Difference Polynomials

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Abstract. For nonzero real h, let Δ_h be the difference operator defined by $\Delta_h f(x) = (f(x+h) - f(x))/h$. The polynomial $\Delta_{h_m} \cdots \Delta_{h_1} x^n$ has degree $n-m$ and is known to have collinear zeros. Using combinatorial analysis, we obtain formulas for the coefficients and the power sums of zeros. A corresponding reduced central difference polynomial is similarly examined. The results are useful in studying the distribution of the zeros.

1991 *Mathematics Subject Classification*. Primary 11B83; Secondary 39A70, 30C15, 30C10.
This paper is in final form and no version of it will be submitted for publication elsewhere.

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0271-4132/93 $1.00 + $.25 per page

1. Introduction

With the shift operator E_h on polynomials $f(x)$, where $h \neq 0$, one can form a (divided) difference operator Δ_h and a central difference operator δ_h defined by

$$\Delta_h f(x) = h^{-1}(E_h - 1) f(x) = h^{-1}(f(x+h) - f(x)) \tag{1.1}$$

and

$$\delta_h f(x) = h^{-1}\left(E_{h/2} - E_{-h/2}\right) f(x) = h^{-1}(f(x+h/2) - f(x-h/2)). \tag{1.2}$$

Let m, n, and d be integers with

$$0 < m < n, \;\; d = n - m, \tag{1.3}$$

and let $\lambda_1, \ldots, \lambda_m$ be nonzero real numbers.

Then

$$D(x) = D_{n,m}(x) = \frac{d!}{n!}\, \Delta_{\lambda_m} \cdots \Delta_{\lambda_1}\, x^n \tag{1.4}$$

is a monic polynomial of degree d over $Q[\lambda_1, \ldots, \lambda_m]$. It will be convenient to relate $D(x)$ to a certain central difference polynomial $C(z)$. Since $\delta_h f(x) = \Delta_h f(x - h/2)$, it is easy to verify that

$$B(x) := \frac{d!}{n!}\, \delta_{i\lambda_m} \cdots \delta_{i\lambda_1} x^n = i^d\, D\left(-\frac{\sigma}{2} - ix\right), \tag{1.5}$$

where

$$\sigma = \lambda_1 + \cdots + \lambda_m \,.$$

Define the reduced central difference polynomial $C(z)$ for $d > 1$ by

$$C(z) = C_{n,m}(z) = \begin{cases} B(\sqrt{z}), & \text{if } d \text{ is even} \\ B(\sqrt{z})/\sqrt{z}, & \text{if } d \text{ is odd.} \end{cases} \tag{1.6}$$

Note that $C(z)$ is a monic polynomial over $\mathbb{Q}[\lambda_1, \ldots, \lambda_m]$ of degree

$$c := \deg(C) = [d/2], \tag{1.7}$$

where the square brackets in (1.7) denote the greatest integer function.

In Theorem 2.3, it is shown that there are c positive numbers $y_1 < y_2 < \cdots < y_c$ such that the d zeros of $D(x)$ are $-\sigma/2 \pm iy_1, \ldots, -\sigma/2 \pm iy_c$, together with $-\sigma/2$ if d is odd. By (1.5)-(1.7), $C(z)$ thus has c distinct positive zeros, $z_1 = y_1^2, \ldots, z_c = y_c^2$. Theorem 2.3 will be applied in §4.

In Theorem 3.1, it is shown that

$$D(x) = \sum_{k=0}^{d} P_k(m) \binom{d}{k} x^{d-k}, \tag{1.8}$$

where $P_k(m)$ is a polynomial in $\sigma_1, \ldots, \sigma_k$ over $\mathbb{Q}$ independent of d, with

$$\sigma_\nu = \lambda_1^\nu + \cdots + \lambda_m^\nu, \quad \sigma = \sigma_1 \qquad (\nu \geq 1). \tag{1.9}$$

In the case $\lambda_1 = \cdots = \lambda_m = 1$ (whereupon $\sigma_\nu = m$ for each ν), $P_k(m)$ reduces to the polynomial $S(m+k,m)/\binom{m+k}{m}$ in m over Q of degree k, where $S(m+k,m)$ is the Stirling number of the second kind.

Harper [4] and Lieb [6] have shown that the polynomials

$$\sum_{k=1}^{N} S(N,k)x^k, \quad \sum_{k=1}^{N} \frac{S(N,k)}{(N-k)!} x^k$$

both have all of their zeros real. Also, it is not difficult to see that $\sum_{k=1}^{N} S(N,k)k!x^k$ has N distinct zeros in the interval $(-1,0]$. In view of (1.8), $D(x)$ provides yet another class of polynomials with collinear zeros whose coefficients naturally involve Stirling numbers.

Theorem 3.2 is the counterpart of Theorem 3.1 for $C(z)$ in place of $D(x)$. It is shown that

$$C(z) = \sum_{k=0}^{c} (-1)^k Q_k(m) \binom{d}{2k} z^{c-k}, \tag{1.10}$$

where $Q_k(m)$ is a polynomial in $\sigma_1, \ldots, \sigma_k$ over $\mathbb{Q}$ independent of d. If $\lambda_1 = \cdots = \lambda_m = 1$, then $Q_k(m)$ reduces to the polynomial $D_{m+2k,m}(-m/2)$ in m over $\mathbb{Q}$ of degree k. The coefficients of this polynomial are calculated in Theorem 3.3 in terms of certain polynomials $E_r(x)$ defined in (3.13).

Let S_k denote the sum of the k-th powers of the d zeros of $D(x)$ and let T_k denote the sum of the k-th powers of the c zeros of $C(z)$. Theorems 4.2 and 4.1 prove the unexpected and useful result that, if $\lambda_1 = \cdots = \lambda_m = 1$, then S_k and T_k are polynomials in $\mathbb{Q}[d,m]$ of total degrees $k+1$ and $2k+1$, respectively.

Tables of $P_k(m)$, $Q_k(m)$, $E_r(x)$, S_k, and T_k are given in §5.

The formulas in this paper can be applied in estimating the zeros of $D(x)$ and $C(z)$. We give here one very simple example. Let z_c denote the largest zero of $C(z)$ (the "spectral radius" of $C(z)$). Since $C(z)$ has c positive zeros, $T_1/c \leq z_c \leq T_1$. By (1.10) and Table 5.2,

$$C(z) = z^c - T_1 z^{c-1} + \ldots = z^c - \frac{\sigma_2(d^2-d)}{24} z^{c-1} + \cdots.$$

Thus, for all $d > 1$,

$$\sigma_2(d-1)/12 \leq z_c \leq \sigma_2(d^2-d)/24. \tag{1.11}$$

Estimate (1.11) is best possible for $d = 2$. For large d, (1.11) is crude, but the estimate can be improved by employing T_k for $k > 1$.

In [3], the distribution of the zeros of $C(z)$ is studied for $\lambda_1 = \cdots = \lambda_m = 1$. For example, it is shown that if $d > 1$, then for any $\epsilon > 0$,

$$nd(m/n)^\epsilon << z_c << nd(md/n)^\epsilon. \tag{1.12}$$

The results of [3] depend heavily on Theorems 3.2 and 4.1 of this paper.

2. Collinearity and simplicity of the zeros of D(x) and C(z)

In this section only, let λ, τ, w, u_k and G be defined as follows: $\lambda = \lambda_m$,

$\tau = \lambda_1 + \cdots + \lambda_{m-1}$, w is any zero of $D(x)$, and $u_1, \dots, u_{d+1}$ are the zeros of $G(x) = D_{n,m-1}(x) = \Pi(x - u_k)$. If $m = 1$, interpret $G(x) = x^n$, $\tau = 0$.

Lemma 2.1. *The d zeros of $D(x)$ lie on the vertical line through $-\sigma/2$.*

Proof. Fix n and use induction on m. Assume that $Re(u_k) = -\tau/2$ for all k; this is valid for $m = 1$. By (1.4),

$$D(x) = (d+1)^{-1}\ \Delta_\lambda\, G(x), \tag{2.1}$$

so

$$\Pi(w + \lambda - u_k) = G(w + \lambda) = G(w) = \Pi(w - u_k). \tag{2.2}$$

Write $w - u_k = r + is_k$, where r and the s_k are real. Then $\Pi(r + is_k) = \Pi(r + \lambda + is_k)$. Consequently, $\Pi((r + \lambda)^2 + s_k^2) = \Pi(r^2 + s_k^2)$, which forces $r = -\lambda/2$. Thus $Re(w) = -\lambda/2 - \tau/2 = -\sigma/2$. □

An alternative proof of Lemma 1 is given in [2], and related results are discussed in [8] and [9].

Lemma 2.2. *The zeros of $D(x)$ are simple.*

Proof. By Lemma 2.1, $Re(u_k) = -\tau/2$ for all k, and $Re(w) = -\sigma/2$, so $w - u_k = -\lambda/2 + is_k$ for real s_k. In particular, $w \neq u_k$ and $w + \lambda \neq u_k$ for all k, and by (2.2),

$$G(w + \lambda) = G(w) \neq 0. \tag{2.3}$$

Since $G'(x)/G(x) = \sum(x - u_k)^{-1}$, it follows from (2.3) that

$$\begin{aligned}\frac{G'(w+\lambda) - G'(w)}{G(w)} &= \frac{G'(w+\lambda)}{G(w+\lambda)} - \frac{G'(w)}{G(w)}\\ &= \sum \frac{1}{w + \lambda - u_k} - \sum \frac{1}{w - u_k} = \sum \frac{-\lambda}{(w - u_k)(w + \lambda - u_k)}\\ &= \sum \frac{\lambda}{s_k^2 + \lambda^2/4} \neq 0.\end{aligned}$$

Thus $G'(w+\lambda) - G'(w) \neq 0$, so by (2.1), $D'(w) \neq 0$. Consequently w is a simple zero of $D(x)$. □

Theorem 2.3. *There are $c = [d/2]$ positive numbers $y_1 < \cdots < y_c$ such that the d zeros of $D(x)$ are $-\sigma/2 \pm iy_1, \ldots, -\sigma/2 \pm iy_c$, together with $-\sigma/2$ if d is odd. The zeros of $C(z)$ are the distinct positive numbers $z_1 = y_1^2, \ldots, z_c = y_c^2$.*

Proof. Since $D(x)$ has real coefficients, its nonreal zeros occur in complex conjugate pairs. The result for $D(x)$ thus follows from Lemmas 2.1 and 2.2. The result for $C(z)$ now follows from (1.6). □

Corollary 2.4. *If $m = 1$, the zeros of $C(z)$ are given by*

$$z_k = \frac{\lambda_1^2}{4}\cot^2(\pi k/n), \qquad k = 1, 2, \ldots, \left[\frac{(n-1)}{2}\right]. \tag{2.4}$$

Proof. The zeros of $D(x)$ are the solutions of $(x + \lambda_1)^n = x^n$, namely

$$x = \lambda_1/(\exp(2\pi ik/n) - 1), \quad 0 < k < n.$$

The result now follows from Theorem 2.3. □

Corollary 2.4 shows that if $m = 1$ and $d = n - 1$, then the spectral radius z_c of $C(z)$ satisfies the asymptotic formula

$$z_c \sim \frac{\lambda_1^2}{4\pi^2}\, d^2, \quad (d \to \infty,\ m = 1). \tag{2.5}$$

This is consistent with the general estimate given in (1.12).

3. Coefficients of D(x) and C(z).

In the following theorem, the coefficients of $D(x)$ are determined in terms of certain polynomials $P_k(m)$. Table 5.1 gives $P_k(m)$ for $k \le 7$.

Theorem 3.1. *We have*

$$D(x) = \sum_{k=0}^{d} P_k(m) \binom{d}{k} x^{d-k}, \tag{3.1}$$

where for each fixed k, $P_k(m) = P_k(m; \lambda_1, \ldots, \lambda_m)$ is a polynomial in $\sigma_1, \ldots, \sigma_k$ over $\mathbb{Q}$ independent of d, and the $P_k(m)$ satisfy the recursion

$$P_0(m) = 1,\ P_k(m) = \frac{1}{k} \sum_{\nu=1}^{k} (-1)^\nu B_\nu \sigma_\nu \binom{k}{\nu} P_{k-\nu}(m) \qquad (k \ge 1), \tag{3.2}$$

where B_ν is the ν-th Bernoulli number. In the case $\lambda_1 = \cdots = \lambda_m = 1$, $P_k(m)$ is the polynomial in $\mathbb{Q}[m]$ with leading term $2^{-k}m^k$ (and trailing term $(-1)^k B_k k^{-1} m$ for $k \ge 1$) given by

$$P_k(m) = S(m+k, m) \Big/ \binom{m+k}{m}, \tag{3.3}$$

where $S(m+k,m)$ is the Stirling number of the second kind.

Proof. By (1.1) and (1.4), $D(x)/d!$ is the coefficient of y^{m+d} in the formal Taylor expansion about $y=0$ of

$$\Delta_{\lambda_m}\cdots\Delta_{\lambda_1} e^{xy} = e^{xy}\prod_{r=1}^{m}\left(e^{\lambda_r y}-1\right)/\lambda_r\,. \tag{3.4}$$

The right side of (3.4) equals

$$e^{xy}\sum_{k=0}^{\infty} P_k(m) y^{k+m}/k!\,, \tag{3.5}$$

where

$$P_k(m) = \binom{m+k}{m}^{-1} S(m+k,m)\,,$$

with

$$\begin{aligned} S(m+k,m) &= S(m+k,m;\,\lambda_1,\ldots,\lambda_m) \\ &= (\lambda_1\cdots\lambda_m m!)^{-1} \sum_{\substack{c_1,\ldots,c_m\geq 1\\ c_1+\cdots+c_m=m+k}} \begin{bmatrix} m+k \\ c_1,\ldots,c_m\end{bmatrix} \lambda_1^{c_1}\cdots\lambda_m^{c_m}\,. \end{aligned}$$

Combinatorially, if $\lambda_1,\ldots,\lambda_m$ are positive integers, $\lambda_1\cdots\lambda_m m!\, S(m+k,m)$ represents the number of ways of placing $m+k$ persons in m houses with $\lambda_1,\ldots,\lambda_m$ rooms respectively, leaving no house empty. If $\lambda_1=\cdots=\lambda_m=1$, $S(m+k,m)$ is the Stirling number of the second kind [**5**, p. 177].

Comparing coefficients of y^{m+d} in (3.4) and (3.5), we obtain (3.1). It remains to prove (3.2).

Since

$$\prod_{r=1}^{m}\left(e^{\lambda_r y}-1\right)/\lambda_r = \sum_{k=0}^{\infty}\frac{P_k(m)}{k!}y^{m+k}\,, \tag{3.6}$$

$P_0(m)=1$. Differentiation in (3.6) yields

$$\sum_{k=0}^{\infty}\frac{(m+k)}{k!}P_k(m)y^{m+k-1} = \sum_{\alpha=1}^{m}\frac{\lambda_\alpha e^{\lambda_\alpha y}}{e^{\lambda_\alpha y}-1}\prod_{r=1}^{m}\left(e^{\lambda_r y}-1\right)/\lambda_r\,. \tag{3.7}$$

Replace the rightmost product in (3.7) by the right side of (3.6) and use the generating function for Bernoulli numbers [**1**, eq. 14a, p. 48] to obtain

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{(m+k)}{k!}P_k(m)y^{m+k-1} &= \sum_{\alpha=1}^{m}\sum_{\nu=0}^{\infty}\frac{B_\nu}{\nu!}(-\lambda_\alpha)^\nu y^{\nu-1}\sum_{\mu=0}^{\infty}\frac{P_\mu(m)}{\mu!}y^{m+\mu} \\ &= \sum_{k=0}^{\infty}\sum_{\nu=0}^{k}\frac{(-1)^\nu B_\nu\sigma_\nu}{\nu!(k-\nu)!}P_{k-\nu}(m)y^{m+k-1}\,, \end{aligned}$$

where the last equality results from putting $k = \mu + \nu$. Comparing coefficients of y^{m+k-1}, we get

$$(m+k)P_k(m) = \sum_{\nu=0}^{k} (-1)^\nu B_\nu \sigma_\nu \binom{k}{\nu} P_{k-\nu}(m),$$

and (3.2) easily follows. □

The counterpart of Theorem 3.1 for $C(z)$ is the following theorem, which gives the coefficients of $C(z)$ in terms of certain polynomials $Q_k(m)$. Table 5.2 gives $Q_k(m)$ for $k \le 5$.

Theorem 3.2. *We have*

$$C(z) = \sum_{k=0}^{c} (-1)^k Q_k(m) \binom{d}{2k} z^{c-k}, \tag{3.8}$$

where, for each fixed k, $Q_k(m) = Q_k(m; \lambda_1, \dots, \lambda_m)$ *is a polynomial in* $\sigma_1, \dots, \sigma_{2k}$ *over* $\mathbb{Q}$ *independent of* d, *and the* $Q_k(m)$ *satisfy the recursion*

$$Q_0(m) = 1,\ Q_k(m) = \frac{1}{2k} \sum_{\nu=1}^{k} B_{2\nu}\sigma_{2\nu} \binom{2k}{2\nu} Q_{k-\nu}(m) \quad (k \ge 1), \tag{3.9}$$

where $B_{2\nu}$ *is the* 2ν*-th Bernoulli number. Moreover,*

$$Q_k(m) = D_{m+2k,m}(-\sigma/2). \tag{3.10}$$

Proof. Applying the Taylor expansion about $x = 0$ in (1.5), we find that

$$i^d D(-\sigma/2 - ix) = \sum_{k \ge 0} \frac{(-1)^k}{(d-2k)!} D^{(d-2k)}(-\sigma/2) x^{d-2k}.$$

Differentiation in (1.4) shows that

$$D^{(d-2k)}(x) = \frac{d!}{(2k)!} D_{m+2k,m}(x),$$

so

$$i^d D(-\sigma/2 - ix) = \sum_{k \ge 0} (-1)^k \binom{d}{2k} D_{m+2k,m}(-\sigma/2) x^{d-2k}.$$

Thus (3.8) and (3.10) hold by (1.5)-(1.6). It remains to prove (3.9).

By (3.10) and (3.4), $Q_k(m)/(2k)!$ is the coefficient of y^{m+2k} in the Taylor expansion about $y = 0$ of

$$e^{-y\sigma/2} \prod_{r=1}^{m} \left(e^{\lambda_r y} - 1\right)/\lambda_r = \prod_{r=1}^{m} 2\lambda_r^{-1} \sinh(\lambda_r y/2).$$

Thus $Q_0(m) = 1$ and

$$\sum_{k=0}^{\infty} \frac{Q_k(m)}{(2k)!}\, y^{m+2k} = \prod_{r=1}^{m} 2\lambda_r^{-1} \sinh(\lambda_r y/2)\,. \tag{3.11}$$

Differentiation in (3.11) yields

$$\sum_{k=0}^{\infty} \frac{Q_k(m)}{(2k)!}(m+2k)y^{m+2k-1} = \sum_{\alpha=1}^{m} \frac{\lambda_\alpha}{2} \coth(\lambda_\alpha y/2) \prod_{r=1}^{m} 2\lambda_r^{-1} \sinh(\lambda_r y/2)\,.$$

Thus, by (3.11),

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{Q_k(m)}{(2k)!}(m+2k)y^{2k} &= \sum_{\alpha=1}^{m} \frac{\lambda_\alpha y}{2} \coth(\lambda_\alpha y/2) \prod_{r=1}^{m} \frac{2\sinh(\lambda_r y/2)}{\lambda_r y} \\ &= \sum_{\alpha=1}^{m} \sum_{\nu=0}^{\infty} \frac{B_{2\nu}}{(2\nu)!}(\lambda_\alpha y)^{2\nu} \sum_{\mu=0}^{\infty} \frac{Q_\mu(m)}{(2\mu)!}\, y^{2\mu} \\ &= \sum_{k=0}^{\infty} \sum_{\nu=0}^{k} \frac{B_{2\nu}\sigma_{2\nu}}{(2\nu)!} \frac{Q_{k-\nu}(m)}{(2k-2\nu)!}\, y^{2k}\,,\end{aligned}$$

where the last equality results from putting $k = \mu + \nu$. Comparing coefficients of y^{2k}, we get

$$(m+2k)Q_k(m) = \sum_{\nu=0}^{k} B_{2\nu}\sigma_{2\nu} \binom{2k}{2\nu} Q_{k-\nu}(m)\,,$$

and (3.9) easily follows. □

For the remainder of this section, let $\lambda_1 = \cdots = \lambda_m = 1$. One can recursively define a unique sequence of polynomials $E_r(x) \in \mathbb{Q}[x]$ with $\deg(E_r(x)) = r$ by

$$E_0(x) = 1,\ E_r(0) = 0 \qquad (r \geq 1)\,, \tag{3.12}$$

$$E_r(x) = \frac{1}{2} \sum_{\nu=1}^{r+1} \frac{24^\nu B_{2\nu}}{(2\nu)!}\, \frac{x-r-1}{x-\nu}\, E_{r+1-\nu}(x-\nu) \qquad (r \geq 1)\,. \tag{3.13}$$

Comparison of the coefficients of x^{r-1} in (3.13) shows that the leading coefficient e_r of $E_r(x)$ satisfies the recursion

$$2re_r = 12\, B_4\, e_{r-1}\,,$$

i.e.,

$$-5re_r = e_{r-1} \qquad (r \geq 1)\,.$$

Thus,

$$E_r(x) \quad \text{has leading term} \quad (-1/5)^r/r!\,. \tag{3.14}$$

Similarly, recursions can be obtained for the other coefficients of $E_r(x)$. The polynomials $E_r(x)$ are tabulated for $r \leq 5$ in Table 5.3.

In the next theorem, we express $Q_k(m)$ as a polynomial in m of degree k with coefficients explicitly given in terms of the $E_r(x)$.

Theorem 3.3. *Let $k \ge 1$, $\lambda_1 = \cdots = \lambda_m = 1$. Then*

$$Q_k(m) = \frac{(2k)!}{k 2 4^k} \sum_{r=1}^{k} E_{k-r}(k) \frac{m^r}{(r-1)!}. \tag{3.15}$$

Furthermore,

$$(-1)^r E_r(k) > 0 \qquad (0 \le r \le k-1), \tag{3.16}$$

so each polynomial $Q_k(m)$ (unlike $P_k(m)$) has alternating signs. Finally,

$$Q_k(m) = \frac{(2k)!}{k! 2 4^k} m^k + \cdots + \frac{B_{2k}}{2k} m. \tag{3.17}$$

Proof. In view of (3.9), (3.17) follows by induction on k. Also by (3.9), it follows that the coefficients of $m, m^2, \ldots, m^k$ in the polynomial expansion of $(-1)^k Q_k(-m)$ are all positive, since [**5**, p. 240] $(-1)^\nu B_{2\nu} < 0$ for $\nu \ge 1$. It remains to prove (3.15). By (3.9) and induction on k,

(3.18)

$$Q_k(m) = \frac{mB_{2k}}{2k} + \frac{m}{2k} \sum_{\nu=1}^{k-1} B_{2\nu} \binom{2k}{2\nu} \frac{(2k-2\nu)!}{(k-\nu) 2 4^{k-\nu}} \sum_{r=1}^{k-\nu} E_{k-\nu-r}(k-\nu) \frac{m^r}{(r-1)!}.$$

By (3.13), the coefficient of m^r in (3.18) is $(2k)! E_{k-r}(k)/((r-1)!\, k\, 2 4^k)$, which proves (3.15). □

4. Power sums for zeros of D(x) and C(z).

Let S_k denote the sum of the k-th powers of the d zeros x_r of $D(x)$, and let T_k denote the sum of the k-th powers of the c zeros z_r of $C(z)$, where $k \ge 1$. Set $S_0 = d$, $T_0 = c$. By Theorem 2.3,

$$\begin{aligned} S_k &= \sum_{r=1}^{d} x_r^k = 2\, Re \sum_{r=1}^{c} (iy_r - \sigma/2)^k + (d-2c)(-\sigma/2)^k \\ &= 2 \sum_{j=0}^{[k/2]} \binom{k}{2j} (-\sigma/2)^{k-2j} (-1)^j \sum_{r=1}^{c} y_r^{2j} + (d-2c)(-\sigma/2)^k. \end{aligned}$$

Thus

$$S_k = d(-\sigma/2)^k + 2 \sum_{j=1}^{[k/2]} \binom{k}{2j} (-\sigma/2)^{k-2j} (-1)^j T_j. \tag{4.1}$$

A similar argument gives the following expression for T_k in terms of S_j:

$$T_k = \frac{(-1)^k}{2} \sum_{j=0}^{2k} \binom{2k}{j} (\sigma/2)^{2k-j} S_j, \qquad k \ge 1.$$

In the notation of (3.1) and (3.8), Newton's identities [**7**, p. 170, line 4] yield, for
$1 \le k \le d$,

$$-k\,P_k(m)\binom{d}{k} = \sum_{\nu=1}^{k} S_\nu P_{k-\nu}(m)\binom{d}{k-\nu} \tag{4.2}$$

and for $1 \le k \le c$,

$$-k\,Q_k(m)\binom{d}{2k} = \sum_{\nu=1}^{k} T_\nu(-1)^\nu Q_{k-\nu}(m)\binom{d}{2k-2\nu}. \tag{4.3}$$

The sums S_k, T_k may be calculated recursively from (4.2) and (4.3); lists of S_k for $k \le 6$ and T_k for $k \le 5$ when $\lambda_1 = \cdots = \lambda_m = 1$ are given in Tables 5.4 and 5.5.

Suppose for the remainder of §4 that $\lambda_1 = \cdots = \lambda_m = 1$. By Theorems 3.1 and 3.3, $P_k(m)$ and $Q_k(m)$ are polynomials in $\mathbb{Q}[m]$ of degree k with no constant terms if $k \ge 1$. It follows from (4.2) and (4.3) that for each fixed $k \ge 1$, there are rational numbers $a_{i,j}(k)$ and $b_{i,j}(k)$ such that

$$S_k = \sum_{j=1}^{k} \sum_{i=1}^{k} a_{i,j}(k) d^i m^j \tag{4.4}$$

and

$$T_k = \sum_{j=1}^{k} \sum_{i=1}^{2k} b_{i,j}(k) d^i m^j. \tag{4.5}$$

Theorems 4.2 and 4.1 show that, surprisingly, S_k and T_k are polynomials in $\mathbb{Q}[d, m]$ of total degrees $k+1$ and $2k+1$, respectively. Theorem 4.1 also shows that the terms in (4.5) "alternate in sign", which is not at all apparent from (4.3).

Theorem 4.1. *Let $\lambda_1 = \cdots = \lambda_m = 1$, and fix $k \ge 1$. Then*

$$T_k = \sum_{j=1}^{k} \sum_{i=1}^{2k+1-j} b_{i,j}(k) d^i m^j, \tag{4.6}$$

where

(4.7) *the coefficient of m in (4.6) is $(-1)^{k+1} B_{2k}\binom{d}{2k}/2$,*

so

$$b_{1,1}(k) = (-1)^k B_{2k}/(4k), \quad b_{2k,1}(k) = \frac{(-1)^{k+1} B_{2k}}{2(2k)!}. \tag{4.8}$$

Thus the polynomial $T_k \in \mathbb{Q}[d,m]$ has total degree $2k+1$. Moreover, for all i,j with $1 \le j \le k$, $1 \le i \le 2k+1-j$,

$$(4.9) \qquad (-1)^{i+j}\, b_{i,j}(k) < 0\,.$$

In particular, all of the terms on the right side of (4.6) are nonzero.

Proof. By (4.3), the coefficient of m in (4.5) equals $(-1)^{k+1}k\binom{d}{2k}$ times the coefficient of m in $Q_k(m)$. Thus, by (3.17), the coefficient of m in (4.5) is $\binom{d}{2k}(-1)^{k+1}\, B_{2k}/2$. This proves (4.7) and (4.8).

By (4.8), a necessary condition for (4.6) to hold is

$$(4.10) \qquad \text{the coefficient of } d^{2k} \text{ in (4.5) is } (-1)^{k+1}\, B_{2k}m/(2(2k)!)\,.$$

Comparing coefficients of d^{2k} in (4.3) and using induction on k, we see that (4.10) is indeed true for all $k \ge 1$, by (3.9). We proceed to prove that T_k has total degree $2k+1$, i.e., that (4.6) holds.

Since $C(z) = C_{m+d,m}(z)$ has zeros $z_1, \dots z_c$,

$$(4.11) \qquad g_d(z) := z^c C(1/z) = \prod_{i=1}^{c} (1 - zz_i)\ .$$

Formally,

$$(4.12) \qquad \log g_d(z) = -\sum_{r=1}^{\infty} T_r z^r/r\,,$$

so

$$(4.13) \qquad g_d'(z)/g_d(z) = -\sum_{r=1}^{\infty} T_r z^{r-1}\,.$$

From (4.11) and (3.8), it can be seen that

$$(4.14) \qquad g_d(z) - 2zg_d'(z)/d = g_{d-1}(z)\,.$$

By (4.13) and (4.14),

$$(4.15) \qquad g_{d-1}(z)/g_d(z) = 1 + 2\sum_{r=1}^{\infty} T_r z^r/d\,.$$

Temporarily write $T_r = T_r(d,m)$ to emphasize the dependence on d and m, and define

$$(4.16) \qquad T_r^* = T_r^*(d,m) = T_r(d,m) - T_r(d-1,m)\,.$$

By (4.12) and (4.16),

$$g_{d-1}(z)/g_d(z) = \exp \sum_{r=1}^{\infty} T_r^* z^r/r. \tag{4.17}$$

Comparing (4.15) and (4.17), we have

$$1 + 2\sum_{r=1}^{\infty} T_r z^r/d = \exp \sum_{r=1}^{\infty} T_r^* z^r/r. \tag{4.18}$$

Thus,

$$T_k = \frac{d}{2} \sum \frac{(T_1^*/1)^{c_1}}{c_1!} \cdots \frac{(T_k^*/k)^{c_k}}{c_k!}, \tag{4.19}$$

where the sum is over all integers $c_1, \ldots, c_k \geq 0$ for which $c_1 + 2c_2 + 3c_3 + \cdots = k$. Consequently,

$$T_k - \frac{d}{2k} T_k^* = \frac{d}{2} \sum \frac{(T_1^*/1)^{c_1}}{c_1!} \cdots \frac{(T_{k-1}^*/(k-1))^{c_{k-1}}}{c_{k-1}!}, \tag{4.20}$$

where the sum is over all integers $c_1, \ldots, c_{k-1} \geq 0$ for which $c_1 + 2c_2 + 3c_3 + \cdots = k$.

By (4.3),

$$T_1 = (d^2m - dm)/24, \tag{4.21}$$

so (4.6) holds for $k = 1$. Assume as induction hypothesis that for $1 \leq r < k$, T_r has total degree $2r + 1$. Then by (4.16), T_r^* has total degree $2r$. Now (4.20) yields that $T_k - \dfrac{d}{2k} T_k^*$ has total degree $2k + 1$. From (4.5) and (4.16),

(4.22)
$$T_k - \frac{d}{2k} T_k^* = \sum_{j=1}^{k} m^j \sum_{i=1}^{2k} \frac{b_{i,j}(k)}{2k} \left\{ (2k - i)d^i - \sum_{\nu=1}^{i-1} \binom{i}{\nu - 1} (-1)^{\nu+i} d^\nu \right\}.$$

Fix i, j with $1 \leq j \leq k$, $2k \geq i > 2k + 1 - j$. If $i = 2k$, then $b_{i,j}(k) = 0$ by (4.10). If $i < 2k$, then $b_{i,j}(k) = 0$ by (4.22), since the members of (4.22) have total degree $2k + 1$. This completes the proof of (4.6).

It remains to prove (4.9). By (4.21), (4.9) is true for $k = 1$. Assume that (4.9) is false, and choose the minimal $k \geq 2$ for which at least one of the coefficients of the monomials $d^i m^j$ in the expansion

$$T_k(-d, -m) = \sum_{j=1}^{k} \sum_{i=1}^{2k+1-j} (-1)^{i+j} b_{i,j}(k) d^i m^j \tag{4.23}$$

is nonnegative. By the minimality of k, (4.9) holds with r in place of k for each r such that $1 \leq r < k$. By (4.16),

$$T_r^*(-d,-m) = \sum_{j=1}^{r} \sum_{i=1}^{2r+1-j} \sum_{\alpha=0}^{i-1} (-1)^{i+j+1} b_{i,j}(r) \binom{i}{\alpha} d^\alpha m^j , \tag{4.24}$$

so for $1 \leq r < k$, all coefficients of $d^\alpha m^j$ on the right of (4.24) are positive. By (4.24), all coefficients of $d^i m^j$ with $2 \leq j \leq k$, $1 \leq i \leq 2k+1-j$ in the polynomial expansion of

$$-dT_1^*(-d,-m)T_{k-1}^*(-d,-m) \in \mathbb{Q}[d,m]$$

are negative. In view of the two statements just above, and (4.20) (with $-d, -m$ in place of d, m), all coefficients of $d^i m^j$ with $2 \leq j \leq k$, $1 \leq i \leq 2k+1-j$ in the polynomial expansion of

$$T_k(-d,-m) + \frac{d}{2k} T_k^*(-d,-m) \in \mathbb{Q}[d,m]$$

are negative.

Applying (4.22) with $-d, -m$ in place of d, m, we have

$$\begin{aligned} &T_k(-d,-m) + \frac{d}{2k} T_k^*(-d,-m) \\ &= \sum_{j=1}^{k} m^j \sum_{i=1}^{2k+1-j} (-1)^{i+j} b_{i,j}(k) \left\{ d^i - \frac{1}{2k} \sum_{\nu=1}^{i} \binom{i}{\nu-1} d^\nu \right\} \\ &= \sum_{j=1}^{k} \sum_{\nu=1}^{2k+1-j} m^j d^\nu \left\{ (-1)^{\nu+j} b_{\nu,j}(k) - \frac{1}{2k} \sum_{i=\nu}^{2k+1-j} (-1)^{i+j} b_{i,j}(k) \binom{i}{\nu-1} \right\} . \end{aligned} \tag{4.25}$$

The coefficient of $d^\nu m^j$ on the right side of (4.25) equals

$$(-1)^{\nu+j} b_{\nu,j}(k) \left(1 - \frac{\nu}{2k}\right) + \frac{1}{2k} \sum_{i=\nu+1}^{2k+1-j} (-1)^{i+j+1} b_{i,j}(k) \binom{i}{\nu-1} . \tag{4.26}$$

Therefore, the expression in (4.26) is negative if

$$2 \leq j \leq k, \ 1 \leq \nu \leq 2k+1-j . \tag{4.27}$$

Since at least one of the coefficients in (4.23) is nonnegative, we can fix a pair ν, j with $1 \leq j \leq k$, $1 \leq \nu \leq 2k+1-j$ and with ν chosen maximal such that

$$(-1)^{\nu+j} b_{\nu,j}(k) \geq 0 . \tag{4.28}$$

By (4.7), the coefficient of m in (4.6) is $(-1)^{k+1} B_{2k} \binom{d}{2k} /2$, so since [**5**, p. 240] $(-1)^k B_{2k} < 0$, (4.28) cannot hold if $j = 1$. Thus the fixed pair ν, j satisfies (4.27); however, by (4.28) and the maximality of ν, the corresponding expression in (4.26) is nonnegative. This contradiction completes the proof of (4.9). □

Theorem 4.2. *Let* $\lambda_1 = \cdots = \lambda_m = 1$, *and fix* $k \geq 1$. *Then*

$$S_k = \sum_{j=1}^{k} \sum_{i=1}^{k+1-j} a_{i,j}(k) d^i m^j \,, \tag{4.29}$$

where the coefficient of m *in (4.29) is* $(-1)^{k+1} B_k \binom{d}{k}$, *and*

$$a_{1,1}(k) = B_k/k,\ a_{1,k}(k) = \left(-\frac{1}{2}\right)^k,\ a_{k,1}(k) = (-1)^{k+1} B_k/k!\,. \tag{4.30}$$

Thus the polynomial $S_k \in \mathbb{Q}[d, m]$ *has total degree* $k+1$.

Proof. This follows directy from Theorem 4.1 and (4.1). □

Acknowledgement. The authors are very grateful to Richard Askey and Ed Bender for helpful comments.

5. Tables

Table 5.1
(Table of $P_k(m)$ with arbitrary λ_i [Theorem 3.1])

$$P_1 = \frac{\sigma_1}{2}$$

$$P_2 = \frac{\sigma_2}{12} + \frac{\sigma_1^2}{4}$$

$$P_3 = \frac{\sigma_1\sigma_2}{8} + \frac{\sigma_1^3}{8}$$

$$P_4 = -\frac{\sigma_4}{120} + \frac{\sigma_2^2}{48} + \frac{\sigma_1^2\sigma_2}{8} + \frac{\sigma_1^4}{16}$$

$$P_5 = -\frac{\sigma_1\sigma_4}{48} + \frac{5\,\sigma_1\sigma_2^2}{96} + \frac{5\,\sigma_1^3\sigma_2}{48} + \frac{\sigma_1^5}{32}$$

$$P_6 = \frac{\sigma_6}{252} - \frac{\sigma_2\sigma_4}{96} - \frac{\sigma_1^2\sigma_4}{32} + \frac{5\,\sigma_2^3}{576} + \frac{5\,\sigma_1^2\sigma_2^2}{64} + \frac{5\,\sigma_1^4\sigma_2}{64} + \frac{\sigma_1^6}{64}$$

$$P_7 = \frac{\sigma_1\sigma_6}{72} - \frac{7\,\sigma_1\sigma_2\sigma_4}{192} - \frac{7\,\sigma_1^3\sigma_4}{192} + \frac{35\,\sigma_1\sigma_2^3}{1152} + \frac{35\,\sigma_1^3\sigma_2^2}{384} + \frac{7\,\sigma_1^5\sigma_2}{128} + \frac{\sigma_1^7}{128}$$

Table 5.1A
(Table of $P_k(m)$ with $\lambda_1 = \cdots = \lambda_m = 1$ (all $\sigma_i = m$))

$$P_1(m) = \frac{m}{2}$$

$$P_2(m) = \frac{m^2}{4} + \frac{m}{12}$$

$$P_3(m) = \frac{m^3}{8} + \frac{m^2}{8}$$

$$P_4(m) = \frac{m^4}{16} + \frac{m^3}{8} + \frac{m^2}{48} - \frac{m}{120}$$

$$P_5(m) = \frac{m^5}{32} + \frac{5\,m^4}{48} + \frac{5\,m^3}{96} - \frac{m^2}{48}$$

$$P_6(m) = \frac{m^6}{64} + \frac{5\,m^5}{64} + \frac{5\,m^4}{64} - \frac{13\,m^3}{576} - \frac{m^2}{96} + \frac{m}{252}$$

$$P_7(m) = \frac{m^7}{128} + \frac{7\,m^6}{128} + \frac{35\,m^5}{384} - \frac{7\,m^4}{1152} - \frac{7\,m^3}{192} + \frac{m^2}{72}$$

Table 5.2
(Table of $Q_k(m)$ with arbitrary λ_i [Theorem 3.2])

$$Q_1 = \frac{\sigma_2}{12}$$

$$Q_2 = \frac{\sigma_2^2}{48} - \frac{\sigma_4}{120}$$

$$Q_3 = \frac{\sigma_6}{252} - \frac{\sigma_2\sigma_4}{96} + \frac{5\,\sigma_2^3}{576}$$

$$Q_4 = -\frac{\sigma_8}{240} + \frac{\sigma_2\sigma_6}{108} + \frac{7\,\sigma_4^2}{2880} - \frac{7\,\sigma_2^2\sigma_4}{576} + \frac{35\,\sigma_2^4}{6912}$$

$$Q_5 = \frac{\sigma_{10}}{132} - \frac{\sigma_2\sigma_8}{64} - \frac{\sigma_4\sigma_6}{144} + \frac{5\,\sigma_2^2\sigma_6}{288} + \frac{7\,\sigma_2\sigma_4^2}{768} - \frac{35\,\sigma_2^3\sigma_4}{2304} + \frac{35\,\sigma_2^5}{9216}$$

Table 5.2A
(Table of $Q_k(m)$ with $\lambda_1 = \cdots = \lambda_m = 1$ (all $\sigma_i = m$))

$$Q_1(m) = \frac{m}{12}$$

$$Q_2(m) = \frac{m^2}{48} - \frac{m}{120}$$

$$Q_3(m) = \frac{5\,m^3}{576} - \frac{m^2}{96} + \frac{m}{252}$$

$$Q_4(m) = \frac{35\,m^4}{6912} - \frac{7\,m^3}{576} + \frac{101\,m^2}{8640} - \frac{m}{240}$$

$$Q_5(m) = \frac{35\,m^5}{9216} - \frac{35\,m^4}{2304} + \frac{61\,m^3}{2304} - \frac{13\,m^2}{576} + \frac{m}{132}$$

Table 5.3
(Table of E_k [Theorem 3.3])

$$E_1 = -\frac{x}{5}$$

$$E_2 = \frac{x^2}{50} + \frac{17\,x}{1050}$$

$$E_3 = -\frac{x^3}{750} - \frac{17\,x^2}{5250}$$

$$E_4 = \frac{x^4}{15000} + \frac{17\,x^3}{52500} + \frac{289\,x^2}{2205000} - \frac{1117\,x}{2425500}$$

$$E_5 = -\frac{x^5}{375000} - \frac{17\,x^4}{787500} - \frac{289\,x^3}{11025000} + \frac{1117\,x^2}{12127500} + \frac{10368\,x}{109484375}$$

Table 5.4
(Table of S_k with $\lambda_1 = \cdots = \lambda_m = 1$ [Section 4])

$$S_1 = -\frac{d\,m}{2}$$

$$S_2 = \frac{d\,m^2}{4} - \frac{d^2m}{12} + \frac{d\,m}{12}$$

$$S_3 = -\frac{d\,m^3}{8} + \frac{d^2m^2}{8} - \frac{d\,m^2}{8}$$

$$S_4 = \frac{d\,m^4}{16} - \frac{d^2m^3}{8} + \frac{d\,m^3}{8} + \frac{d^3m^2}{72} - \frac{5\,d^2m^2}{144} + \frac{d\,m^2}{48} + \frac{d^4m}{720} - \frac{d^3m}{120} + \frac{11\,d^2m}{720} - \frac{d\,m}{120}$$

$$S_5 = -\frac{d\,m^5}{32} + \frac{5\,d^2m^4}{48} - \frac{5\,d\,m^4}{48} - \frac{5\,d^3m^3}{144} + \frac{25\,d^2m^3}{288} - \frac{5\,d\,m^3}{96} - \frac{d^4m^2}{288} + \frac{d^3m^2}{48} - \frac{11\,d^2m^2}{288} + d\,\frac{m^2}{48}$$

$$S_6 = \frac{d\,m^6}{64} - \frac{5\,d^2m^5}{64} + \frac{5\,d\,m^5}{64} + \frac{5\,d^3m^4}{96} - \frac{25\,d^2m^4}{192} + \frac{5\,d\,m^4}{64} + \frac{d^4m^3}{432} - \frac{d^3m^3}{54} + \frac{67\,d^2m^3}{1728} - \frac{13\,d\,m^3}{576} - \frac{d^5m^2}{1440} + \frac{17\,d^4m^2}{2880} - \frac{13\,d^3m^2}{720} + \frac{67\,d^2m^2}{2880} - \frac{d\,m^2}{96} - \frac{d^6m}{30240} + \frac{d^5m}{2016} - \frac{17\,d^4m}{6048} + \frac{5\,d^3m}{672} - \frac{137\,d^2m}{15120} + \frac{d\,m}{252}$$

Table 5.5
(Table of T_k with $\lambda_1 = \cdots = \lambda_m = 1$ [Section 4])

$$T_1 = \frac{d^2m}{24} - \frac{dm}{24}$$

$$T_2 = \frac{d^3m^2}{144} - \frac{5d^2m^2}{288} + \frac{dm^2}{96} + \frac{d^4m}{1440} - \frac{d^3m}{240} + \frac{11d^2m}{1440} - \frac{dm}{240}$$

$$T_3 = \frac{5d^4m^3}{3456} - \frac{11d^3m^3}{1728} + \frac{d^2m^3}{108} - \frac{5dm^3}{1152} + \frac{d^5m^2}{2880} - \frac{17d^4m^2}{5760} + \frac{13d^3m^2}{1440}$$
$$- \frac{67d^2m^2}{5760} + \frac{dm^2}{192} + \frac{d^6m}{60480} - \frac{d^5m}{4032} + \frac{17d^4m}{12096} - \frac{5d^3m}{1344} + \frac{137d^2m}{30240} - \frac{dm}{504}$$

$$T_4 = \frac{7d^5m^4}{20736} - \frac{31d^4m^4}{13824} + \frac{13d^3m^4}{2304} - \frac{65d^2m^4}{10368} + \frac{35dm^4}{13824} + \frac{7d^6m^3}{51840} - \frac{79d^5m^3}{51840}$$
$$+ \frac{703d^4m^3}{103680} - \frac{191d^3m^3}{12960} + \frac{533d^2m^3}{34560} - \frac{7dm^3}{1152}$$
$$+ \frac{d^7m^2}{67200} - \frac{971d^6m^2}{3628800} + \frac{2363d^5m^2}{1209600}$$
$$- \frac{13327d^4m^2}{1814400} + \frac{3589d^3m^2}{241920} - \frac{54563d^2m^2}{3628800} + \frac{101dm^2}{17280} + \frac{d^8m}{2419200} - \frac{d^7m}{86400}$$
$$+ \frac{23d^6m}{172800} - \frac{7d^5m}{8640} + \frac{967d^4m}{345600} - \frac{469d^3m}{86400} + \frac{121d^2m}{22400} - \frac{dm}{480}$$

$$T_5 = \frac{7d^6m^5}{82944} - \frac{193d^5m^5}{248832} + \frac{725d^4m^6}{248832} - \frac{1375d^3m^5}{248832}$$
$$+ \frac{863d^2m^5}{165888} - \frac{35dm^5}{18432} + \frac{d^7m^4}{20736}$$
$$- \frac{43d^6m^4}{62208} + \frac{515d^5m^4}{124416} - \frac{3275d^4m^4}{248832} + \frac{721d^3m^4}{31104} - \frac{1751d^2m^4}{82944} + \frac{35dm^4}{4608}$$
$$+ \frac{17d^8m^3}{1935360} - \frac{163d^7m^3}{870912} + \frac{493d^6m^3}{290304} - \frac{73513d^5m^3}{8709120} + \frac{142811d^4m^3}{5806080} - \frac{361843d^3m^3}{8709120}$$
$$+ \frac{107731d^2m^3}{2903040} - \frac{61dm^3}{4608} + \frac{d^9m^2}{1741824} - \frac{103d^8m^2}{5806080} + \frac{257d^7m^2}{1088640} - \frac{5093d^6m^2}{2903040}$$
$$+ \frac{13763d^5m^2}{1741824} - \frac{14123d^4m^2}{645120} + \frac{11213d^3m^2}{311040} - \frac{46171d^2m^2}{1451520} + \frac{13dm^2}{1152} + \frac{d^{10}m}{95800320}$$
$$- \frac{d^9m}{2128896} + \frac{29d^8m}{3193344} - \frac{5d^7m}{50688} + \frac{3013d^6m}{4561920} - \frac{95d^5m}{33792} + \frac{4523d^4m}{598752} - \frac{6515d^3m}{532224}$$
$$+ \frac{7129d^2m}{665280} - \frac{dm}{264}$$

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