Classification of certain types of maximal matrix subalgebras

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Abstract

Let $\mathcal{M}_n(K)$ denote the algebra of $n \times n$ matrices over a field $K$ of characteristic zero. A nonunital subalgebra $\mathcal{N} \subset \mathcal{M}_n(K)$ will be called a nonunital intersection if $\mathcal{N}$ is the intersection of two unital subalgebras of $\mathcal{M}_n(K)$. Appealing to recent work of Agore, we show that for $n \geq 3$, the dimension (over $K$) of a nonunital intersection is at most $(n-1)(n-2)$, and we completely classify the nonunital intersections of maximum dimension $(n-1)(n-2)$. We also classify the unital subalgebras of maximum dimension properly contained in a parabolic subalgebra of maximum dimension in $\mathcal{M}_n(K)$.

1 Introduction

Let $\mathcal{M}_n(F)$ denote the algebra of $n \times n$ matrices over a field $F$. For some interesting sets $\Lambda$ of subspaces $S \subset \mathcal{M}_n(F)$, those $S \in \Lambda$ of maximum dimension over $F$ have been completely classified. For example, a theorem of Gerstenhaber and Serezhkin [7, Theorem 1] states that when $\Lambda$ is the set of subspaces $S \subset \mathcal{M}_n(F)$ for which every matrix in $S$ is nilpotent, then each $S \in \Lambda$ of maximum dimension is conjugate to the algebra of all strictly upper triangular matrices in $\mathcal{M}_n(F)$. For another example, it is shown in [1, Prop. 2.5] that when $\Lambda$ is the set of proper unital subalgebras $S \subset \mathcal{M}_n(F)$ and $F$ is an algebraically closed field of characteristic zero, then each $S \in \Lambda$ of maximum dimension is a parabolic subalgebra of maximum dimension in $\mathcal{M}_n(F)$.

The goal of this paper is to classify the elements in $\Lambda$ of maximum dimension in the cases $\Lambda = \Gamma$ and $\Lambda = \Omega$, where the sets $\Gamma$ and $\Omega$ are defined below. First we need some definitions.

Write $\mathcal{M} = \mathcal{M}_n = \mathcal{M}_n(K)$, where $K$ is a field of characteristic zero. (It would be interesting to know if this restriction on the characteristic can be relaxed for the results in this paper.) In the spirit of [3, p. viii], we define a subalgebra of $\mathcal{M}$ to be a vector subspace of $\mathcal{M}$ over $K$ closed under the multiplication of $\mathcal{M}$ (cf. [3, p. 2]); thus a subalgebra need not have a unity, and the unity of a unital subalgebra need not be a unity of the parent algebra. Subalgebras $\mathcal{A}, \mathcal{B} \subset \mathcal{M}$ are said to be similar if $\mathcal{A} = \{S^{-1}BS : B \in \mathcal{B}\}$ for some invertible $S \in \mathcal{M}$.

In Isaac’s text [4, p. 161], every ring is required to have a unity, but the unity in a subring need not be the same as the unity in its parent ring. Under
this definition, a ring may have subrings whose intersection is not a subring. This motivated us to study examples of pairs of unital subalgebras in $\mathcal{M}$ whose intersection $\mathcal{N}$ is nonunital. We call such $\mathcal{N}$ a nonunital intersection and we let $\Gamma$ denote the set of all nonunital intersections $\mathcal{N} \subset \mathcal{M}$. Note that $\Gamma$ is closed under transposition and conjugation, i.e., if $\mathcal{N} \in \Gamma$, then $\mathcal{N}^T \in \Gamma$ and $S^{-1}\mathcal{N}S \in \Gamma$ for any invertible $S \in \mathcal{M}$.

In order to define $\Omega$, we need to establish additional notation. Let $\mathcal{M}[R_n]$ denote the subalgebra of $\mathcal{M}$ consisting of those matrices whose $n$-th row is zero. Similarly, $\mathcal{M}[R_n, C_n]$ indicates that the $n$-th row and $n$-th column are zero, etc. For $1 \leq i, j \leq n$, let $E_{i,j}$ denote the elementary matrix in $\mathcal{M}$ with a single entry 1 in row $i$, column $j$, and 0 in each of the other $n^2 - 1$ positions. The identity matrix in $\mathcal{M}$ will be denoted by $I$. For the maximal parabolic subalgebra $\mathcal{P} := \mathcal{M}[R_n] + KE_{n,n}$ in $\mathcal{M}$, define $\Omega$ to be the set of proper subalgebras $\mathcal{B}$ of $\mathcal{P}$ with $\mathcal{B} \neq \mathcal{M}[R_n]$.

We now describe Theorems 3.1–3.3, our main results. Theorem 3.1 shows that $\dim \mathcal{N} \leq (n - 1)(n - 2)$ for each $\mathcal{N} \in \Gamma$. Theorem 3.2 shows that up to similarity, $\mathcal{W} := \mathcal{M}[R_n, R_{n-1}, C_n]$ and $\mathcal{W}^T := \mathcal{M}[R_n, C_{n-1}, C_n]$ are the only subalgebras in $\Gamma$ having maximum dimension $(n - 1)(n - 2)$. In Theorem 3.3, we show that $\dim \mathcal{B} \leq n^2 - 2n + 3$ for each $\mathcal{B} \in \Omega$, and we classify all $\mathcal{B} \in \Omega$ of maximum dimension $n^2 - 2n + 3$.

The proofs of our theorems depend on four lemmas, which are proved in Section 2. Lemma 2.1 shows that $\mathcal{W}$ (and hence also $\mathcal{W}^T$) is a nonunital intersection of dimension $(n - 1)(n - 2)$ when $n \geq 3$. Lemmas 2.2 and 2.3 show that $\dim \mathcal{L} \leq n(n - 1)$ for any nonunital subalgebra $\mathcal{L} \subset \mathcal{M}$, and when equality holds, $\mathcal{L}$ must be similar to $\mathcal{M}[R_n]$ or $\mathcal{M}[C_n]$. (Thus if $\Lambda$ denotes the set of nonunital subalgebras $\mathcal{L} \subset \mathcal{M}$, Lemmas 2.2 and 2.3 classify those $\mathcal{L} \in \Lambda$ of maximum dimension.) Lemma 2.4 shows that if $\mathcal{U} \subset \mathcal{M}$ is a subalgebra with unity different from $I$, then some conjugate of $\mathcal{U}$ is contained in $\mathcal{M}[R_n, C_n]$.

## 2 Lemmas

Recall the definition $\mathcal{W} := \mathcal{M}[R_n, R_{n-1}, C_n]$.

**Lemma 2.1.** For $n \geq 3$, $\mathcal{W} \in \Gamma$ and $\dim \mathcal{W} = (n - 1)(n - 2)$.

**Proof.** For $n > 1$, define $A \in \mathcal{M}$ by $A = I + E_{n,n-1}$. Note that $A^{-1} = I - E_{n,n-1}$. A computation shows that for $M \in \mathcal{M}[R_n, C_n]$, the conjugate
$AMA^{-1}$ is obtained from $M$ by replacing the (zero) bottom row of $M$ by the $(n-1)$-th row of $M$. Since the bottom two rows of $AMA^{-1}$ are identical, it follows that

$$AMA^{-1} \in \mathcal{M}[R_n, C_n] \cap \mathcal{M}[R_n, C_n]A^{-1}$$

if and only if $AMA^{-1} \in \mathcal{W}$.

Since $\mathcal{W} = A^{-1}WA$, this shows that $\mathcal{W}$ is the intersection of the unital subalgebras $A^{-1}\mathcal{M}[R_n, C_n]A$ and $\mathcal{M}[R_n, C_n]$. To see that $\mathcal{W}$ is nonunital, note that $E_{1,n-1}$ is a nonzero matrix in $\mathcal{W}$ for which $E_{1,n-1}\mathcal{W}$ is the zero matrix for each $W \in \mathcal{W}$; thus $\mathcal{W}$ cannot have a right identity, so $\mathcal{W} \in \Gamma$. Finally, it follows from the definition of $\mathcal{W}$ that $\dim \mathcal{W} = (n-1)(n-2)$.

**Remark:** The same proof shows that $\mathcal{W} \in \Gamma$ when the field $K$ is replaced by an arbitrary ring $R$ with $1 \neq 0$. If $R$ happens to be commutative, then the dimension of the algebra $\mathcal{W}$ over $R$ is well defined [8, p. 483] and it equals $(n-1)(n-2)$.

**Lemma 2.2.** For any nonunital subalgebra $\mathcal{L} \subset \mathcal{M}$, $\dim \mathcal{L} \leq n(n-1)$.

**Proof.** If $\mathcal{L} + KI = \mathcal{M}$, then $\mathcal{L}$ would be a two-sided proper ideal of $\mathcal{M}$, contradicting the fact that $\mathcal{M}$ is a simple ring [8, p. 280]. Fergus Gaines [2, Lemma 4] proved that for any field $F$, the $F$-dimension of a proper unitary subalgebra of $\mathcal{M}_n(F)$ is at most $n^2 - n + 1$. (Agore [1, Cor. 2.6] proved this only for fields of characteristic zero.) Since $\mathcal{L} + KI$ is a proper subalgebra of $\mathcal{M}$ containing the unity $I$, it follows that

$$\dim \mathcal{L} = -1 + \dim (\mathcal{L} + KI) \leq n(n-1).$$

□

**Lemma 2.3.** Any nonunital subalgebra $\mathcal{L} \subset \mathcal{M}$ with $\dim \mathcal{L} = n(n-1)$ must be similar to either $\mathcal{M}[R_n]$ or $\mathcal{M}[C_n] = \mathcal{M}[R_n]^T$.

**Proof.** Consider the two parabolic subalgebras $\mathcal{P}, \mathcal{P}' \subset \mathcal{M}$ defined by

$$\mathcal{P} = \mathcal{P}_K = \mathcal{M}[R_n] + KE_{n,n}, \quad \mathcal{P}' = \mathcal{P}'_K = \mathcal{M}[C_1] + KE_{1,1}.$$ 

Note that $\mathcal{P}'$ is similar to the transpose $\mathcal{P}^T$. Since $\mathcal{L} + KI$ is a proper subalgebra of $\mathcal{M}$ of dimension $n(n-1)+1$, it follows from Agore [1, Prop. 2.5] that $\mathcal{L} + KI$ is similar to $\mathcal{P}$ or $\mathcal{P}'$, under the condition that $K$ is algebraically closed. However, Nolan Wallach [9] has proved that this condition can be
dropped; see the Appendix. Thus, replacing $L$ by a conjugate if necessary, we may assume that $L + KI = P$ or $L + KI = PT$. We will assume that $L + KI = P$, since the proof for $PT$ is essentially the same. It suffices to show that $L$ is similar to $M[R_n]$ or $M[C_1]$, since $M[C_1]$ is similar to $M[C_n]$.

Assume temporarily that each $L \in \mathcal{L}$ has all entries 0 in its upper left $(n-1) \times (n-1)$ corner. Then $n = 2$, because if $n \geq 3$, then every matrix in $P$ would have a zero entry in row 1, column 2, contradicting the definition of $P$. Since $\mathcal{L} \subset M_2[C_1]$ and both sides have dimension 2, we have $\mathcal{L} = M_2[C_1]$, which proves the theorem under our temporary assumption.

When the temporary assumption is false, there exists $L \in \mathcal{L}$ with the entry 1 in row $i$, column $j$ for some fixed pair $i, j$ with $1 \leq i, j \leq n-1$. Since $E_{i,i}$ and $E_{j,j}$ are in $P = \mathcal{L} + KI$ and $\mathcal{L}$ is a two-sided ideal of $P$, we have $E_{i,j} = E_{i,i}LE_{j,j} \in \mathcal{L}$. Consequently, $E_{a,b} = E_{a,i}E_{i,j}E_{j,b} \in \mathcal{L}$ for all pairs $a, b$ with $1 \leq a \leq n - 1$ and $1 \leq b \leq n$. Therefore

$$\mathcal{M}[R_n] = \sum_{a=1}^{n-1} \sum_{b=1}^{n} KE_{a,b} \subset \mathcal{L},$$

and since both $\mathcal{M}[R_n]$ and $\mathcal{L}$ have the same dimension $n(n-1)$, we conclude that $\mathcal{L} = \mathcal{M}[R_n]$. □

Remark: Any subalgebra $B \subset \mathcal{M}$ properly containing $\mathcal{M}[R_n]$ must also contain $I$. To see this, note that $B$ contains a nonzero matrix of the form

$$B := \sum_{i=1}^{n} c_i E_{n,i}, \quad c_i \in K.$$ 

If $c_j = 0$ for all $j < n$, then $E_{n,n} \in B$, so $I \in B$. On the other hand, if $c_j \neq 0$ for some $j < n$, then $E_{n,n} = c_j^{-1}BE_{j,n} \in B$, so again $I \in B$.

Lemma 2.4. Suppose that a subalgebra $U \subset \mathcal{M}$ has a unity $e \neq I$. Then $S^{-1}US \subset \mathcal{M}[R_n, C_n]$ for some invertible $S \in \mathcal{M}$.

Proof. Let $r$ be the rank of the matrix $e$. Note that $e$ is idempotent, so by [6, p. 27], there exists an invertible $S \in \mathcal{M}$ for which $S^{-1}eS = D_r$, where $D_r$ is a diagonal matrix with entries 1 in rows 1 through $r$, and entries 0 elsewhere. Replacing $U$ by $S^{-1}US$ if necessary, we may assume that $e = D_r$. Since $r \leq n - 1$, we have

$$U = e U e \subset e \mathcal{M} e = D_r \mathcal{M} D_r \subset D_{n-1} \mathcal{M} D_{n-1} = \mathcal{M}[R_n, C_n].$$

□
3 Theorems

Recall that $\Gamma$ is the set of all nonunital intersections in $\mathcal{M}$.

**Theorem 3.1.** If $\mathcal{N} \in \Gamma$, then $\dim \mathcal{N} \leq (n-1)(n-2)$.

*Proof.* Let $\mathcal{N} \in \Gamma$, so that $\mathcal{N} = \mathcal{U} \cap \mathcal{V}$ for some pair of unital subalgebras $\mathcal{U}, \mathcal{V} \subset \mathcal{M}$. Since $\mathcal{N}$ is nonunital, one of $\mathcal{U}, \mathcal{V}$, say $\mathcal{U}$, does not contain $I$. Thus $\mathcal{U}$ contains a unity $e \neq I$. Define $S$ as in Lemma 2.4. Replacing $\mathcal{U}, \mathcal{V}, \mathcal{N}$ by $S^{-1}\mathcal{U}S$, $S^{-1}\mathcal{V}S$, $S^{-1}\mathcal{N}S$, if necessary, we deduce from Lemma 2.4 that $\mathcal{U}$ is contained in $\mathcal{M}[R_n, C_n]$. Since $\mathcal{N}$ is a nonunital subalgebra of $\mathcal{U} \subset \mathcal{M}[R_n, C_n]$, it follows from Lemma 2.2 with $(n-1)$ in place of $n$ that $\dim \mathcal{N} \leq (n-1)(n-2)$. \qed

**Theorem 3.2.** Let $n \geq 3$. Then up to similarity, $\mathcal{W}$ and $\mathcal{W}^T$ are the only subalgebras of $\mathcal{M}$ in $\Gamma$ having dimension $(n-1)(n-2)$.

*Proof.* By Lemma 2.1, every subalgebra of $\mathcal{M}$ similar to $\mathcal{W}$ or $\mathcal{W}^T$ lies in $\Gamma$ and has dimension $(n-1)(n-2)$. Conversely, let $\mathcal{N} \in \Gamma$ with $\dim \mathcal{N} = (n-1)(n-2)$. We must show that $\mathcal{N}$ is similar to $\mathcal{W}$ or $\mathcal{W}^T$.

We may assume, as in the proof of Theorem 3.1, that $\mathcal{N}$ is a nonunital subalgebra of $\mathcal{M}[R_n, C_n]$. Let $\mathcal{L}$ be the subalgebra of $\mathcal{M}_{n-1}$ consisting of those matrices in the upper left $(n-1) \times (n-1)$ corners of the matrices in $\mathcal{N}$. Since $\dim \mathcal{L} = \dim \mathcal{N} = (n-1)(n-2)$, it follows from Lemma 2.3 that $\mathcal{L}$ is similar to $\mathcal{M}_{n-1}[R_{n-1}]$ or $\mathcal{M}_{n-1}[C_{n-1}]$. Thus $\mathcal{N}$ is similar to $\mathcal{W} = \mathcal{M}[R_n, R_{n-1}, C_n]$ or $\mathcal{W}^T = \mathcal{M}[R_n, C_{n-1}, C_n]$. \qed

Recall that $\Omega$ denotes the set of proper subalgebras $\mathcal{B} \neq \mathcal{M}[R_n]$ in $\mathcal{P}$.

**Theorem 3.3.** Let $\mathcal{B} \in \Omega$. Then $\dim \mathcal{B} \leq n^2 - 2n + 3$. If $\mathcal{B}$ has maximum dimension $n^2 - 2n + 3$, then $\mathcal{B}$ is similar to one of

$$\mathcal{M}E_{n,n} + \mathcal{M}[R_n, C_1] + KE_{1,1}, \quad \mathcal{M}E_{n,n} + \mathcal{M}[R_n, R_{n-1}] + KE_{n-1,n-1}.$$

*Proof.* Let $e \in \mathcal{M}$ denote the diagonal matrix of rank $n-1$ with entry 0 in row $n$ and entries 1 in the remaining rows. Because $e$ is a left identity in $\mathcal{M}[R_n]$ and $\mathcal{B}e \subset \mathcal{M}[R_n, C_n]$, it follows that $\mathcal{B}e$ is an algebra.

First suppose that $\mathcal{B}e = \mathcal{M}[R_n, C_n]$. Then $\mathcal{P} = \mathcal{C} + \mathcal{D}$, where

$$\mathcal{C} = \mathcal{B} + KE_{n,n}, \quad \mathcal{D} = \mathcal{M}[R_n]E_{n,n}.$$
We proceed to show that \( C \cap D \) is zero. Assume for the purpose of contradiction that there exists a nonzero matrix \( B \in C \cap D \). Then \( B \in B \). We have \( BB = D \), since the matrices in \( B \) have all possible submatrices in their upper left \((n - 1)\) by \((n - 1)\) corners. Thus \( D \subseteq B \subseteq C \), which implies that \( M[R_n] \subseteq B \) and \( P = C = B + KE_{n,n} \). If \( KE_{n,n} \subseteq B \), then \( B = P \), and if \( KE_{n,n} \) is not contained in \( B \), then \( B = M[R_n] \); either case contradicts the fact that \( B \in \Omega \).

Since \( C \cap D \) is zero,

\[
\dim B \leq \dim C = \dim P - \dim D = (n^2 - n + 1) - (n - 1) = n^2 - 2n + 2.
\]

Thus

\[
\dim B < n^2 - 2n + 3,
\]

so the desired upper bound for \( \dim B \) holds when \( B = M[R_n, C_n] \).

Next suppose that \( B \) is a proper subalgebra of \( M[R_n, C_n] \). We proceed to show that

\[
d := \dim B \leq (n - 1)(n - 2) + 1,
\]

by showing that

\[
\dim L \leq (n - 1)(n - 2) + 1
\]

for every proper subalgebra \( L \) of \( M_{n-1} \). If \( L \) is nonunital, then

\[
\dim L \leq (n - 1)(n - 2) < (n - 1)(n - 2) + 1
\]

by Lemma 2.2 (with \( n - 1 \) in place of \( n \)). If \( L \) contains a unit different from the identity of \( M_{n-1} \), then by Lemma 2.4 (with \( L \) in place of \( U \)),

\[
\dim L \leq \dim M[R_{n-1}, C_{n-1}] = (n - 2)^2 < (n - 1)(n - 2) + 1.
\]

If \( L \) contains the identity of \( M_{n-1} \), then by [2, Lemma 4],

\[
\dim L \leq (n - 1)(n - 2) + 1.
\]

This completes the demonstration that \( d \leq (n - 1)(n - 2) + 1 \).

Let \( B_1 e, B_2 e, \ldots, B_d e \) be a basis for \( B e \), with \( B_i \in B \). Since \( B \) is a subspace of the vector space spanned by the \( d + n \) matrices

\[
B_1, \ldots, B_d, E_{1,n}, \ldots, E_{n,n},
\]
it follows that
\[ \dim B \leq d + n \leq (n - 1)(n - 2) + 1 + n = n^2 - 2n + 3. \]
Thus the desired upper bound for \( \dim B \) holds in all cases.

The argument above shows that when we have the equality
\[ \dim B = d + n = (n - 1)(n - 2) + 1 + n = n^2 - 2n + 3, \]
then
\[ B = Be + ME_{n,n}. \]
Moreover, from the equality \( d = \dim Be = (n - 1)(n - 2) + 1 \), it follows from Agore [1, Prop. 2.5] (again appealing to the Appendix to dispense with the condition of algebraic closure) that there is an invertible matrix \( S \) in the set \( M[R_n, C_n] + E_{n,n} \) such that \( S^{-1}BeS \) is equal to one of
\[ M[R_n, C_n, C_1] + KE_{1,1}, \quad M[R_n, C_n, R_{n-1}] + KE_{n-1,n-1}. \]
Since \( S^{-1}ME_{n,n}S = ME_{n,n} \), we achieve the desired classification of \( \Omega \).

4 Appendix

Let \( F \) be a field of characteristic 0 with algebraic closure \( \overline{F} \). Given a proper subalgebra \( C \subset M_n(F) \) of maximum dimension, Agore [1, Prop. 2.5] proved that the \( \overline{F} \)-span of \( C \) is similar over \( \overline{F} \) to the \( \overline{F} \)-span of some parabolic subalgebra \( D \) of maximum dimension in \( M_n(F) \). The purpose of this Appendix is to deduce that \( C \) is similar over \( F \) to \( D \).

Lemma 4.1. (Wallach) Let \( A \) be a subspace of \( M_n(F) \) of dimension \( n - 1 \) such that \( A \otimes_F \overline{F} \) has basis of one of the following two forms:

a) \( x_1 \otimes \lambda_1, x_2 \otimes \lambda_1, \ldots, x_{n-1} \otimes \lambda_1, \) with \( \lambda_1 \in (F^n)^*, x_j \in F^n \) and \( \lambda_1(x_j) = 0, \)

b) \( x_1 \otimes \lambda_1, x_1 \otimes \lambda_2, \ldots, x_1 \otimes \lambda_{n-1}, \) with \( \lambda_j \in (F^n)^*, x_1 \in F^n \) and \( \lambda_j(x_1) = 0. \)

Then in case a) \( A \) is \( F \)-conjugate (i.e., under \( GL(n, F) \)) to the span of the matrices \( E_{i,n} \) with \( i = 1, \ldots, n - 1, \) and in case b) \( A \) is \( F \)-conjugate to the span of the matrices \( E_{n,i} \) with \( i = 1, \ldots, n - 1. \)

Proof. In either case, if \( X, Y \in A \) then \( XY = 0 \) and \( X \) has rank 1. For \( X \) of rank 1, we have \( XF^n = Fy \) for some \( y \neq 0. \) Thus there exists \( \mu \in (F^n)^* \)
with \( Xz = \mu(z)y = (y \otimes \mu)(z) \) for all \( z \). We conclude that \( \mathcal{A} \) has a basis over \( F \) of the form \( X_i = y_i \otimes \mu_i \) for \( i = 1, \ldots, n - 1 \).

We now assume that case a) is true (the argument for the other case is essentially the same). In case a), there exists \( z \in \bar{F}^n \) such that

\[
\{X_1(z), \ldots, X_{n-1}(z)\}
\]

is linearly independent over \( \bar{F} \). This implies that

\[
\mu_1(z) \cdots \mu_{n-1}(z) y_1 \wedge \cdots \wedge y_{n-1} \neq 0.
\]

Thus \( y_1, \ldots, y_{n-1} \) are linearly independent. But \( 0 = X_iX_j = \mu_i(y_j)y_i \otimes \mu_j \). Thus \( \mu_i(y_j) = 0 \) for all \( j = 1, \ldots, n - 1 \). Let \( \nu \) be a non-zero element of \((F^n)^*\) such that \( \nu(y_i) = 0 \) for all \( i = 1, \ldots, n - 1 \). Then \( \nu \) is unique up to non-zero scalar multiple. Thus \( y_i \otimes \nu, i = 1, \ldots, n - 1 \) is an \( F \)-basis of \( \mathcal{A} \). There exists \( g \in \text{GL}(n, F) \) such that if \( e_1, \ldots, e_n \) is the standard basis and \( \xi_1, \ldots, \xi_n \) is the dual basis then \( gy_i = e_i \) and \( \nu \circ g = \xi_n \). This completes the proof in case a).

\[ \text{Proposition 4.2. (Wallach) Suppose that } \mathcal{L} \subset \mathcal{M}_n(F) \text{ is a subalgebra such that } \mathcal{L} \otimes_F \bar{F} \text{ is either:}
\]

a) conjugate to the parabolic subalgebra \( \mathcal{P}_F \),

b) conjugate to the parabolic subalgebra \( (\mathcal{P}_F)^T \).

\[ \text{In case a) } \mathcal{L} \text{ is } F \text{-conjugate to } \mathcal{P}_F. \text{ In case b) } \mathcal{L} \text{ is } F \text{-conjugate to } \mathcal{P}_F^T. \]

\[ \text{Proof. We just do case a) as case b) is proved in the same way. We look upon } \mathcal{L} \text{ as a Lie algebra over } F. \text{ Then Levi's theorem [5, p. 91] implies that}
\]

\[ \mathcal{L} = S \oplus R \text{ with } S \text{ a semi-simple Lie algebra and } R \text{ the radical (the maximal solvable ideal). Thus } \mathcal{L} \otimes_F \bar{F} = S \otimes_F \bar{F} \oplus R \otimes_F \bar{F}. \text{ Therefore } R \otimes_F \bar{F} \text{ is the radical of } \mathcal{L} \otimes_F \bar{F}. \text{ If we conjugate } \mathcal{L} \otimes_F \bar{F} \text{ to } \mathcal{P}_F \text{ via } h \in \text{GL}(n, \bar{F}), \text{ then we see that}
\]

\[ h[R \otimes_F \bar{F}, R \otimes_F \bar{F}]h^{-1}
\]

has basis \( E_{i,n}, i = 1, \ldots, n - 1 \). Thus hypothesis a) of Lemma 4.1 is satisfied for \( \mathcal{A} = [R, R] \). There exists therefore \( g \in \text{GL}(n, F) \) such that \( g\mathcal{A}g^{-1} \) has basis \( E_{i,n}, i = 1, \ldots, n - 1 \). Assume that we have replaced \( \mathcal{L} \) with \( g\mathcal{L}g^{-1} \).

Then \( \mathcal{A} \) has basis \( E_{i,n}, i = 1, \ldots, n - 1 \). Since \( [\mathcal{L}, \mathcal{A}] \subset \mathcal{A} \) and \( \mathcal{P}_F \) is exactly the set of elements \( X \) of \( \mathcal{M}_n(F) \) such that \( [X, \mathcal{A}] \subset \mathcal{A} \), we have \( \mathcal{L} \subset \mathcal{P}_F \).

Thus \( \mathcal{L} = \mathcal{P}_F \), as both sides have the same dimension. \( \square \)
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