4. [10 points] Consider the yaw-pitch-roll values \( y = 0, p = 90^\circ, \) and \( r = -90^\circ. \) Express this orientation as a \( 3 \times 3 \) rotation matrix.

1. [15 points] For each of the following, state whether it is true or false by writing "T" or "F" on the line. Then, if it is false, give a counterexample. (In order to be true, it must be true for all quaternions \( q_1, q_2. \))

   a. \( q_1q_2 = q_2q_1. \)

   b. \( q_1q_2 = \pm q_2q_1. \) (That is, either they are equal, or they are opposite in sign.)

   c. \( q_1^*q_1 = q_1q_1^*. \)

   d. \( q_1^*q_2 = q_2^*q_1. \)

   e. \( q_1 + q_1^* \) is a scalar. (That is, with no \( i,j,k \) components)

   f. \( q_1^*q_1 \) is a scalar.

2. [10 points] Compute the following:
   a. \((1 + j + k)^{-1}.\)

   b. \((1 + j + k)^*.\)

   c. \(||1 + j + k||.\)

3. [25 points] Let \( q_1 = i + k.\)
   a. What rotation \( R_{\theta, u} \) does \( q_1 \) represent? Express your answer with \( u \) a unit vector.

   b. Find a quaternion \( q_2 \) such that \( q_1 = (q_2)^2. \) (So \( q_2 \) is a square root of \( q_1. \))

   c. Find a quaternion \( q_3 \) such that \( q_1 = (q_3)^3. \) (So \( q_3 \) is a cube root of \( q_1. \))

   d. In part (b), you found a square root \( q_2 \) of \( q_1. \) Of course, \(-q_2 \) is also a square root. Give two more square roots of \( q_1. \)
5. [20 points] Consider the yaw-pitch-roll values $y_1 = 90^\circ$, $p_1 = -90^\circ$, and $r_1 = 0$.

a. Find two other yaw-pitch-roll values which represent the same orientation.

b. Are there yaw-pitch-roll values $q_2, p_2, r_2$ with $p_2 = 90^\circ$ which represent the same orientation as $y_1, p_1, r_2$? If so, give one example.

6. [20 points] Conversion of yaw-pitch-roll to rotation matrices and to quaternions. Convention: $x$-axis is leftward, $y$-axis is upward, and $z$-axis is forward.

a. Let Yaw be $-90^\circ$, Pitch be $90^\circ$ and Roll be $-90^\circ$. Express this orientation as (i) a rotation matrix, and (ii) a quaternion.

b. Let Yaw be $-90^\circ$, Pitch be $90^\circ$ and Roll be $90^\circ$. (Change in sign for Roll.) Express this orientation as (i) a rotation matrix, and (ii) a quaternion.

6. [15 points] Let $x = \langle 0, \sqrt{2}/2, \sqrt{2}/2 \rangle$ and $y = \langle 1, 0, 0 \rangle$.

a. What is $\text{LERP}(x, y, \frac{1}{3})$ equal to?

b. What is $\text{SLERP}(x, y, \frac{1}{3})$ equal to?

6. [28 points] Answer the following questions about quaternions.

a. Compute $(1 + j)^2$.

b. Compute $(1 + j)(i + k)$.

c. Compute $(i + k)^{-1}$.

d. Give an example of two unit quaternions such that $q_1q_2 \neq q_2q_1$.

e. Let $q$ be the unit quaternion $q = \frac{\sqrt{2}}{2} i + \frac{1}{2} k$. This defines a rotation $R_{\theta, u}$. Give the values of $\theta$ and $u$.

f. Now let $q$ be the unit quaternion $q = \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2\sqrt{2}} i + \frac{1}{2\sqrt{2}} k$. This also defines a rotation $R_{\theta, u}$. Give its values of $\theta$ and $u$.

7. [10 points] A quaternion $q = d + ai + bj + ck$ defines a rotation $R$ in $\mathbb{R}^3$. Give the formula for $R(k)$, that is, the result of applying the rotation to $k = \langle 0, 0, 1 \rangle$. 


9. [10 points] Let \( \vec{u} = (1, 0, 0) \) and \( \vec{v} = (-\frac{1}{2}, \frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}) \) be unit vectors in \( \mathbb{R}^3 \).
Compute \( \text{Slerp}(\frac{1}{3}, \vec{u}, \vec{v}) \).

1. A airplane is flying northward, oriented horizontally as usual. This is the orientation for yaw, pitch and roll all equal to zero.

   a. The airplane changes to flying south and flipped upside down. Characterize this new orientation in terms of yaw, pitch and roll.

Quaternions may be written as either \( (d, a, b, c) \) or \( d + ai + bj + ck \). You may use \( \arctan \), \( \arcsin \) or \( \arccos \) in your answers *if* necessary.

For these problems, let \( q_1, q_2 \) and \( q_3 \) be the quaternions \( q_1 = \langle 1, 0, 0, 0 \rangle \), \( q_2 = \langle 0, 0, 1, 0 \rangle \) and \( q_3 = \langle 2, 0, 0, 1 \rangle \).

1. What are \( q_1^*, ||q_1|| \) and \( q_1^{-1} \)?

2. What are \( q_2^*, ||q_2|| \) and \( q_2^{-1} \)?

3. What are \( q_3^*, ||q_3|| \) and \( q_3^{-1} \)?

4. What rotation \( R_{\theta, v} \) is represented by \( q_1 \)? (Give \( \theta \) and \( v \). For these three questions, multiple answers are possible.)

5. What rotation \( R_{\theta, v} \) is represented by \( q_2 \)?

6. What rotation \( R_{\theta, v} \) is represented by \( q_3 \)?
1. For each of the following, answer “T” (True) if the equation holds for all quaternions; and answer “F” if the equation is (sometimes) false.

   - a. $q_1 q_2 = q_2 q_1$  [T/F]
   - b. $q_1 + q_2 = q_2 + q_1$  [T/F]
   - c. $q_1^* q_1 = q_1 q_1^*$  [T/F]
   - d. $q_1^{-1} q_1 = 1.$  [T/F]
   - e. $(q^*)^* = q$  [T/F]
   - f. $q_1^* q_2 = (q_2 q_1)^*$  [T/F]
   - g. $(q_2^{-1} q_1)^{-1} = (q_1^{-1} q_2)$  [T/F]
   - h. $(q_2^{-1} q_1)^{-1} = (q_2^{-1} q_1)$  [T/F]
   - i. $j^{-1} = -j.$  [T/F]
   - j. $ik = ki.$  [T/F]

2. Suppose $q_1$ and $q_2$ are quaternions that represent rotations $Q$ and $R$. Let $S$ be the rotation which is the composition of $Q$ and $R$ so $S(x) = Q(R(x))$. What quaternion represents the rotation $S$?

3. What quaternion represents the orientation corresponding to Yaw 90°, and Pitch −90°, and Roll −90°? (Recall that yaw-pitch-roll has the $y$ axis pointing forwards, the $x$-axis pointing leftwards, and the $z$ axis pointing upwards.)
1. Give the rotation $R_{\theta,u}$ which corresponds to yaw = $90^\circ$, pitch = $0^\circ$, and roll = $0^\circ$. (For your answers to questions 1 and 2, give the values of $\theta$ and $u$.)

2. Give the rotation $R_{\theta,u}$ which corresponds to yaw = $0^\circ$, pitch = $0^\circ$, and roll = $45^\circ$.

3. Let $\Omega$ denote the orientation yaw = $180^\circ$, pitch = $0^\circ$, and roll = $180^\circ$. Give a second representation of $\Omega$ in terms of different yaw, pitch, roll values, with yaw equal to zero.

1. Evaluate the following quaternion products:
   a. $jk =$
   b. $kj =$
   c. $(1 + 2j)(3i + 4k) =$
   d. $kji =$

2. Let $q = 3i + 4k$.
   a. What is its norm, $||q||$?
   b. What is its conjugate, $q^*$?
   c. What is its inverse, $q^{-1}$?

3. Give two unit quaternions which represent the rotation $R_{\pi,k}$.

4. What rotation $R_{\theta,u}$ is represented by the quaternion $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$? (Give $\theta$ and $u$.)