1. (1 point) Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

2. Let

\[ A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 3 \\ 2 & 5 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 2 \end{pmatrix}. \]

(a) (3 points) Find \( A^{-1} \), showing your work.

(b) (3 points) Find a \( 3 \times 2 \) matrix \( X \) that satisfies \( AX = B \), showing your work.
3. The matrix

\[ A = \begin{pmatrix}
2 & 6 & 0 & 0 & 2 \\
1 & 5 & 4 & 6 & 3 \\
0 & 1 & 2 & 3 & 1 \\
1 & 4 & 2 & 5 & 2
\end{pmatrix}. \]

is row equivalent to

\[ \begin{pmatrix}
1 & 0 & -6 & 0 & -2 \\
0 & 1 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}. \]

(a) (3 points) Find a basis for Col(\(A\)).

(b) (3 points) Find a basis for \(\text{Row}(A)^\perp\).

(c) (2 points) What is the rank of \(A^T\)? Justify your answer.
4. In each of the following examples, a vector space $V$ is given, along with a subset $S \subset V$. Check whether the set $S$ is a subspace of $V$ or not. If $S$ is a subspace of $V$, find $\dim(S)$. Justify your answers.

(a) (3 points) $V = \mathbb{P}_3$ is the space of polynomials of degree $\leq 3$, and 
$$S = \{ p \in \mathbb{P}_3 : p(0) = p(1) - 1 \}.$$ 

(b) (3 points) $V = M_{2 \times 2}$ is the space of $2 \times 2$ matrices, and 
$$S = \{ A \in M_{2 \times 2} : \det(A) = 2 \}.$$
(c) (3 points) \( V = \mathbb{R}^3 \), and

\[
S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = y + 2 \right\}.
\]
5. The sets 
\[ B = \left\{ \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}, \quad C = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \]

are bases to \( \mathbb{R}^3 \).

(a) (3 points) Assume \([v]_B = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}\) and find \([v]_C\).

(b) (3 points) Define a linear map \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) by \( T(v) = [v]_B \) for any \( v \in \mathbb{R}^3 \). Is \( T \) one-to-one? Justify your answer.
6. Let
\[
A = \begin{pmatrix}
2 & -1 & 1 \\
-1 & 2 & -1 \\
1 & -1 & 2
\end{pmatrix}.
\]

(a) (3 points) Find the eigenvalues of $A$.

(b) (2 points) Is $A$ diagonalizable? Justify your answer.

(c) (2 points) What are the eigenvalues of $A^2$? Justify your answer.
7. (a) (3 points) For $x = (x_1, x_2), y = (y_1, y_2)$ let

$$< x, y >= x_1 y_2 + x_2 y_1.$$ 

Is $< x, y >$ an inner product? Justify your answer.

(b) (2 points) Assume $\{v_1, v_2\}$ is an orthonormal set (for the dot product). Is the set $\{v_1 + v_2, v_1 - v_2\}$ orthogonal? Justify your answer.
8. (a) (3 points) Find an orthonormal basis for
\[
W = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\},
\]
showing your work.

(b) (3 points) Find the orthogonal projection of the vector \( \mathbf{y} = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 2 \end{pmatrix} \) onto \( W \), showing your work.
(c) (2 points) Assume $A = (a_1, a_2, a_3)$ where $\{a_1, a_2, a_3\}$ is an orthonormal set in $\mathbb{R}^4$ ($A$ is a $4 \times 3$ matrix). Find a $QR$ factorization for $A$ ($Q$ and $R$ should be constructed using $\{a_1, a_2, a_3\}$ and numbers).

9. (5 points) **bonus question** Compute $\det(A^5)$ for

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
1 & 1 & 2 \\
1 & 2 & 1
\end{pmatrix}.
\]

Justify your answer.