Representation of p-adic $\mathrm{GL}(n)$

Finn McGlade

UCSD

Administration

Welcome to RTG Grad Colloquium!

- Student run (organisers Kiran[1], Nandagopal and myself).
- The goal is to get graduate students in the RTG (algebra, number theory, algebraic geometry) group to give talks about whatever.

  We don't really have expectations for the talks, but we'd like to encourage early career grad students to participate.
- The target audience should be graduate students in the RTG group.
- The talks should be self contained and can be reasonably technical.

The seminar will not be able to run if we don't have volunteer speakers. So far there are two volunteers, but they cannot speak until November 18th.

- 10/21 needs filling.
- 10/28 needs filling.
- 11/04 needs filling.
- 11/11 needs filling.

[1] Kiran is not a student

The p-adic numbers

Definition

Take $p \geq 2$ prime. The p-adic numbers $\mathbb{Q}_p$ is the field of fractions of

$$\mathbb{Z}_p := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}.$$

topologized so that $\{p^n\mathbb{Z}_p\}_{n\in\mathbb{Z}}$ is a neighbourhood basis of 0.

A p-adic number $a \in \mathbb{Q}_p$ is a p-adic decimal expansion

$$a_{-\ell}a_{-\ell+1}\cdots a_0.a_1\cdots = a_{-\ell}p^{-\ell} + a_{-\ell+1}p^{-\ell+1} + \cdots + a_0p^0 + a_1p^1 + \cdots$$

where $\ell \in \mathbb{Z}_{\geq 0}$ and $a_{-\ell}, a_{-\ell+1}, \ldots \{0, 1, \ldots, p-1\}$.
Compare this with p-adic decimal expansion of $b \in \mathbb{R}$,

$$\pm b_\ell b_{\ell-1}\cdots b_0.b_{-1}\cdots = \pm(b_\ell p^\ell + b_{\ell-1}p^{\ell-1} + \cdots + b_0p^0 + b_{-1}p^{-1} + \cdots)$$

where $\ell \in \mathbb{Z}_{\geq 0}$ and $b_\ell, b_{\ell-1}, \ldots \{0, 1, \ldots, p-1\}$.

p-adic Arithmetic

In $\mathbb{Q}_p$ the addition and multiplication of p-adic decimal expansions

$$a_{-\ell}a_{-\ell+1}\cdots a_0.a_1\cdots$$

is similar to in $\mathbb{R}$, except the "carry operations" move left to right.

Example

In $\mathbb{Q}_3$, $\begin{array}{r} 1.000\ldots \\ +\ 2.222\ldots \\ \hline 0.000\ldots \end{array}$ whereas in base 3 real arithmetic $\begin{array}{r} 1.000\ldots \\ +\ 2.222\ldots \\ \hline 10.222\ldots \end{array}$

This is no small difference, $\mathrm{Gal}(\overline{\mathbb{R}}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ whereas $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a huge infinite group with many non-abelian quotients.

Example (Jones-Roberts [JR06])

The number N_d of isomorphism classes of degree d-extensions of $\mathbb{Q}_2$.

d	2	3	4	5	7	8	10	13	15	16	17	21	24	25
N_d	7	2	59	2	2	1823	158	2	4	!!![a]	2	6	!!!	3

[a] the data base crashes

Smooth Representations of $\mathrm{GL}_n(\mathbb{Q}_p)$

Goal

The representation theory of $\mathrm{GL}_n(\mathbb{Q}_p)$ gives arithmetic information of $\mathbb{Q}_p$. We'll use $\mathrm{GL}_p(\mathbb{Q}_p)$ to study examples of non-abelian degree $p^3(p+1)$ extensions of $\mathbb{Q}_p$.

All representations are complex, V is always a complex vector space.

Definition

Let (π, V) be a complex $\mathrm{GL}_n(\mathbb{Q}_p)$-representation.
The representation (π, V) is *smooth* if (π, V) satisfies: if $v \in V$ then

$$\mathrm{Stab}_G(v) := \{g \in G \colon \pi(g)v = v\} \quad \text{is an open subgroup.}$$

Often, this condition is naturally true of the representations (π, V) occuring in the wild. The situation is a little funny because we're studying representations of the p-adic object $\mathrm{GL}_n(\mathbb{Q}_p)$ on a complex vectorspace V.

- There is no naive definition of "(π, V) algebraic" or "(π, V) rational".
- If V is smooth and irreducible such that $\dim V > 1$ then $\dim V = \infty$.

Principal Series for GL_2.

Let $B = \left\{ \begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix} : t_1, t_2 \in \mathbb{Q}_p^\times, x \in \mathbb{Q}_p \right\}$ and suppose

$$\mu \colon B \to \mathbb{C}^\times \quad \text{is a smooth character.}$$

Define

$$\mathcal{B}_\mu = \left\{ \begin{array}{c} \text{locally constant functions} \\ f \colon \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{C} \end{array} : \begin{array}{c} \text{if } x \in \mathrm{GL}_2(\mathbb{Q}_p) \text{ and } b \in B \\ \text{then } f(bx) = \mu(b) f(x) \end{array} \right\}.$$

Then $\mathcal{B}_\mu$ is a smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representation ρ_μ via right translations i.e. if $g, x \in \mathrm{GL}_2(\mathbb{Q}_p)$ and $f \in \mathcal{B}_\mu$ then

$$\rho_\mu(g) f(x) = f(xg).$$

With minor tweaking, *smooth induction* $\mathrm{Ind}_H^{\mathrm{GL}_n(\mathbb{Q}_p)}(\chi)$ makes sense for $H \leq \mathrm{GL}_n(\mathbb{Q}_p)$ any closed subgroup and $\chi \colon H \to \mathbb{C}$ a smooth character. But

$$(\rho_\mu, \mathcal{B}_\mu) = \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\mu)$$

is particularly special, in part because $B \backslash \mathrm{GL}_2(\mathbb{Q}_p)$ is compact.

Parabolic Induction.

$P \leq \mathrm{GL}_n(\mathbb{Q}_p)$ a closed subgroup such that $P\backslash\mathrm{GL}_n(\mathbb{Q}_p)$ is compact.

Definition

Let (σ, W) be a smooth representation of P. *The parabolic induction* $\mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{Q}_p)}\sigma$ is the $\mathrm{GL}_n(\mathbb{Q}_p)$-representation (X, Σ) where

$$X = \left\{ \begin{array}{c} \text{locally constant functions} \\ f\colon \mathrm{GL}_n(\mathbb{Q}_p) \to W \end{array} : \begin{array}{c} \text{if } x \in \mathrm{GL}_n(\mathbb{Q}_p) \text{ and } p \in P \\ \text{then } f(px) = \sigma(p)f(x) \end{array} \right\}$$

and $\Sigma\colon \mathrm{GL}_n(\mathbb{Q}_p) \to \mathrm{GL}(X)$ is given by right translations.

Let $P = \{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}\} = \{\begin{pmatrix} \mathrm{GL}_2(\mathbb{Q}_p) & \begin{matrix}\mathbb{Q}_p \\ \mathbb{Q}_p\end{matrix} \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}\} \simeq \mathbb{Q}_p^2 \rtimes (\mathrm{GL}_2(\mathbb{Q}_p) \times \mathbb{Q}_p^\times)$.

$P\backslash G$ is compact and if $\mu\colon B \to \mathbb{C}^\times$ is a smooth character then

$$\sigma\colon P \to \mathrm{GL}_2(\mathbb{Q}_p) \xrightarrow{\rho_\mu} \mathrm{GL}(\mathcal{B}_\mu) \quad \text{is a smooth representation of } P$$

and $\mathrm{Ind}_P^{\mathrm{GL}_3(\mathbb{Q}_p)}\sigma$ is smooth $\mathrm{GL}_3(\mathbb{Q}_p)$-representation.

Parabolic Induction.

$P \leq \mathrm{GL}_n(\mathbb{Q}_p)$ a closed subgroup such that $P\backslash\mathrm{GL}_n(\mathbb{Q}_p)$ is compact.

Definition

Let (σ, W) be a smooth representation of P. *The parabolic induction* $\mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{Q}_p)}\sigma$ is the $\mathrm{GL}_n(\mathbb{Q}_p)$-representation (X, Σ) where

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and $\Sigma\colon \mathrm{GL}_n(\mathbb{Q}_p) \to \mathrm{GL}(X)$ is given by right translations.

Question

Let (π, V) be an irreducible smooth representation of $\mathrm{GL}_n(\mathbb{Q}_p)$. Does there exists a proper parabolic subgroup $P < \mathrm{GL}_n(\mathbb{Q}_p)$ and a smooth P-representation (σ, W) such that π is isomorphic to a subrepresentation of $\mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{Q}_p)}\sigma$? Is every irreducible smooth (π, V) isomorphic to a subrepresentation of a non-trivial parabolic induction $\mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{Q}_p)}\sigma$?

Matrix Coefficients.

Let (π, V) be a smooth representation of GL_n. Write (π^*, V^*) for the $\mathrm{GL}_n(\mathbb{Q}_p)$ representation on $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ with

$$\langle \pi^*(g)\lambda, v\rangle := \langle \lambda, \pi(g^{-1})v\rangle \qquad v \in V, \quad \lambda \in V^* \quad g \in \mathrm{GL}_n(\mathbb{Q}_p).$$

The *smooth dual* $(\pi^\vee, V^\vee)$ *of* π is the $\mathrm{GL}_n(\mathbb{Q}_p)$-representation

$$V^\vee = \bigcup_{U \leq \mathrm{GL}_n(\mathbb{Q}_p) \text{ compact open}} (V^*)^U$$

where $(V^*)^U = \{\lambda \in V^* \colon \text{if } u \in U \text{ then } \pi^*(u)\lambda = \lambda\}$.

Theorem (Harish-Chandra [BZ76])

Let (π, V) be an irreducible smooth representation of $\mathrm{GL}_n(\mathbb{Q}_p)$. TFAE.

- *There does not exists a pair $(P, (\sigma, W))$ consisting of a proper parabolic subgroup $P < \mathrm{GL}_n(\mathbb{Q}_p)$ and smooth P-representation (σ, V) such that (π, V) is isomorphic to a subrepresentation of $\mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{Q}_p)}(\sigma)$.*
- *If $v \in V$ and $\lambda \in V^\vee$ then the function $\gamma_{v\otimes\lambda} \colon \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{C}$, $g \mapsto \langle \pi(g)v, \lambda\rangle$ is compactly supported modulo the center Z of $\mathrm{GL}_n(\mathbb{Q}_p)$.*

Supercuspidal Representations

Theorem (Harish-Chandra [BZ76])

Let (π, V) be an irreducible smooth representation of $\mathrm{GL}_n(\mathbb{Q}_p)$. TFAE.

- *There does not exists a pair $(P, (\sigma, W))$ consisting of a proper parabolic subgroup $P < \mathrm{GL}_n(\mathbb{Q}_p)$ and smooth P-representation (σ, V) such that (π, V) is isomorphic to a subrepresentation of $\mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{Q}_p)}(\sigma)$.*
- *If $v \in V$ and $\lambda \in V^\vee$ then the function $\gamma_{v\otimes\lambda} \colon \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{C}$, $g \mapsto \langle \pi(g)v, \lambda \rangle$ is compactly supported modulo the center Z of $\mathrm{GL}_n(\mathbb{Q}_p)$.*

Definition

An irreducible smooth $\mathrm{GL}_n(\mathbb{Q}_p)$-representation (π, V) is *supercuspidal* if the "matrix coefficients" $\{\gamma_{v\otimes\lambda} \colon (v, \lambda) \in V \times V^\vee\}$ of π are compactly supported modulo the center Z of $\mathrm{GL}_n(\mathbb{Q}_p)$.

There are a lot of supercuspidal representation of $\mathrm{GL}_n(\mathbb{Q}_p)$. Supercuspidal representation have an arithmetic intepretation, known as the Local Langlands Correspondence for $\mathrm{GL}_n(\mathbb{Q}_p)$.

The Local Langlands Correspondence for $\mathrm{GL}_n(\mathbb{Q}_p)$.

Definition

An irreducible smooth $\mathrm{GL}_n(\mathbb{Q}_p)$-representation (π, V) is *supercuspidal* if the "matrix coefficients" $\{\gamma_{v\otimes\lambda} \colon (v,\lambda) \in V \times V^\vee\}$ of π are compactly supported modulo the center Z of $\mathrm{GL}_n(\mathbb{Q}_p)$.

Theorem (Harris-Taylor, Henniart)

(Roughly), there is a canonical bijection $\pi \mapsto \rho_\pi$

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{supercuspidal representations} \\ \textit{of } \mathrm{GL}_n(\mathbb{Q}_p) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{continuous} \\ n\textit{-dimensional irreducible} \\ \mathbb{C}\textit{-representations of} \\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \textit{ up to equivalence.} \end{array} \right\}.$$

If (π, V) a supercupsidal of $\mathrm{GL}_n(\mathbb{Q}_p)$, then there is an continuous homomorphism $\rho_\pi \colon \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_n(\mathbb{C})$, well defined up to conjugacy on $\mathrm{GL}_n(\mathbb{C})$. In particular, there is a finite extension $E_\pi/\mathbb{Q}_p$ such that $\ker(\rho_\pi) = \mathrm{Gal}(\overline{\mathbb{Q}}_p/E_\pi)$ and ρ_π embedds $\mathrm{Gal}(E_\pi/\mathbb{Q}_p)$ into $\mathrm{GL}_n(\mathbb{C})$ as a finite subgroup.

The Local Langlands Correspondence for $\mathrm{GL}_n(\mathbb{Q}_p)$.

Theorem (Harris-Taylor, Henniart)

(Roughly), there is a canonical bijection $\pi \mapsto \rho_\pi$

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If (π, V) *a supercupsidal of* $\mathrm{GL}_n(\mathbb{Q}_p)$, *then there is a finite extension* $E_\pi/\mathbb{Q}_p$ *such that* $\ker(\rho_\pi) = \mathrm{Gal}(\overline{\mathbb{Q}}_p/E_\pi)$ *and* ρ_π *embedds* $\mathrm{Gal}(E_\pi/\mathbb{Q}_p)$ *into* $\mathrm{GL}_n(\mathbb{C})$ *as a finite subgroup (well defined up to conjugacy).*

1. Define $p-1$ finite subgroups $G_1, \dots, G_{p-1}$ of $\mathrm{GL}_p(\mathbb{C})$.
2. Describe $p-1$ supercuspidal representations $\pi_1, \dots, \pi_{p-1}$ of $\mathrm{GL}_p(\mathbb{Q}_p)$.
3. Quote a theorem of Bushnell and Henniart saying $\mathrm{Gal}(E_{\pi_i}/\mathbb{Q}_p) = G_i$.

We cannot describe the fields E_{π_i} themselves, only the Galois groups G_i and their images under ρ_{π_i} in $\mathrm{GL}_p(\mathbb{C})$. We also have good information of the "ramification filtration of $\mathrm{Gal}(E_{\pi_i}/\mathbb{Q}_p)$". Assume $p \neq 2$.

The Heisenberg Group

The Heisenberg group $H(p) = \{\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_p\}$ has center $\mathbb{F}_p \simeq \{\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$ which fits into an extension $0 \to \mathbb{F}_p \to H(p) \to \mathbb{F}_p^2 \to 0$.

Proposition (Classical, Chan [Cha])

(i) The $\mathrm{SL}_2(\mathbb{F}_p)$ action $\mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_p^2)$ lifts uniquely to an action $\mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{Aut}(H(p))$ such that $\mathrm{SL}_2(\mathbb{F}_p)$ fixes the center $\mathbb{F}_p \subseteq H(p)$.

(ii) There is a natural bijection $\psi \mapsto \pi_\psi$

$$\left\{ \begin{array}{c} \text{non-trivial character} \\ \psi\colon \mathbb{F}_p \to \mathbb{C}^\times \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{isom. classes of } p\text{-dimensional} \\ \text{irreducible } \mathbb{C}\text{-representations of } H(p) \end{array} \right\}$$

where π_ψ is the representation of $H(p)$ on the $\mathbb{C}$-vector space $L^2(\mathbb{F}_p)$ of set maps $f\colon \mathbb{F}_p \to \mathbb{C}$ with

$$\pi_\psi \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} f(x) = \psi(-bx + c) f(x - a)$$

for $x, a, b, c \in \mathbb{F}_p$ and $f \in L^2(\mathbb{F}_p)$.

The Weil Representation

Let $\psi\colon \mathbb{F}_p \to \mathbb{C}^\times$ be non-trivial and write $\pi_\psi\colon H(p) \to \mathrm{GL}(L^2(\mathbb{F}_p)) \simeq \mathrm{GL}_p(\mathbb{C})$. Given $g \in \mathrm{SL}_2(\mathbb{F}_p)$, define

$$\pi_\psi \circ g\colon H(p) \to \mathrm{GL}(L^2(\mathbb{F}_p)), \quad h \mapsto \pi_\psi(g \cdot h).$$

Then $\pi_\psi \circ g$ is p-dimensional irreducible with central character ψ. So

$$\pi_\psi \simeq \pi_\psi \circ g \qquad \text{as } H(p)\text{-representation on } L^2(\mathbb{F}_p).$$

By Schur's lemma, $g \in \mathrm{SL}_2(\mathbb{F}_p)$ determines an element $W_\psi(g) \in \mathrm{PGL}(L^2(\mathbb{F}_p))$.

Proposition (Classical, Chan [Cha])

Let $\psi\colon \mathbb{F}_p \to \mathbb{C}^\times$ be non-trivial.

(i) The projective $\mathbb{C}$-representation $W_\psi\colon \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{PGL}(L^2(\mathbb{F}_p))$ lifts uniquely to the Weil-representation

$$\widetilde{W}_\psi\colon \mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{GL}(L^2(\mathbb{F}_p)) \simeq \mathrm{GL}_p(\mathbb{C})$$

such that trivial central character.

The groups $G_1, \ldots, G_{p-1} \leq \mathrm{GL}_p(\mathbb{C})$

The $p-1$ non-trivial character $\psi\colon \mathbb{F}_p \to \mathbb{C}^\times$ are ennumerated $\psi_1, \ldots, \psi_{p-1}$.

Definition

Define $G_i \leq \mathrm{GL}_p(\mathbb{C})$ as the subgroup

$$G_i = \langle \pi_{\psi_i}(H(p)), \widetilde{W}_{\psi_i}(U) \rangle \leq \mathrm{GL}_p(L^2(\mathbb{F}_p)) \simeq \mathrm{GL}_p(\mathbb{C}).$$

where $U \in \mathrm{SL}_2(\mathbb{F}_p)$ is a generator of the *Coxeter Torus* $(d \in \mathbb{F}_p^\times \backslash \mathbb{F}_p^{\times 2})$

$$T_d = \{ \begin{pmatrix} t & du \\ u & t \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_p) \colon t, u \in \mathbb{F}_p, t^2 - du^2 = 1 \},$$

which is cyclic of order $p+1$, and well defined up to conjugacy in $\mathrm{SL}_2(\mathbb{F}_p)$.

Proposition (Bushnell-Henniart [BH14])

$G_i \leq \mathrm{GL}_p(\mathbb{C})$ acts irreducibly on $\mathbb{C}^p$, the conjugacy class of G_i in $\mathrm{GL}_p(\mathbb{C})$ depends only on ψ_i, and G_i is an extension

$$1 \to H(p) \to G_i \to \mathbb{Z}/(p+1)\mathbb{Z} \to 0.$$

Affine Generic Characters

The supercuspidals π_i such that $\mathrm{Gal}(E_{\pi_i}/\mathbb{Q}_p) \simeq G_i$ are induced off

$$I^+ = \left\{ \begin{pmatrix} 1+pa_1 & & b_{ij} & \\ & 1+pa_2 & & \\ & pc_{ij} & \ddots & \\ & & & 1+pa_p \end{pmatrix} \in \mathrm{GL}_p(\mathbb{Q}_p) \;\middle|\; a_\ell, b_{ij}, c_{ji} \in \mathbb{Z}_p \right\}.$$

For $\psi_i \colon \mathbb{F}_p \to \mathbb{C}^\times$ a non-trivial additive character define $\chi_i \colon I^+ \to \mathbb{C}^\times$ by

$$\chi_i \begin{pmatrix} 1+pa_1 & & b_{ij} & \\ & 1+pa_2 & & \\ & pc_{ij} & \ddots & \\ & & & 1+pa_n \end{pmatrix} = \psi_i(b_{1,2} + \ldots + b_{p-1,p} + c_{p,1}).$$

where $b_{1,2} + \ldots + b_{p-1,p} + c_{p,1} \in \mathbb{Z}_p$ is taken modulo p.

Theorem (Reeder [GR10])

Extend χ_i trivially to the subgroup ZI^+ where Z denotes the center of $\mathrm{GL}_p(\mathbb{Q}_p)$. The compactly induced representation

$$\pi_i := \text{c-Ind}_{ZI^+}^{\mathrm{GL}_p(\mathbb{Q}_p)} \chi_i \quad \textit{is irreducible and supercuspidal.}$$

and $\pi_i \not\simeq \pi_j$ for $i \neq j$.

Simple Supercuspidals

Theorem (Reeder [GR10])

Extend χ_i trivially to the subgroup ZI^+ where Z denotes the center of $\mathrm{GL}_p(\mathbb{Q}_p)$. The compactly induced representation

$$\pi_i := \text{c-Ind}_{ZI^+}^{\mathrm{GL}_p(\mathbb{Q}_p)} \chi_i$$ *is irreducible and supercuspidal.*

$$\text{c-Ind}_{ZI^+}^{\mathrm{GL}_p(\mathbb{Q}_p)} \chi_i = \left\{ \begin{array}{c} \text{locally constant functions} \\ f\colon \mathrm{GL}_p(\mathbb{Q}_p) \to \mathbb{C} \end{array} : \begin{array}{c} \text{if } x \in \mathrm{GL}_n(\mathbb{Q}_p) \text{ and } k \in ZI^+ \text{ then} \\ f(kx) = \chi_i(k) f(x) \text{ and } f \text{ is compactly} \\ \text{supported modulo } ZI^+ \end{array} \right\}$$

with $\mathrm{GL}_p(\mathbb{Q}_p)$ acting by right translations.

Remark

(σ, W) an irreducible smooth ZI^+-representation. By Harish Chandra's result

$$\text{c-Ind}_{ZI^+}^{\mathrm{GL}_n(\mathbb{Q}_p)} \sigma \text{ is irreducible} \implies \text{c-Ind}_{ZI^+}^{\mathrm{GL}_n(\mathbb{Q}_p)} \sigma \text{ is supercuspidal}.$$

Compact induction is the main tool for constructing supercuspidal representations.

The Langlands Correspondence for π_i

Theorem (Bushnell-Henniart [BH14])

Let $\psi_i \colon \mathbb{F}_p \to \mathbb{C}^\times$ be a non-trivial character. Write

$$\pi_i = \text{c-Ind}_{ZI^+}^{\mathrm{GL}_p(\mathbb{Q}_p)} \chi_i$$

and

$$1 \to H(p) \to G_i \to \mathbb{Z}/(p+1)\mathbb{Z} \to 0.$$

- *If $\rho_{\pi_i} \colon \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_p(\mathbb{C})$ is the Langlands parameter of π_i then $\mathrm{Im}(\rho_{\pi_i}) = G_i$ (up to conjugacy).*
- *If E_{π_i} is the kernel field of ρ_{π_i} then $\mathrm{Gal}(E_{\pi_i}/\mathbb{Q}_p) \simeq G_i$. In particular, $E_{\pi_i}/\mathbb{Q}_p$ is a non-abelian Galois extension of degree $p^3(p+1)$.*
- *The extension $E_{\pi_i}/\mathbb{Q}_p$ is totally ramified with non-abelian wild inertia subgroup*

$$\mathrm{Gal}(E_{\pi_i}/\mathbb{Q}_p)_1 \simeq H(p).$$

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