Hiroshima Math. J.
39 (2009), 217–235

Evaluations of hypergeometric functions over finite fields

Ron Evans and John Greene

(Received October 15, 2008)

Abstract. We prove two general formulas for a two-parameter family of hypergeometric ${}_3F_2(z)$ functions over a finite field $\mathbf{F}_q$, where q is a power of an odd prime. Each formula evaluates a ${}_3F_2$ in terms of a ${}_2F_1$ over $\mathbf{F}_{q^2}$. As applications, we evaluate infinite one-parameter families of ${}_3F_2(\frac{1}{4})$ and ${}_3F_2(-1)$, thereby extending results of J. Greene–D. Stanton and K. Ono, who gave evaluations in special cases.

1. Introduction and main theorems

Let $\mathbf{F}_q$ be a field of q elements, where q is a power of an odd prime p. Throughout this paper, A, B, C, D, E, R, S, χ, ψ, ε, ϕ will denote complex multiplicative characters on $\mathbf{F}_q^*$, extended to map 0 to 0. The notation ε and ϕ will always be reserved for the trivial and quadratic characters, respectively. Write $\bar{A}$ for the inverse (complex conjugate) of A. For $y \in \mathbf{F}_q$, let ζ^y denote the additive character

$$\zeta^y := \exp\left(\frac{2\pi i}{p}(y^p + y^{p^2} + \cdots + y^q)\right). \tag{1.1}$$

Recall the definitions of the Gauss sum

$$G(A) = \sum_{y \in \mathbf{F}_q} A(y)\zeta^y \tag{1.2}$$

and the Jacobi sum

$$J(A, B) = \sum_{y \in \mathbf{F}_q} A(y)B(1 - y). \tag{1.3}$$

Note that

$$G(\varepsilon) = -1, \qquad J(\varepsilon, \varepsilon) = q - 2,$$

2000 *Mathematics Subject Classification*. 11T24, 11L05, 33C20.
Key words and phrases. Hypergeometric functions over finite fields, Gauss sums, Jacobi sums, Davenport–Hasse formulas, lifted characters, Stickelberger's congruence.

and for nontrivial A,

$$G(A)G(\bar{A}) = A(-1)q, \qquad J(A,\bar{A}) = -A(-1).$$

Gauss and Jacobi sums are related by [1, p. 59]

$$J(A,B) = \frac{G(A)G(B)}{G(AB)}, \qquad \text{if } AB \neq \varepsilon. \tag{1.4}$$

The following formulas are special cases of the Davenport–Hasse product relation for Gauss sums [1, p. 351]:

$$A(4)G(A)G(A\phi) = G(A^2)G(\phi) \tag{1.5}$$

and

$$A(27)G(A)G(A\psi)G(A\bar{\psi}) = qG(A^3), \tag{1.6}$$

where ψ is a cubic character on $\mathbf{F}_q$.

For $x \in \mathbf{F}_q$, define the hypergeometric ${}_2F_1$ function over $\mathbf{F}_q$ by [7, p. 82]

$${}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix}\,\middle|\, x\right) = \frac{\varepsilon(x)}{q} \sum_{y \in \mathbf{F}_q} B(y)\bar{B}C(y-1)\bar{A}(1-xy) \tag{1.7}$$

and the hypergeometric ${}_3F_2$ function over $\mathbf{F}_q$ by [7, p. 83]

$${}_3F_2\left(\begin{matrix} A, B, C \\ D, E \end{matrix}\,\middle|\, x\right) = \frac{\varepsilon(x)}{q^2} \sum_{y,z \in \mathbf{F}_q} C(y)\bar{C}E(y-1)B(z)\bar{B}D(z-1)\bar{A}(1-xyz). \tag{1.8}$$

The "binomial coefficient" over $\mathbf{F}_q$ is defined by [7, p. 80]

$$\binom{A}{B} = \frac{B(-1)}{q} J(A,\bar{B}). \tag{1.9}$$

We will need the function

$$F(A,B;x) := \frac{q}{q-1} \sum_{\chi} \binom{A\chi^2}{\chi}\binom{A\chi}{B\chi}\chi\left(\frac{x}{4}\right), \qquad x \in \mathbf{F}_q, \tag{1.10}$$

and its normalization

$$F^*(A,B;x) := F(A,B;x) + AB(-1)\frac{\bar{A}(x/4)}{q}. \tag{1.11}$$

Another character sum that we will need is

$$g(R,S;x) := \sum_{t \in \mathbf{F}_q} R(1-t)S(1-xt^2), \qquad x \in \mathbf{F}_q. \tag{1.12}$$

It is not hard to show that $g(R,S;x)$ is related to the Gegenbauer function P_R^S [3, (4.1)] by

$$g\left(\bar{R},\bar{S};\frac{x^2}{x^2-1}\right) = qR(x)S(1-x^2)P_R^S(x), \qquad x^2 \notin \{0,1\}. \tag{1.13}$$

For $u \in \mathbf{F}_{q^2}$, let $N(u) = u^{1+q}$ denote the norm map from $\mathbf{F}_{q^2}$ to $\mathbf{F}_q$. For any character χ on $\mathbf{F}_q$, the composition χN is a character on $\mathbf{F}_{q^2}$ called the lift of χ from $\mathbf{F}_q$ to $\mathbf{F}_{q^2}$. We place a hat on a character sum function over $\mathbf{F}_q$ to indicate that the sum is taken over $\mathbf{F}_{q^2}$ instead of $\mathbf{F}_q$. For example, for $u \in \mathbf{F}_{q^2}$,

$$\hat{g}(RN,SN;u) := \sum_{t\in\mathbf{F}_{q^2}} RN(1-t)SN(1-ut^2).$$

Our main results are Theorems 1.1 and 1.2 below, which evaluate an infinite two-parameter class of ${}_3F_2$ functions in terms of ${}_2\hat{F}_1$. This class of ${}_3F_2$ functions has also been evaluated in terms of F^* (see, e.g., (2.1)). In one respect, F^* has an advantage over ${}_2\hat{F}_1$, since F^* is defined over $\mathbf{F}_q$ while ${}_2\hat{F}_1$ is defined over the quadratic extension $\mathbf{F}_{q^2}$. However, there is a tradeoff, since a ${}_2F_1$ over any finite field is generally better understood than F^*. We are unable to express F^* in terms of a ${}_2F_1$, in general. However, because every element of $\mathbf{F}_{q^2}$ is a square, it is always possible to express $\hat{F}^*$ in terms of a ${}_2\hat{F}_1$; see (3.2)–(3.3). This has been our motivation for passing to the extension field $\mathbf{F}_{q^2}$ in this paper.

Theorem 1.1. *Let $C \neq \phi$, $A \notin \{\varepsilon, C, C^2\}$. For $x \in \mathbf{F}_q$, $x \notin \{0,1\}$, define $u \in \mathbf{F}_{q^2}$ by $u^2 = (x-1)/x$. Then*

$${}_3F_2\left(\begin{matrix} A, \bar{A}C^2, C\phi \\ C^2, C \end{matrix}\middle| x\right) = \frac{\bar{C}(x)\phi(1-x)}{q} - Z_1\, {}_2\hat{F}_1\left(\begin{matrix} \bar{C}\phi N, C\phi N \\ \bar{A}C\phi N \end{matrix}\middle| \frac{1-u}{2}\right),$$

where

$$Z_1 := C(-1)\bar{A}(1-x)AN(u)\bar{A}C\phi N(1-u)\frac{J(\bar{A}C^2, A\bar{C})\hat{J}(A\phi N, \bar{A}CN)}{J(A,\bar{A}C)\hat{J}(\phi N, A\phi N)}.$$

Theorem 1.2. *Let $C \neq \phi$, $A \notin \{\varepsilon, C, C^2\}$, $x \in \mathbf{F}_q$, $x \notin \{0,1\}$. Define a character α on $\mathbf{F}_{q^2}$ by $\alpha^2 = \bar{A}N$. Then*

$${}_3F_2\left(\begin{matrix} A, \bar{A}C^2, C\phi \\ C^2, C \end{matrix}\middle| x\right) = \frac{\bar{C}(x)\phi(1-x)}{q} - Z_2\, {}_2\hat{F}_1\left(\begin{matrix} \bar{\alpha}\phi N, \bar{\alpha} \\ CN \end{matrix}\middle| \frac{x}{x-1}\right),$$

where

$$Z_2 := C(-1)A\bar{C}(4)\bar{A}(1-x)\frac{J(\bar{A}C^2, A\bar{C})\hat{J}(\phi N, \bar{A}CN)}{J(A,\bar{A}C)\hat{J}(\alpha CN, \alpha\phi N)}.$$

Observe that the argument of the ${}_2\hat{F}_1$ in Theorem 1.2 lies in $\mathbf{F}_q$, while that in Theorem 1.1 may lie in the less accomodating field $\mathbf{F}_{q^2}$. Here too there is a tradeoff, however, because Theorem 1.1 does not require the introduction of the extra character α on $\mathbf{F}_{q^2}$ that occurs in Theorem 1.2.

As an application of Theorem 1.1, we evaluate the following one-parameter family of ${}_3F_2(\frac{1}{4})$.

Theorem 1.3. *Let S be a character on $\mathbf{F}_q$ which is not trivial, cubic, or quartic, i.e.* $\mathrm{ord}(S) \notin \{1,3,4\}$. *Then*

$$ {}_3F_2\left(\begin{matrix} \bar{S}, S^3, S \\ S^2, S\phi \end{matrix} \middle| \frac{1}{4}\right) = \begin{cases} \phi(-1)S(4)/q, & \text{if } q \equiv 2 \pmod 3 \\ \frac{\phi(-1)S(4)}{q^2}(q + 2\,\mathrm{Re}(J(S,\psi)J(\bar{S},\psi))), & \text{if } q \equiv 1 \pmod 3, \end{cases} $$

where ψ denotes a cubic character on $\mathbf{F}_q$ when $q \equiv 1 \pmod 3$.

The special case $S = \phi$ of Theorem 1.3 was proved in 1998 by K. Ono [10, Theorem 6(v)], [11], thus solving a problem posed in 1992 by Koike [9, p. 465]. We remark that in view of [7, Theorem 4.2], there is a result similar to Theorem 1.3 in which the argument $1/4$ is replaced by its reciprocal 4.

As an application of Theorem 1.2, we evaluate the following one-parameter family of ${}_3F_2(-1)$.

Theorem 1.4. *Let C be a character on $\mathbf{F}_q$ which is not quadratic or quartic, i.e.,* $\mathrm{ord}(C) \notin \{2,4\}$. *Then*

$$ {}_3F_2\left(\begin{matrix} \phi, C^2\phi, C\phi \\ C^2, C \end{matrix} \middle| -1\right) $$

$$ = \begin{cases} \frac{-1}{q}, & \text{if } q \equiv 3 \pmod 4 \text{ and } \phi(2) = C(-1) \\ \frac{-1}{q} - \frac{2\phi(2)}{q^2}\,\mathrm{Re}\,\hat{J}(\beta\eta, \beta^6), & \text{if } q \equiv 3 \pmod 4 \text{ and } \phi(2) = -C(-1) \\ \frac{1}{q}, & \text{if } q \equiv 1 \pmod 4 \text{ and } C\chi \text{ is not a square} \\ \frac{1}{q} + \frac{2}{q^2}\,\mathrm{Re}(J(D,\phi)J(\bar{D}\chi,\phi)), & \text{if } q \equiv 1 \pmod 4 \text{ and } C\chi = D^2, \end{cases} $$

where χ, D are characters on $\mathbf{F}_q$ with $\mathrm{ord}(\chi) = 4$ *when* $q \equiv 1 \pmod 4$*, and β, η are characters on $\mathbf{F}_{q^2}$ with* $\mathrm{ord}(\beta) = 8$, $\eta^2 = CN$ *when* $q \equiv 3 \pmod 4$.

The special case $C = \varepsilon$ of Theorem 1.4 was proved in 1986 by Greene and Stanton [8], thus solving a problem posed in 1981 [5]. The case $q \equiv 1 \pmod 4$ of Theorem 1.4 (which does not require passing to the extension field $\mathbf{F}_{q^2}$) is equivalent to another result of Greene and Stanton [8, (5.1)]. To see the equivalence, observe that the ${}_3F_2$ in Theorem 1.4 can be transformed via [7, Theorem 4.2(i)] to

$$C(-1)\,{}_3F_2\left(\begin{matrix}\phi, \bar{C}\phi, C\phi \\ C, \bar{C}\end{matrix}\middle| -1\right),$$

and this ${}_3F_2$ can in turn be converted to that in [8, (5.1)] by changing the order of the numerator parameters in accordance with [7, Theorem 3.20(i)(ii)]. As Greene and Stanton only sketched a proof of [8, (5.1)], our proof of Theorem 1.4 will treat the case $q \equiv 1 \pmod 4$ in full detail.

The evaluations in Theorems 1.3, 1.4 have analogues over the reals which are much easier to prove. An analogue for Theorem 1.3 over the reals is

$${}_3F_2\left(\begin{matrix}1-s, 3s-1, s \\ 2s, s+\frac{1}{2}\end{matrix}\middle| \frac{1}{4}\right) = \left(\frac{16}{27}\right)^{s-1}\left(\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(s+\frac{1}{3}\right)}\right)^2.$$

An analogue for Theorem 1.4 is

$${}_3F_2\left(\begin{matrix}\frac{1}{2}, 2c-\frac{1}{2}, c \\ 2c, c+\frac{1}{2}\end{matrix}\middle| -1\right) = 2^{-2c+1/2}\left(\frac{\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{4c+3}{8}\right)\Gamma\left(\frac{4c+5}{8}\right)}\right)^2.$$

These two analogues may be proved, for appropriate values of the parameters, by applying Pfaff's transformation [7, (4.5)] to the ${}_2F_1$ in Clausen's theorem [8, (4.1)] and then evaluating the resulting ${}_2F_1\left(-\frac{1}{3}\right)$ and ${}_2F_1\left(\frac{1}{2}\right)$ in terms of gamma functions via [2, p. 104, (51) and (53)].

We prove Theorems 1.1 and 1.2 in Section 2, after presenting seven prerequisite lemmas and theorems. Theorems 1.3 and 1.4 are proved in Sections 3 and 4, respectively.

For further ${}_3F_2$ evaluations, see [4] and [3]. In [3, (1.14)], for $\operatorname{ord}(S) \notin \{1,3,4\}$, we proved the evaluation

$${}_3F_2\left(\begin{matrix}\bar{S}, S^3, S \\ S^2, S\phi\end{matrix}\middle| \frac{-1}{8}\right)$$

$$= \begin{cases} -\phi(-1)S(-8)/q, & \text{if } S \text{ is not a square} \\ \frac{\phi(-1)S(8)}{q^2}\left(q + 2\operatorname{Re}(J(S,D)J(\bar{S},\bar{D}\phi))\right), & \text{if } S = D^2, \end{cases}$$

thus extending a result of K. Ono, who obtained the special case $S = \phi$ [10, Theorem 6(ii)]. Our evaluations of ${}_3F_2\left(\frac{1}{4}\right)$ and ${}_3F_2(-1)$ in Theorems 1.3 and 1.4 are more complicated to prove, in that they require passing to a quadratic extension of $\mathbf{F}_q$. Our attempts to extend Ono's ${}_3F_2\left(\frac{1}{64}\right)$ evaluation [10, Theorem 6(vii)] using Theorems 1.1 or 1.2 have been unsuccessful, because we have been unable to evaluate the corresponding ${}_2\hat{F}_1$ functions.

We close this section with an example of a ${}_3F_2(-1)$ evaluation where the three numerator parameters are each ϕ, the two denominator parameters are each ε, and q is a prime congruent to 1 or 3 (mod 8) so that $q = x^2 + 2y^2$ for integers x, y. First consider the case $q \equiv 3 \pmod 8$. By Theorem 1.4 with $C = \varepsilon$,

$$ {}_3F_2\left(\begin{matrix}\phi,\phi,\phi\\ \varepsilon,\varepsilon\end{matrix}\,\middle|\,-1\right) = -\frac{1}{q} + \frac{2}{q^2}\,\mathrm{Re}\,\hat{J}(\beta^5,\beta^6). \tag{1.14} $$

To simplify this formula, first observe that

$$ \hat{J}(\beta^5,\beta^6) = \frac{\hat{G}(\beta^5)\hat{G}(\beta^6)}{\hat{G}(\bar{\beta}^5)} = \frac{\beta(-1)\hat{G}(\beta^5)^2\hat{G}(\beta^6)}{q^2}. \tag{1.15} $$

By [1, Theorem 11.6.1], $\hat{G}(\beta^6) = -q$. Since the restriction of β to $\mathbf{F}_q$ is the quadratic character ϕ, we have $\beta(-1) = -1$. Thus (1.15) becomes

$$ \hat{J}(\beta^5,\beta^6) = \frac{\hat{G}(\beta^5)^2}{q}. \tag{1.16} $$

By [1, Theorems 12.1.1 and 12.7.1(b)],

$$ \hat{G}(\beta^5)^2 = (x + iy\sqrt{2})^2 G(\phi)^2 = -q(x + iy\sqrt{2})^2. \tag{1.17} $$

By (1.16)–(1.17), the evaluation (1.14) becomes

$$ {}_3F_2\left(\begin{matrix}\phi,\phi,\phi\\ \varepsilon,\varepsilon\end{matrix}\,\middle|\,-1\right) = \frac{q - 4x^2}{q^2}, \tag{1.18} $$

in agreement with [10, Theorem 6(iii)].

Now suppose that $q \equiv 1 \pmod 8$. By Theorem 1.4 with $C = \varepsilon$,

$$ {}_3F_2\left(\begin{matrix}\phi,\phi,\phi\\ \varepsilon,\varepsilon\end{matrix}\,\middle|\,-1\right) = \frac{1}{q} + \frac{2}{q}\,\mathrm{Re}\,J(D,\phi)^2, $$

where D is an octic character on $\mathbf{F}_q$. By [1, Theorems 3.3.1 and 2.1.4],

$$ \mathrm{Re}\,J(D,\phi)^2 = \mathrm{Re}(x + iy\sqrt{2})^2 = x^2 - 2y^2. $$

Thus

$$ {}_3F_2\left(\begin{matrix}\phi,\phi,\phi\\ \varepsilon,\varepsilon\end{matrix}\,\middle|\,-1\right) = \frac{4x^2 - q}{q^2}, $$

again in agreement with [10, Theorem 6(iii)].

2. Proof of Theorems 1.1 and 1.2

Our first lemma gives an alternative representation for the function F^* defined in (1.11), by employing Gauss sums in lieu of Jacobi sums.

LEMMA 2.1. *If $A \neq C$, then*

$$F^*(A, C; x) = \frac{C(-1)}{q(q-1)G(A\overline{C})} \sum_{\chi} G(A\chi^2)G(\bar{\chi})G(\overline{C}\bar{\chi})\chi\left(\frac{x}{4}\right).$$

PROOF. First suppose that A is nontrivial. By (1.11),

$$F^*(A, C; x) = \frac{AC(-1)\bar{A}\left(\frac{x}{4}\right)}{q} + \frac{C(-1)}{q(q-1)} \sum_{\chi} J(A\chi^2, \bar{\chi})J(A\chi, \overline{C}\bar{\chi})\chi\left(\frac{x}{4}\right)$$

$$= \frac{AC(-1)\bar{A}\left(\frac{x}{4}\right)}{q} + \frac{C(-1)}{q(q-1)} J(\bar{A}, A)J(\varepsilon, \overline{C}A)\bar{A}\left(\frac{x}{4}\right)$$

$$+ \frac{C(-1)}{q(q-1)G(A\overline{C})} \sum_{\chi \neq \bar{A}} G(A\chi^2)G(\bar{\chi})G(\overline{C}\bar{\chi})\chi\left(\frac{x}{4}\right)$$

$$= \frac{AC(-1)\bar{A}\left(\frac{x}{4}\right)}{q-1} + \frac{C(-1)}{q(q-1)G(A\overline{C})} \sum_{\chi \neq \bar{A}} G(A\chi^2)G(\bar{\chi})G(\overline{C}\bar{\chi})\chi\left(\frac{x}{4}\right),$$

and the result follows. When A is trivial, the first equality above still holds, and after separating out the summand for trivial χ, we arrive at the desired result.

The next theorem represents F^* in terms of the function g defined in (1.12).

THEOREM 2.2. *If $A \neq C$ and $x \notin \{0, 1\}$, then*

$$F^*\left(A, C; \frac{x}{x-1}\right) = \frac{A(2)A\overline{C}(1-x)}{q} g(A\overline{C}^2, \bar{A}C; 1-x).$$

PROOF. Let $y \neq 0$. By Lemma 2.1,

$$L := F^*(A, C; y)q(q-1)C(-1)G(A\overline{C}) = \sum_{\chi} G(A\chi^2)G(\bar{\chi})G(\overline{C}\bar{\chi})\chi\left(\frac{y}{4}\right)$$

$$= \sum_{t,u,v \neq 0} \zeta^{t+u+v} A(t)\overline{C}(v) \sum_{\chi} \chi\left(\frac{yt^2}{4uv}\right) = (q-1) \sum_{t,v \neq 0} \zeta^{t+v+yt^2/4v} A(t)\overline{C}(v).$$

Replacing t by $2vt$, we obtain

$$L = (q-1) \sum_{t,v \neq 0} \zeta^{v(yt^2+2t+1)} A(2t)A\overline{C}(v)$$

$$= (q-1)A(2)G(A\overline{C}) \sum_{t \neq 0} A(t)\bar{A}C(yt^2 + 2t + 1),$$

since $A \neq C$. Replace t by $1/t$ to obtain

$$L = (q-1)A(2)G(A\overline{C})\sum_t A\overline{C}^2(t)\bar{A}C(t^2+2t+y)$$

$$= (q-1)A(2)G(A\overline{C})\sum_t A\overline{C}^2(t-1)\bar{A}C(t^2+y-1).$$

Putting $y = \frac{x}{x-1}$, we get

$$F^*\left(A, C; \frac{x}{x-1}\right) = \frac{A(2)C(-1)}{q}\sum_t A\overline{C}^2(t-1)\bar{A}C\left(t^2+\frac{1}{x-1}\right)$$

$$= \frac{A\overline{C}(1-x)A(2)}{q}\sum_t A\overline{C}^2(1-t)\bar{A}C(1-(1-x)t^2),$$

the desired result.

Corollary 2.3. *If $A \neq C$ and $u \notin \{0,1\}$, then*

$$F^*(A, C; u) = \frac{A(2)}{q}\sum_t A\overline{C}^2(1-t)\bar{A}C(1-u-t^2).$$

The next theorem gives a transformation formula for F^* akin to Euler's transformation for ${}_2F_1$ [7, (4.6)].

Theorem 2.4. *Let $u \neq 1$, $A \notin \{\varepsilon, C, C^2\}$. Then*

$$F^*(A, C; u) = \bar{A}C\phi(1-u)\frac{J(A, \bar{A}C)}{J(\bar{A}C^2, A\overline{C})}F^*(\bar{A}C^2, C; u).$$

Proof. Define the function L_1 by

$$L_1(A) := G(\bar{A}C^2)G(A\overline{C})F^*(A, C; u).$$

By Corollary 2.3,

$$L_1(A) = \frac{A(2)}{q}\sum_t \sum_{w,z\neq 0} A\overline{C}^2\left(\frac{1-t}{w}\right)\bar{A}C^2\left(\frac{1-u-t^2}{z}\right)\zeta^{w+z}$$

$$= \frac{A(2)}{q}\sum_t \sum_{w,z\neq 0} \bar{A}C^2(w)A\overline{C}(z)\zeta^{w(1-t)+z(1-u-t^2)}$$

$$= \frac{C(4)}{q}\sum_{w,z\neq 0} \bar{A}C^2(w)C(z)\sum_t \zeta^{z(2w-2wt+1-u-t^2)}$$

$$= \frac{C(4)}{q}\sum_{w,z\neq 0} \bar{A}C^2(w)C(z)\zeta^{z((w+1)^2-u)}\sum_t \zeta^{-z(t+w)^2}$$

$$= L_2(A),$$

where

$$L_2(A) := \frac{\phi(-1)C(4)G(\phi)}{q} \sum_{w,z\neq 0} \bar{A}C^2(w)C\phi(z)\zeta^{z((w+1)^2-u)}.$$

Replacing w by $1/w$ and z by zw^2, we have $L_2(A) = L_3(A)$, where

$$L_3(A) := \frac{\phi(-1)C(4)G(\phi)}{q} \sum_{w,z\neq 0} A(w)C\phi(z)\zeta^{z((w+1)^2-uw^2)}.$$

Replacing w by $w(1-u)$ and z by $\frac{z}{(1-u)}$ in the definition of $L_2(A)$, we obtain

$$L_2(A) = \bar{A}C\phi(1-u)L_3(\bar{A}C^2) = \bar{A}C\phi(1-u)L_1(\bar{A}C^2),$$

which is equivalent to the desired result.

The next theorem expresses a ${}_3F_2$ in terms of the square of the function g defined in (1.12).

Theorem 2.5. *Let $C \neq \phi$, $A \notin \{\varepsilon, C, C^2\}$, $x \neq 1$. Then*

$$\begin{aligned} {}_3F_2\left(\begin{matrix} A, \bar{A}C^2, C\phi \\ C^2, C \end{matrix}\middle| x\right) &= -\frac{\overline{C}(x)\phi(1-x)}{q} \\ &\quad + C(-1)A\overline{C}(4)A\overline{C}^2(1-x) \\ &\quad \times \frac{J(\bar{A}C^2, A\overline{C})}{q^2 J(A, \bar{A}C)} g(A\overline{C}^2, \bar{A}C; 1-x)^2. \end{aligned}$$

Proof. By [3, Theorem 1.1],

$$\begin{aligned} {}_3F_2\left(\begin{matrix} A, \bar{A}C^2, C\phi \\ C^2, C \end{matrix}\middle| x\right) &= -\frac{\overline{C}(x)\phi(1-x)}{q} \\ &\quad + \overline{C}(-4)\overline{C}\phi(1-x)F^* \\ &\quad \times \left(A, C; \frac{x}{x-1}\right)F^*\left(\bar{A}C^2, C; \frac{x}{x-1}\right). \end{aligned} \tag{2.1}$$

Substitute for the rightmost factor using Theorem 2.4, and then apply Theorem 2.2 to obtain the desired result.

The next theorem provides us with the means of passing to the extension field $\mathbf{F}_{q^2}$ in this paper. It expresses g^2 in terms of the function $\hat{g}$ defined over $\mathbf{F}_{q^2}$, in analogy with the Davenport–Hasse lifting theorem [1, Theorem 11.5.2].

Theorem 2.6. *Suppose that R and S are nontrivial, $x \in \mathbf{F}_q$, and $x \notin \{0,1\}$. Then*

$$g(R,S;x)^2 = -\hat{g}(RN,SN;x) + \frac{2S(-4)\bar{R}\phi(x)RS(x-1)J(R,S^2)G(S)^2}{G(S^2)}.$$

Proof. Let $p(x)$ be a polynomial over $\mathbf{F}_q$, and for a character E on $\mathbf{F}_q$, define

$$H(E) = \sum_{u \in \mathbf{F}_q} E(p(u)). \tag{2.2}$$

Choose $\gamma \in \mathbf{F}_{q^2}$ so that γ^2 generates $\mathbf{F}_q^*$. We have

$$H(E)^2 = \sum_{y,z \in \mathbf{F}_q} E(f(y,z)) \tag{2.3}$$

and

$$\hat{H}(EN) = \sum_{y,z \in \mathbf{F}_q} EN(p(y+\gamma z)) = \sum_{y,z \in \mathbf{F}_q} E(f(y,\gamma z)), \tag{2.4}$$

where

$$f(y,z) := p(y+z)p(y-z). \tag{2.5}$$

Since $f(y,z)$ is an even function of z, there exists a two-variable polynomial h over $\mathbf{F}_q$ such that

$$f(y,z) = h(y,z^2), \qquad f(y,\gamma z) = h(y,\gamma^2 z^2). \tag{2.6}$$

By (2.3)–(2.6),

$$\begin{aligned} H(E)^2 + \hat{H}(EN) &= \sum_{y,z} E(h(y,z^2)) + \sum_{y,z} E(h(y,\gamma^2 z^2)) \\ &= \sum_{y,z} (1+\phi(z))E(h(y,z)) + \sum_{y,z} (1-\phi(z))E(h(y,z)) \\ &= 2\sum_{y,z} E(h(y,z)). \end{aligned} \tag{2.7}$$

Applying (2.7) to the function g, we have

$$
\begin{aligned}
&\frac{g(R,S;x)^2+\hat{g}(RN,SN;x)}{2}\\
&\quad=\sum_{y,z}R(1-2y+y^2-z)S(1-2x(y^2+z)+x^2(y^2-z)^2)\\
&\quad=\sum_{y,z}R(1-2y+z)S(1-2x(2y^2-z)+x^2z^2)\\
&\quad=\sum_{y,z}R(1+y+z)S((1+xz)^2-xy^2)\\
&\quad=S(x)\sum_{y,z}R\left(1-\frac{1}{x}+y+z\right)S(xz^2-y^2)\\
&\quad=S(x)RS^2\left(\frac{x-1}{x}\right)\sum_{y,z}R(1+y+z)S(xz^2-y^2)\\
&\quad=\bar{R}\bar{S}(x)RS^2(x-1)\left\{S(x)J(R,S^2)+\sum_{y,z}R(1+y(1+z))S^2(y)S(xz^2-1)\right\}\\
&\quad=\bar{R}\bar{S}(x)RS^2(x-1)\{S(x)J(R,S^2)+(q-1)S(x-1)\delta(S^2)+J(R,S^2)W_1\}
\end{aligned}
$$

where

$$W_1:=\sum_z \bar{S}^2(1+z)S(xz^2-1),$$

and $\delta(S^2)=1$ if $S=\phi$ and $\delta(S^2)=0$ otherwise. Now,

$$
\begin{aligned}
W_1&=\sum_{z\neq 0}\bar{S}^2(z)S(x(z-1)^2-1)=\sum_{z\neq 0}S\left(\frac{x(z-1)^2-1}{z^2}\right)\\
&=\sum_{z\neq 0}S(x(1-z)^2-z^2)=-S(x)+W_2,
\end{aligned}
$$

where

$$W_2:=\sum_z S(x(1-z)^2-z^2).$$

Thus

$$
\begin{aligned}
&\frac{g(R,S;x)^2+\hat{g}(RN,SN;x)}{2}\\
&\qquad=\bar{R}\bar{S}(x)RS^2(x-1)\{(q-1)\phi(x-1)\delta(S^2)+J(R,S^2)W_2\}. \qquad (2.8)
\end{aligned}
$$

We have

$$W_2 = S(x-1)\sum_z S\left(z^2 - \frac{2xz}{x-1} + \frac{x}{x-1}\right) = S(x-1)\sum_z S\left(z^2 - \frac{x}{(x-1)^2}\right)$$
$$= \bar{S}(x-1)\sum_z (1+\phi(z))S(z-x) = \bar{S}(1-x)S\phi(x)J(S,\phi), \tag{2.9}$$

since S is nontrivial. By (2.8)–(2.9),

$$\frac{g(R,S;x)^2 + \hat{g}(RN,SN;x)}{2} = \bar{R}\phi(x)R\phi(x-1)(q-1)\delta(S^2)$$
$$+ \bar{R}\phi(x)RS(x-1)S(-1)J(S,\phi)J(R,S^2). \tag{2.10}$$

By (1.5), when $S \neq \phi$,

$$J(S,\phi) = \frac{S(4)G(S)^2}{G(S^2)}. \tag{2.11}$$

The result now follows from (2.10)–(2.11) when $S \neq \phi$, and it follows easily from (2.10) when $S = \phi$.

The following theorem expresses a ${}_3F_2$ in terms of the function $\hat{F}^*$ defined over $\mathbf{F}_{q^2}$.

Theorem 2.7. *Let* $C \neq \phi$, $A \notin \{\varepsilon, C, C^2\}$, $x \in \mathbf{F}_q$, $x \notin \{0,1\}$. *Then*

$$ {}_3F_2\left(\begin{matrix} A, \bar{A}C^2, C\phi \\ C^2, C \end{matrix} \middle| x\right) = \frac{\bar{C}(x)\phi(1-x)}{q}$$
$$- \frac{\bar{C}(-4)\bar{A}(1-x)J(\bar{A}C^2, A\bar{C})}{J(A,\bar{A}C)}\hat{F}^*$$
$$\times \left(AN, CN; \frac{x}{x-1}\right). \tag{2.12}$$

Proof. Combining Theorems 2.5 and 2.6, we have, after simplification,

$$ {}_3F_2\left(\begin{matrix} A, \bar{A}C^2, C\phi \\ C^2, C \end{matrix} \middle| x\right) = \frac{\bar{C}(x)\phi(1-x)}{q} - V_1, \tag{2.13}$$

where

$$V_1 := C(-1)A\bar{C}(4)A\bar{C}^2(1-x)\frac{J(\bar{A}C^2, A\bar{C})}{q^2 J(A,\bar{A}C)}\hat{g}(A\bar{C}^2N, \bar{A}CN; 1-x).$$

By Theorem 2.2,

$$V_1 = V_2 \hat{F}^*\left(AN, CN; \frac{x}{x-1}\right), \tag{2.14}$$

where

$$V_2 := \frac{C(-1)A\overline{C}(4)A\overline{C}^2(1-x)J(\bar{A}C^2, A\overline{C})}{AN(2)A\overline{C}N(1-x)J(A, \bar{A}C)}.$$

Since $AN(2) = A(4)$ and $A\overline{C}N(1-x) = A^2\overline{C}^2(1-x)$,

$$V_2 = \frac{\overline{C}(-4)\bar{A}(1-x)J(\bar{A}C^2, A\overline{C})}{J(A, \bar{A}C)}. \tag{2.15}$$

The result now follows from (2.13)–(2.15).

We are now in a position to prove Theorems 1.1 and 1.2. Let $C \neq \phi$, $A \notin \{\varepsilon, C, C^2\}$, $x \in \mathbf{F}_q$, $x \notin \{0, 1\}$. By [3, Theorem 1.2],

$$\hat{F}^*\left(AN, CN; \frac{x}{x-1}\right) = \frac{A(4)\hat{J}(\phi N, \bar{A}CN)}{\hat{J}(\alpha CN, \alpha\phi N)}\, {}_2\hat{F}_1\left(\begin{matrix} \bar{\alpha}\phi N, \bar{\alpha} \\ CN \end{matrix} \,\middle|\, \frac{x}{x-1}\right), \tag{2.16}$$

where α is a character on $\mathbf{F}_{q^2}$ defined by $\alpha^2 = \bar{A}N$. Also, by [3, Theorem 1.6],

$$\begin{aligned} &\hat{F}^*\left(AN, CN; \frac{x}{x-1}\right) \\ &\quad = C(4)AN(u)\bar{A}C\phi N(1-u)\frac{\hat{J}(A\phi N, \bar{A}CN)}{\hat{J}(\phi N, A\phi N)}\, {}_2\hat{F}_1\left(\begin{matrix} \overline{C}\phi N, C\phi N \\ \bar{A}C\phi N \end{matrix} \,\middle|\, \frac{1-u}{2}\right), \end{aligned} \tag{2.17}$$

where $u \in \mathbf{F}_{q^2}$ is defined by $u^2 = (x-1)/x$. Using (2.17) in (2.12) we obtain Theorem 1.1. Using (2.16) in (2.12), we obtain Theorem 1.2.

3. Proof of Theorem 1.3

Let $\operatorname{ord}(S) \notin \{1, 3, 4\}$. We first consider the case $q \equiv 1 \pmod 3$, so that there exists a cubic character ψ on $\mathbf{F}_q$ and an element $u = \sqrt{-3} \in \mathbf{F}_q$. Write $\omega = \frac{-1+u}{2}$ (a primitive cube root of unity in $\mathbf{F}_q$).

By [3, Theorem 1.7] with $A = \overline{S}$, $C = S\phi$,

$${}_3F_2\left(\begin{matrix} \overline{S}, S^3, S \\ S^2, S\phi \end{matrix} \,\middle|\, \frac{1}{4}\right) = \frac{-\phi(-1)S(4)}{q} + \phi(-1)S(4\omega)\frac{J(\overline{S}, S^3)}{J(S, S)}\, {}_2F_1\left(\begin{matrix} \overline{S}, S \\ S^2 \end{matrix} \,\middle|\, -\omega\right)^2.$$

By (1.7),

$${}_2F_1\left(\begin{matrix} \overline{S}, S \\ S^2 \end{matrix} \,\middle|\, -\omega\right) = \frac{1}{q}\sum_y S(y)S(y-1)S(1+\omega y).$$

With the substitution $y = \frac{(2x+1)u-3}{6\omega}$, we have

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}\bar{S}, S\\ S^2\end{matrix}\middle| -\omega\right) &= \frac{1}{q} S\left(\frac{-3-u}{18}\right) \sum_x S(1-x^3)\\
&= \frac{1}{q} S\left(\frac{-3-u}{18}\right) \sum_x S(1-x)(1+\psi(x)+\bar{\psi}(x))\\
&= \frac{1}{q} S\left(\frac{-3-u}{18}\right) (J(S,\psi)+J(S,\bar{\psi})). \qquad (3.1)
\end{aligned}
$$

Therefore

$$
{}_3F_2\left(\begin{matrix}\bar{S}, S^3, S\\ S^2, S\phi\end{matrix}\middle| \frac{1}{4}\right) = \frac{-\phi(-1)S(4)}{q} + T_1 + 2T_2,
$$

where

$$
T_1 := \phi(-1)S(-4)\frac{J(\bar{S}, S^3)}{J(S,S)} \frac{\bar{S}(27)}{q^2} (J^2(S,\psi)+J^2(S,\bar{\psi})) \qquad (3.2)
$$

and

$$
T_2 := \phi(-1)S(-4)\frac{J(\bar{S}, S^3)}{J(S,S)} \frac{\bar{S}(27)}{q^2} J(S,\psi)J(S,\bar{\psi}). \qquad (3.3)
$$

Using (1.6), one obtains the simplification $T_2 = \frac{\phi(-1)S(4)}{q}$. Therefore

$$
{}_3F_2\left(\begin{matrix}\bar{S}, S^3, S\\ S^2, S\phi\end{matrix}\middle| \frac{1}{4}\right) = \frac{\phi(-1)S(4)}{q}\left(1+\frac{T_1}{T_2}\right), \qquad (3.4)
$$

and the result readily follows from (3.2)–(3.4) in the case $q \equiv 1 (\mathrm{mod}\ 3)$.

Now let $q \equiv 2 (\mathrm{mod}\ 3)$. In this case $\phi(-3) = -1$. Define the elements $u = \sqrt{-3}$, $\omega = \frac{-1+u}{2}$ in $\mathbf{F}_{q^2}$, and let λ denote a cubic character on $\mathbf{F}_{q^2}$. As in (3.1), we have

$$
\begin{aligned}
{}_2\hat{F}_1\left(\begin{matrix}\bar{S}N, SN\\ S^2N\end{matrix}\middle| -\omega\right) &= \frac{1}{q^2} SN\left(\frac{-3-u}{18}\right)(\hat{J}(SN,\lambda)+\hat{J}(SN,\bar{\lambda}))\\
&= \frac{2\bar{S}(27)}{q^2}\hat{J}(SN,\lambda),
\end{aligned}
$$

where the last equality holds because

$$
\hat{J}(SN,\lambda) = \hat{J}(S^qN,\lambda^q) = \hat{J}(SN,\bar{\lambda}).
$$

We now apply Theorem 1.1 with $A = \bar{S}$, $C = S\phi$, $x = \frac{1}{4}$ to obtain

$$ {}_3F_2\left(\begin{matrix}\bar{S}, S^3, S \\ S^2, S\phi\end{matrix}\,\middle|\,\frac{1}{4}\right) = \frac{S(4)\phi(3)}{q} - \frac{2\bar{S}(27)}{q^2}\hat{J}(SN,\lambda)Z_1 $$

$$ = \frac{S(4)\phi(3)}{q} - \frac{2\phi(-1)S\left(\frac{-4}{27}\right)}{q^2}\,\frac{G(S^3)G(\bar{S}^2\phi)\hat{G}(S^2\phi N)\hat{G}(\bar{S}N)\hat{G}(\lambda)}{G(\bar{S})G(S^2\phi)\hat{G}(\lambda SN)\hat{G}(\phi N)}. \tag{3.5} $$

By the Davenport–Hasse lifting theorem [1, Theorem 11.5.2], $\hat{G}(\chi N) = -G(\chi)^2$ for any character χ on $\mathbf{F}_q$. Also, $\hat{G}(\lambda) = q$, by [1, Theorem 11.6.1]. Thus (3.5) becomes

$$ {}_3F_2\left(\begin{matrix}\bar{S}, S^3, S \\ S^2, S\phi\end{matrix}\,\middle|\,\frac{1}{4}\right) = \frac{S(4)\phi(3)}{q} + \frac{2\phi(-1)S\left(\frac{-4}{27}\right)G(S^3)G(\bar{S})}{q\hat{G}(\lambda SN)} $$

$$ = \frac{S(4)\phi(-1)}{q}(-1 + 2Q_1), $$

where

$$ Q_1 := \frac{\bar{S}(-27)G(S^3)G(\bar{S})}{\hat{G}(\lambda SN)}. \tag{3.6} $$

It remains to prove that $Q_1 = 1$.

Since $q \equiv 2 (\mathrm{mod}\ 3)$,

$$ \hat{G}(\lambda SN) = \hat{G}(\lambda^q S^q N) = \hat{G}(\bar{\lambda} SN), $$

so by (1.6),

$$ Q_1^2 = \frac{\bar{S}^2(27)G(S^3)^2 G(\bar{S})^2}{\hat{G}(\lambda SN)\hat{G}(\bar{\lambda}SN)} = 1. $$

Thus $Q_1 = \pm 1$. To determine the correct sign, first consider the case $S = \phi$. By [1, Theorem 11.6.1] with $m = 6$,

$$ \hat{G}(\lambda SN) = \hat{G}(\lambda\phi N) = -\phi(-1)q. $$

Thus, from (3.6), $Q_1 = -\phi(-27) = 1$ when $S = \phi$. Since $\mathrm{ord}(S) \notin \{1, 3, 4\}$, we may now assume that $\mathrm{ord}(S) \geq 5$.

Choose a character χ on $\mathbf{F}_q$ of order $q - 1$ in such a way that

$$ \bar{S} = \chi^w, \qquad \text{with } 1 \leq w \leq \frac{q-1}{5}. $$

Let $v(w)$ denote the fraction

$$ v(w) = \frac{(-27)^w\left(w + \frac{q-2}{3}\right)!\left(w + \frac{2q-1}{3}\right)!}{(q-1-w)!(3w)!}, $$

reduced to lowest terms. To show that the expression Q_1 in (3.6) equals 1, it suffices by Stickelberger's congruence [1, Theorem 11.2.1] to prove that

$$v(m) \equiv -1 (\mathrm{mod}\, p), \qquad 0 \le m \le \frac{q-1}{5}. \tag{3.7}$$

(To see more clearly the connection with Stickelberger's congruence, simplify the left side of (3.7) using Anton's congruence [6, (2)].) We have

$$v(0) = \frac{\left(\frac{q-2}{3}\right)!\left(\frac{2q-1}{3}\right)!}{(q-1)!} \equiv \frac{-\left\{(q-1)(q-2)\dots\left(q-\frac{q-2}{3}\right)\right\}\left(\frac{2q-1}{3}\right)!}{(q-1)!}$$

$$\equiv -\frac{(q-1)!}{(q-1)!} = -1 (\mathrm{mod}\, p).$$

Now (3.7) follows easily by induction on m, so the proof that $Q_1 = 1$ is complete.

4. Proof of Theorem 1.4

Let $\mathrm{ord}(C) \notin \{2, 4\}$. We first consider the case $q \equiv 1 (\mathrm{mod}\, 4)$, so that there exists a character χ on $\mathbf{F}_q$ of order 4.

By [7, Corollary 4.30], if S, $T\phi$, and $\bar{S}T^2\phi$ are all nontrivial,

$${}_3F_2\left(\begin{matrix}\phi, S, T^2 \\ T^2\phi, \bar{S}T^2\end{matrix}\middle| -1\right) = \frac{S\phi(-1)}{q} + ST(-1)\,{}_3F_2\left(\begin{matrix}T\phi, T, \bar{S}T^2\phi \\ T^2\phi, \bar{S}T^2\end{matrix}\middle| 1\right). \tag{4.1}$$

Apply (4.1) with $S = C\phi$, $T = C\chi$ to get

$${}_3F_2\left(\begin{matrix}\phi, C\phi, C^2\phi \\ C^2, C\end{matrix}\middle| -1\right) = \frac{C(-1)}{q} + \chi(-1)\,{}_3F_2\left(\begin{matrix}C\bar{\chi}, C\chi, C\phi \\ C^2, C\end{matrix}\middle| 1\right). \tag{4.2}$$

Thus

$${}_3F_2\left(\begin{matrix}\phi, C^2\phi, C\phi \\ C^2, C\end{matrix}\middle| -1\right) = \frac{1}{q} + C\chi(-1)\,{}_3F_2\left(\begin{matrix}C\bar{\chi}, C\phi, C\chi \\ C, C^2\end{matrix}\middle| 1\right), \tag{4.3}$$

because by [7, Theorem 3.20(ii)], the effect of transposing the two rightmost numerator parameters in the ${}_3F_2$ on the left side of (4.2) is to multiply that ${}_3F_2$ times the factor $C(-1)$.

By [7, Theorem 4.38(i)],

$${}_3F_2\left(\begin{matrix}C\bar{\chi}, C\phi, C\chi \\ C, C^2\end{matrix}\middle| 1\right) = \begin{cases} 0, & \text{if } C\chi \text{ is not a square} \\ \frac{J(D,\phi)J(\bar{D}\chi,\phi)+J(D\phi,\phi)J(\bar{D}\chi\phi,\phi)}{q^2}, & \text{if } C\chi = D^2. \end{cases} \tag{4.4}$$

The desired result in the case $q \equiv 1 (\mathrm{mod}\, 4)$ follows easily from (4.3) and (4.4).

Now let $q \equiv 3 \pmod 4$, and let β, η be characters on $\mathbf{F}_{q^2}$ with $\mathrm{ord}(\beta) = 8$, $\eta^2 = CN$. Write $\alpha = \beta^2$.

By [7, (4.15)],

$$ {}_2\hat{F}_1\left(\begin{matrix}\alpha, \bar{\alpha}\\ CN\end{matrix}\,\middle|\,\frac{1}{2}\right) = \frac{\alpha(2)}{q^2}(\hat{J}(\beta\eta, \bar{\alpha}) + \hat{J}(\beta^5\eta, \bar{\alpha})) = \frac{\alpha(2)}{\hat{G}(\alpha)} Q_2, \tag{4.5} $$

where

$$ Q_2 := \frac{\hat{G}(\beta\eta)}{\hat{G}(\bar{\beta}\eta)} + \frac{\hat{G}(\beta^5\eta)}{\hat{G}(\bar{\beta}^5\eta)}. \tag{4.6} $$

By Theorem 1.2 with $x = -1$ and $A = \phi$,

$$ \begin{aligned} {}_3F_2\left(\begin{matrix}\phi, C^2\phi, C\phi\\ C^2, C\end{matrix}\,\middle|\,-1\right) &= \frac{C(-1)\phi(2)}{q} - \frac{\alpha(2)}{\hat{G}(\alpha)} Q_2 Z_2 = \frac{C(-1)\phi(2)}{q} \\ &\quad - \frac{\bar{C}(-4)\phi(2)\alpha(2)Q_2}{\hat{G}(\alpha)} \frac{G(C^2\phi)G(\bar{C}\phi)\hat{G}(\phi N)\hat{G}(C\phi N)}{G(\phi)G(C\phi)\hat{G}(\alpha CN)\hat{G}(\bar{\alpha})} \\ &= \frac{C(-1)\phi(2)}{q} - \frac{Q_2 Q_3}{q}, \end{aligned} \tag{4.7} $$

where

$$ Q_3 := \frac{\bar{C}(4)\phi(-2)\alpha(2)G(C^2\phi)G(\phi)}{\hat{G}(\alpha CN)}. \tag{4.8} $$

We proceed to simplify Q_2. First suppose that $C(-1) = \phi(2) = 1$. Then $q \equiv 7 \pmod 8$ and $C^{(q-1)/2} = \varepsilon$, so $\eta^q = \eta$. Thus

$$ \hat{G}(\beta\eta) = \hat{G}(\beta^q\eta^q) = \hat{G}(\bar{\beta}\eta), $$

so the first term in (4.6) equals 1. The second term also equals 1, so $Q_2 = 2$. Next suppose that $C(-1) = \phi(2) = -1$. Then $\eta^q = \beta^4\eta$ and $q \equiv 3 \pmod 8$, and it follows similarly that $Q_2 = 2$. Next suppose that $C(-1) = -\phi(2)$. Then $\hat{G}(\beta\eta) = \hat{G}(\beta^3\eta)$ and $\hat{G}(\bar{\beta}\eta) = \hat{G}(\bar{\beta}^3\eta)$. It follows in this case that

$$ Q_2 = 2\,\mathrm{Re}\,\frac{\hat{G}(\beta\eta)}{\hat{G}(\bar{\beta}\eta)}. $$

Since $\hat{G}(\bar{\alpha}) = \phi(2)q$ by [1, Theorem 11.6.1], we have shown that

$$ Q_2 = \begin{cases} 2, & \text{if } C(-1) = \phi(2) \\ \frac{2\phi(2)}{q}\,\mathrm{Re}\,\hat{J}(\beta\eta, \bar{\alpha}), & \text{if } C(-1) = -\phi(2). \end{cases} \tag{4.9} $$

In view of (4.7) and (4.9), it remains to show that $Q_3 = 1$.

Since $q \equiv 3 \pmod 4$,

$$\hat{G}(\bar{\alpha}CN) = \hat{G}(\bar{\alpha}^q C^q N) = \hat{G}(\alpha CN),$$

so by (1.5) and (4.8),

$$Q_3^2 = \frac{\alpha(4)\bar{C}^2(4)G(C^2\phi)^2 G(\phi)^2}{\hat{G}(\alpha CN)\hat{G}(\bar{\alpha}CN)} = 1.$$

Thus $Q_3 = \pm 1$.

Since $q \equiv 3 \pmod 4$, we have $\phi(-1) = -1$, and the restriction of α to $\mathbf{F}_q$ is trivial. Thus (4.8) becomes

$$Q_3 = -\frac{\bar{C}^2\phi(2)G(C^2\phi)G(\phi)}{\hat{G}(\alpha CN)}. \tag{4.10}$$

Let χ be a character on $\mathbf{F}_q$ of order $q-1$, chosen so that

$$\bar{C}^2\phi = \chi^w, \qquad \text{with } 1 \le w \le \frac{q-1}{2}.$$

Since w must be odd, we write

$$w = 2k+1, \qquad \text{with } 0 \le k \le \frac{q-3}{4}.$$

Let $u(k)$ denote the fraction

$$u(k) = \frac{2^{2k+1}k!\left(k+\frac{q+1}{2}\right)!}{(2k+1)!\left(\frac{q-1}{2}\right)!},$$

reduced to lowest terms. To show that the expression Q_3 in (4.10) equals 1, it suffices again by congruences of Stickelberger [1, Theorem 11.2.1] and Anton [6, (2)] to prove that

$$u(k) \equiv 1 \pmod p, \qquad 0 \le k \le \frac{q-3}{4}.$$

This congruence is trivially true for $k=0$, and it follows for general k by induction. This completes the proof that $Q_3 = 1$.

References

[1] B. C. Berndt, R. J. Evans, and K. S. Williams, Gauss and Jacobi sums, Wiley-Interscience, New York, 1998.

[2] A. Erdélyi, et al., Higher Transcendental Functions, Vol. I, McGraw-Hill, New York, 1953.

[3] R. J. Evans and J. Greene, Clausen's theorem and hypergeometric functions over finite fields, Finite Fields Appl. **15** (2009), 97–109.

[4] R. J. Evans and F. Lam, Special values of hypergeometric functions over finite fields, Ramanujan J. **19** (2009), 151–162.

[5] R. J. Evans, J. Pulham, and J. Sheehan, On the number of complete subgraphs contained in certain graphs, J. Combin. Theory Ser. B **30** (1981), 364–371.

[6] A. Granville, Arithmetic Properties of Binomial Coefficients I: Binomial coefficients modulo prime powers, Canadian Math. Soc. Conference Proceedings, vol 20 (1997), pp. 253–275.

[7] J. Greene, Hypergeometric functions over finite fields, Trans. Amer. Math. Soc. **301** (1987), 77–101.

[8] J. Greene and D. Stanton, A character sum evaluation and Gaussian hypergeometric series, J. Number Theory **23** (1986), 136–148.

[9] M. Koike, Hypergeometric series over finite fields and Apéry numbers, Hiroshima Math. J. **22** (1992), 461–467.

[10] K. Ono, Values of Gaussian hypergeometric series, Trans. Amer. Math. Soc. **350** (1998), 1205–1223.

[11] K. Ono, The web of modularity: Arithmetic of the coefficients of modular forms and q-series, CBMS No. 102, Amer. Math. Soc., Providence, RI, 2004.

Ron Evans
Department of Mathematics
University of California at San Diego
La Jolla, CA 92093-0112*, USA*
E-mail: revans@ucsd.edu

John Greene
Department of Mathematics and Statistics
University of Minnesota–Duluth
Duluth, MN 55812*, USA*
E-mail: jgreene@d.umn.edu