question carefully and answer each question completely.
6. Write your solutions clearly in the spaces provided; only work on this exam booklet will be graded.
7. Show all of your work. No credit will be given for unsupported answers, even if correct.

1. (1 point) Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

2. Let
$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 3 \\ 2 & 5 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 2 \end{pmatrix}.$$

   (a) (3 points) Find A^{-1}, showing your work.

   (b) (3 points) Find a 3×2 matrix X that satisfies $AX = B$, showing your work.

3. The matrix

$$A = \begin{pmatrix} 2 & 6 & 0 & 0 & 2 \\ 1 & 5 & 4 & 6 & 3 \\ 0 & 1 & 2 & 3 & 1 \\ 1 & 4 & 2 & 5 & 2 \end{pmatrix}.$$

is row equivalent to

$$\begin{pmatrix} 1 & 0 & -6 & 0 & -2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) (3 points) Find a basis for $\mathrm{Col}(A)$.

(b) (3 points) Find a basis for $\mathrm{Row}(A)^{\perp}$.

(c) (2 points) What is the rank of A^T? Justify your answer.

4. In each of the following examples, a vector space V is given, along with a subset $S \subset V$. Check whether the set S is a subspace of V or not. If S is a subspace of V, find $\dim(S)$. Justify your answers.

   (a) (3 points) $V = \mathbb{P}_3$ is the space of polynomials of degree ≤ 3, and

   $$S = \{p \in \mathbb{P}_3 \,:\, p(0) = p(1) - 1\}.$$

   (b) (3 points) $V = M_{2\times 2}$ is the space of 2×2 matrices, and

   $$S = \{A \in M_{2\times 2} \,:\, \det(A) = 2\}.$$

(c) (3 points) $V = \mathbb{R}^3$, and

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : \ x = y + 2 \right\}.$$

5. The sets

$$B = \left\{ \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}, \quad C = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

are bases to $\mathbb{R}^3$.

(a) (3 points) Assume $[v]_B = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ and find $[v]_C$.

(b) (3 points) Define a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(v) = [v]_B$ for any $v \in \mathbb{R}^3$. Is T one-to-one? Justify your answer.

6. Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

(a) (3 points) Find the eigenvalues of A.

(b) (2 points) Is A diagonalizable? Justify your answer.

(c) (2 points) What are the eigenvalues of A^2? Justify your answer.

7. (a) (3 points) For $x = \left(x_1, x_2\right), y = \left(y_1, y_2\right)$ let

$$< x, y >= x_1 y_2 + x_2 y_1.$$

Is $< x, y >$ an inner product? Justify your answer.

(b) (2 points) Assume $\{v_1, v_2\}$ is an orthonormal set (for the dot product). Is the set $\{v_1 + v_2, v_1 - v_2\}$ orthogonal? Justify your answer.

8. (a) (3 points) Find an orthonormal basis for

$$W = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\},$$

showing your work.

(b) (3 points) Find the orthogonal projection of the vector $\underline{y} = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 2 \end{pmatrix}$ onto W, showing your work.

(c) (2 points) Assume $A = (\underline{a}_1\ \underline{a}_2\ \underline{a}_3)$ where $\{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$ is an orthonormal set in $\mathbb{R}^4$ (A is a 4×3 matrix). Find a QR factorization for A (Q and R should be constructed using $\{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$ and numbers).

9. (5 points) **bonus question** Compute $\det(A^5)$ for

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

Justify your answer.