UNRAMIFIED FORCING[1]

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1. **Introduction.** The method of forcing was invented by Cohen in order to solve some classical independence problems. As it became apparent that the method was applicable in much more general circumstances, it was simplified and generalized by set theorists. One result of these efforts is the theory of Boolean models, discussed in the article by Scott and Solovay, part II of these Proceedings.

A feature of the Boolean approach is that the use of the ramified hierarchy of constructible sets has disappeared. Since forcing models can be obtained from Boolean models, it is apparent that this hierarchy is not needed for forcing models either.

One purpose of the present article is to give a direct construction of forcing models which does not use the ramified hierarchy. In addition, I have tried to present a summary of some of the simplifications and generalizations of forcing theory mentioned above. Many of these are only contained in the folk literature at present.

It would be an impossible task to list all the people who have contributed to each advance in the subject. The historical notes should at least show who the main contributors have been. The overall exposition owes much to notes by Scott and by Silver for this institute and to lectures by Rowbottom at UCLA in the fall of 1967. Conversations with Chang and Rowbottom have also been very helpful.

2. **Background.** We review here the principal facts about set theory which we need.

We use ZF for the Zermelo-Fraenkel axiom system (extensionality, regularity, infinity, union, replacement, and power set) and ZFC for ZF plus the axiom of

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choice. In considering a function as a set of ordered pairs, we put the value first and the argument second in the ordered pair. Thus the range (domain) of f is the set of first (second) elements of ordered pairs in f. Notation: $U(x)$ for the union of the sets in x, $S(x)$ for the power set of x, $\langle x, y\rangle$ for an ordered pair, $\mathrm{Ra}(x)$ for the range of x, $\mathrm{Do}(x)$ for the domain of x, ${}^{x}y$ for the set of mappings from x to y.

We identify an ordinal with the set of previous ordinals and a cardinal with the smallest ordinal having that cardinal (so that $\aleph_\alpha = \omega_\alpha$). We use Greek letters for ordinals and German letters for infinite cardinals (or infinite cardinals of a model). We write $|x|$ for the cardinal of x and $\mathfrak{m}^+$ for the next cardinal after $\mathfrak{m}$. Recall that $\mathrm{cf}(\alpha)$ is the smallest β such that there is a mapping of β onto a cofinal subset of α. An infinite cardinal $\mathfrak{m}$ is *regular* if $\mathrm{cf}(\mathfrak{m}) = \mathfrak{m}$ and *singular* if $\mathrm{cf}(\mathfrak{m}) < \mathfrak{m}$. Then $\mathrm{cf}(\alpha)$ is always 1 or a regular cardinal. If $\mathfrak{m}$ is regular, then the union of $<\mathfrak{m}$ sets each of cardinal $<\mathfrak{m}$ has cardinal $<\mathfrak{m}$. We use $S_\mathfrak{m}(x)$ for $[y : y \subset x \;\&\; |y| < \mathfrak{m}]$.

The $V(\alpha)$ (sometimes written $R(\alpha)$) are defined by

$$V(\alpha) = \bigcup_{\beta<\alpha} S(V(\beta)),$$

using transfinite induction. The *rank* of x, designated by $\mathrm{rk}(x)$, is the smallest α such that $x \in V(\alpha + 1)$. This is well defined, and $x \in y$ implies $\mathrm{rk}(x) < \mathrm{rk}(y)$. Moreover, $\mathrm{rk}(\alpha) = \alpha$.

We assume some elementary cardinal arithmetic. Recall that

$$|{}^{x}y| = |y|^{|x|}, \qquad |S(x)| = 2^{|x|}.$$

König's theorem says that $\mathrm{cf}(2^\mathfrak{m}) > \mathfrak{m}$. We write GCH for the generalized continuum hypothesis: $\forall\mathfrak{n}(2^\mathfrak{n} = \mathfrak{n}^+)$. *Weak powers* are defined by

$$\mathfrak{m}^{\underset{\smile}{\mathfrak{n}}} = \sum_{\mathfrak{p}<\mathfrak{n}} \mathfrak{m}^\mathfrak{p}.$$

(Here we allow $\mathfrak{p}$ to be finite; but these terms can be omitted if $\mathfrak{n} > \aleph_0$.) Then $\mathfrak{m}^{\underset{\smile}{\mathfrak{n}}^+} = \mathfrak{m}^\mathfrak{n}$. Since every subset of x of cardinal $\mathfrak{p}$ is the range of an element of ${}^{\mathfrak{p}}x$, we have $|S_\mathfrak{n}(x)| \leqq |x|^{\underset{\smile}{\mathfrak{n}}}$.

We have

$$\mathfrak{n} < \mathrm{cf}(\mathfrak{m}) \;\&\; (\forall\mathfrak{p} < \mathfrak{m})(2^\mathfrak{p} \leqq \mathfrak{m}) \to \mathfrak{m}^\mathfrak{n} = \mathfrak{m}. \tag{3.1}$$

For $\mathfrak{n} < \mathrm{cf}(\mathfrak{m})$ implies ${}^{\mathfrak{n}}\mathfrak{m} = \bigcup_{\alpha<\mathfrak{m}} {}^{\mathfrak{n}}\alpha$. But $|{}^{\mathfrak{n}}\alpha| = |\alpha|^\mathfrak{n} \leqq 2^{|\alpha|\cdot\mathfrak{n}} \leqq \mathfrak{m}$ by hypothesis; so $\mathfrak{m}^\mathfrak{n} \leqq \mathfrak{m}\cdot\mathfrak{m} = \mathfrak{m}$. If the GCH holds, this reduces to $\mathfrak{n} < \mathrm{cf}(\mathfrak{m}) \to \mathfrak{m}^\mathfrak{n} = \mathfrak{m}$. It follows from (3.1) that

$$\mathfrak{m} \text{ regular } \& \; (\forall\mathfrak{p} < \mathfrak{m})(2^\mathfrak{p} \leqq \mathfrak{m}) \to \mathfrak{m}^{\underset{\smile}{\mathfrak{m}}} = \mathfrak{m}.$$

Thus $\mathfrak{m}^{\underset{\smile}{\mathfrak{m}}} = \mathfrak{m}$ if $\mathfrak{m}$ is $\aleph_0$ is strongly inacessible; and $\mathfrak{m}^{\underset{\smile}{\mathfrak{m}}} = \mathfrak{m}$ for all regular $\mathfrak{m}$ if the GCH holds.

We now turn to models of set theory. A model is *transitive* if its universe is a transitive set and its membership relation is the usual membership relation restricted to its universe. The collapsing technique of Mostowski [8] shows that every well-founded model of the extensionality axiom is isomorphic to a transitive model.

Henceforth, we shall take *model* to mean *transitive model*, and identify a model with its universe.

Let M be a model. To indicate that something is being considered in the model M, we append the phrase 'in M' or a superscript M. Thus if Φ is a sentence (possibly containing names of elements in M), we abbreviate 'Φ is true when interpreted in M' to 'Φ holds in M' or simply Φ^M. A *cardinal in M* is an element a of M such that 'a is a cardinal' holds in M. We designate by ω_1^M the element a of M such that $\Phi(a)^M$, where $\Phi(x)$ is the sentence of set theory saying that x is ω_1. Other examples are interpreted similarly.

A *class in M* is a set $[a : a \in M \ \&\ \Phi(a)^M]$, where $\Phi(x)$ contains only symbols from the language of ZFC and symbols for sets in M. Every set in M is a class in M. If M is a model of ZFC, then every class in M which is included in a set in M is itself a set in M. A *functional in M* is a function F such that for some formula $\Phi(x, y)$ of the type described above, $F(a) = b$ if and only if $a, b \in M$ and $\Phi(a, b)^M$. If M is a model of ZFC, then the image of a set in M under a functional in M is a set in M.

LEMMA 2.1. *Every transitive set M satisfying the following four conditions is a model of* ZF.

(a) $\omega \in M$.

(b) *Every class in M which is included in a set in M is itself a set in M.*

(c) *For every functional F in M and every set a in M included in the domain of F,* $U([F(b) : b \in a])$ *is included in a set in M.*

(d) *For every set a in M,* $S(a) \cap M$ *is included in a set in M.*

A relation symbol P defined in ZFC is *absolute* if for every model M of ZFC, P^M coincides with P on arguments in M. A similar definition holds for operation symbols (including constants, which are operation symbols with zero arguments). Most of the symbols introduced through the development of ordinals are absolute; S is an exception. For details, see [9] or [10]. If F is absolute, then F takes elements of M into elements of M (since F^M does). The absoluteness of 'is an ordinal' implies that the ordinals in M are the real ordinals which belong to M.

The *axiom of constructibility* [5] states that every set is constructible. It implies that there is a definable well-ordering of the universe and that the GCH holds. If M is a model of ZFC, the constructible sets in M form a model of ZFC plus the axiom of constructibility. The canonical function mapping the ordinals onto the constructible sets is absolute; so the constructible sets in a model M of ZFC are the images under this function of the ordinals in M. Thus if M and N have the same ordinals, then they have the same constructible sets.

3. **Notions of forcing.** Suppose that M is a model of ZFC, and that $a, b \in M$. We wish to extend M to a model N in which there is a mapping F of a onto b. To avoid obvious difficulties, we suppose that a is infinite and $b \neq 0$.

Let C be the set of all mappings form a finite subset of a into b. Then $C \in M$ by absoluteness. The set G of all finite subsets of F will be a subset of C; but it will not necessarily be in M. Our idea is first to select G, and then use G to build N.

Each $p \in C$ gives a condition which F must satisfy if p is to be in G; namely, we must have $p \subset F$. If $p \subset q$, the condition q gives more information than the condition p. We then say that q is an extension of p, and write $q \leqq p$. (This notation is to suggest that q allows fewer models N than p.) Then C is a partially ordered set with 0 as the largest element.

There are three obvious conditions which G must satisfy: (a) $0 \in G$; (b) if $p \in G$ and $p \leqq q$, then $q \in G$; (c) any two elements in G have a common extension in G. If G satisfies these conditions, $F = U(G)$ will be a mapping from a subset of a onto a subset of b.

In order to have $\mathrm{Do}(F) = a$ and $\mathrm{Ra}(F) = b$, G must also intersect certain sets, viz., the sets $[p : x \in \mathrm{Do}(p)]$ for $x \in a$ and the sets $[p : y \in \mathrm{Ra}(p)]$ for $y \in b$. These sets are in M and have the following property: every condition in C has an extension in the set. We shall therefore require that G intersect every set in M having this property.

We now generalize these notions. A *notion of forcing* is a partially ordered set C having a largest element. We write $\leqq_C$ or $\leqq$ for the ordering and 1_C or 1 for the largest element. The elements of C are called *conditions*. Conditions are generally designated by p, q, and r. If $p \leqq q$, we say p is an *extension* of q. A subset D of C is *C-dense* (or simply *dense*) if every condition in C has an extension in D.

Let C be a notion of forcing and let M be a set. A subset G of C is *C-generic* over M (or simply *generic*) if the following conditions hold.

(G1) $1 \in G$.

(G2) For all $p \in G$ and $q \geqq p$, $q \in G$.

(G3) For all $p, q \in G$, p and q have a common extension in G.

(G4) For all dense sets D in M, $G \cap D \neq 0$.

EXISTENCE THEOREM. *Let C be a notion of forcing; M a countable set; $p \in C$. Then there is a set G which is C-generic over M and contains p.*

PROOF. Let $a_0, a_1, \ldots$ be the elements of M. Choose p_n inductively as follows: $p_0 = p$; p_{n+1} is an extension of p_n in a_n if such an extension exists; $p_{n+1} = p_n$ otherwise. Then $G = [q : \exists n(p_n \leqq q)]$ has the required properties.

REMARK. This is the only place in which we make direct use of the countability of M. It would obviously suffice to assume that $M \cap S(C)$ is countable.

We let $H_{\mathfrak{m}}(A, B)$ be the set of all mappings p from an element of $S_{\mathfrak{m}}(A)$ to B. For $p, q \in H_{\mathfrak{m}}(A, B)$, we write $p \leqq q$ for $q \subset p$. Then $H_{\mathfrak{m}}(A, B)$ is a notion of forcing with largest element 0. We write $H(A, B)$ for $H_{\aleph_0}(A, B)$. Note that $H(A, B)$ is absolute.

Now suppose that M is a model of ZFC; $a, b \in M$; $b \neq 0$; $\mathfrak{m}$ is an infinite cardinal in M such that $\mathfrak{m} \leqq |a|$ in M; $C = H_{\mathfrak{m}}^M(a, b)$; G is C-generic over M; and $F = U(G)$. Then the remarks at the beginning of the section show that F is a mapping of a onto b. (We need $\mathfrak{m} \leqq |a|$ in M to guarantee that $[p : y \in \mathrm{Ra}(p)]$ is dense for $y \in b$.) Thus if we can construct an extension N of M containing G, we will obtain an extension in which there is a mapping of a onto b.

REMARK. This shows that if a is countably infinite and b is uncountable, then

no C-generic set over M exists. Thus the requirement of countability in the existence theorem cannot be dropped.

HISTORICAL NOTE. The basic ideas of this section are due to Cohen [1], [2]. The notion of a dense set is due to Solovay.

4. **The model.** Now suppose that C is a notion of forcing in[2] a countable model M of ZFC and that G is C-generic over M. We are going to construct an extension of M containing G.

We shall first define a structure which has universe M but has a new membership relation $\in_G$ defined by

$$a \in_G b \leftrightarrow (\exists p \in G)(\langle a, p\rangle \in b).$$

(Here and in what follows, a, b, c, and d represent elements of M.) We then use the collapsing technique to convert $\langle M, \in_G\rangle$ into a transitive model. We first note that

$$a \in_G b \rightarrow a \in \mathrm{Ra}(b) \tag{4.1}$$

and hence

$$a \in_G b \rightarrow \mathrm{rk}(a) < \mathrm{rk}(b). \tag{4.2}$$

We then define

$$K_G(b) = [K_G(a) : a \in_G b].$$

By (4.2), this is a legitimate definition by induction on $\mathrm{rk}(b)$. Finally, we define

$$M[G] = [K_G(a) : a \in M].$$

PRINCIPAL THEOREM. *Let M be a countable model of* ZFC; *C a notion of forcing in M; G a set which is C-generic over M. Then $M[G]$ is a countable model of* ZFC *which includes M and contains G; and it is the smallest such model.*

The proof of the fundamental theorem will be given in this and the next two sections. Throughout the rest of the paper, M, C, and G are assumed to satisfy the hypotheses of the fundamental theorem unless otherwise indicated. We shall write $\bar{a}$ for $K_G(a)$.

We begin with some simple observations. From the definition of $M[G]$ and the countability of M, we see that $M[G]$ is countable and transitive. Now define

$$\hat{b} = [\langle \hat{a}, 1\rangle : a \in b]$$

by induction on $\mathrm{rk}(b)$. This definition can be given in M; so the mapping from a to $\hat{a}$ is a functional in M. In particular, $\hat{a} \in M$. An easy induction (using (G1)) shows that $K_G(\hat{b}) = b$; so $M \subset M[G]$. Now set (with an abuse of notation)

$$\hat{G} = [\langle \hat{p}, p\rangle : p \in C].$$

Then $\hat{G} \in M$ and $K_G(\hat{G}) = G$. Hence $G \in M[G]$. In summary, $M[G]$ is a countable

[2] A notion of forcing C is in a model M if both the set C and the relation $\leq$ are in M.

transitive set including M and containing G. The verification that it is a model of ZFC requires a new notion, which we now turn to.

5. **Forcing.** We introduce a language, called the *forcing language*, which is suitable for discussing $M[G]$. The symbols of the forcing language are the symbols of ZFC and the elements[3] of M. Each element a of M is regarded as a constant which designates the element $\bar{a}$ of $M[G]$; we say a is a *name* of $\bar{a}$. If Φ is a sentence of the forcing language, $\vdash_G \Phi$ means that Φ is true in $M[G]$.

Let $p \in C$, and let Φ be a sentence of the forcing language. We say that p *forces* Φ, and write $p \Vdash \Phi$, if $\vdash_G \Phi$ for every set G which is C-generic over M and contains p.

Our immediate object is to prove three lemmas about forcing.

DEFINABILITY LEMMA. *If* $\Phi(x_1, \ldots, x_n)$ *is a formula of* ZFC *containing only the free variables shown, then*

$$[\langle p, a_1, \ldots, a_n\rangle : p \Vdash \Phi(a_1, \ldots, a_n)]$$

is a class in M.

EXTENSION LEMMA. *If* $p \Vdash \Phi$ *and* $q \leqq p$, *then* $q \Vdash \Phi$.

TRUTH LEMMA. *If* G *is generic, then* $\vdash_G \Phi$ *if and only if* $(\exists p \in G)(p \Vdash \Phi)$.

Our procedure is as follows. We define a modified notion of forcing, designated by $p \Vdash^* \Phi$. We then prove that the three lemmas hold when $\Vdash$ is replaced by $\Vdash^*$. From this, we show that

$$p \Vdash \Phi \leftrightarrow p \Vdash^* \neg\neg\, \Phi. \tag{5.1}$$

Then by taking the three lemmas for $\Vdash^*$ and replacing Φ by $\neg\neg\Phi$, we obtain the lemmas for $\Vdash$.

We first specify the undefined symbols of the forcing language. We let $\neg$ and $\vee$ be the undefined propositional connectives and let $\exists$ be the undefined quantifier. As undefined relation symbols, we take $\in$ and $\neq$. Of course, $x \notin y$ is defined as $\neg(x \in y)$ and $x = y$ is defined as $\neg(x \neq y)$.

We now define $p \Vdash^* \Phi$ by the following five clauses.

(a) $p \Vdash^* a \in b$ if $\exists c(\exists q \geqq p)(\langle c, q\rangle \in b \;\&\; p \Vdash^* a = c)$.

(b) $p \Vdash^* a \neq b$ if $\exists c(\exists q \geqq p)(\langle c, q\rangle \in a \;\&\; p \Vdash^* c \notin b)$ or
$\exists c(\exists q \geqq p)(\langle c, q\rangle \in b \;\&\; p \Vdash^* c \notin a)$.

(c) $p \Vdash^* \neg\Phi$ if $(\forall q \leqq p)\neg(q \Vdash^* \Phi)$.

(d) $p \Vdash^* \Phi \vee \Psi$ if $p \Vdash^* \Phi$ or $p \Vdash^* \Psi$.

(e) $p \Vdash^* \exists x\Phi(x)$ if $\exists b(p \Vdash^* \Phi(b))$.

We must first straighten out the circularities in the definition. If we trace back the definition of $p \Vdash^* a \neq b$, we find that it is defined in terms of certain $p' \Vdash^* a' \neq b'$ and $p' \Vdash^* b' \neq a'$ where $\mathrm{rk}(a') < \mathrm{rk}(a)$ and $\mathrm{rk}(b') < \mathrm{rk}(b)$. Thus we may define

[3] Those who do not like sets to be used as symbols can introduce a set of symbols which is in one-one correspondence with M.

$p \Vdash^* a \neq b$ by induction on max (rk(a), rk(b)). We may then use (c) to define $p \Vdash^* a = b$, (a) to define $p \Vdash^* a \in b$, and (c) to define $p \Vdash^* a \notin b$. We can then prove (b). This defines $p \Vdash^* \Phi$ for atomic Φ. We then use (c), (d) and (e) to define $p \Vdash^* \Phi$ for all Φ by induction on the length of Φ.

It is trivial to prove the definability lemma for $\Vdash^*$ by induction on the length of $\Phi(x_1, \ldots, x_n)$ (noting that the quantifiers in (a)–(e) vary through M or through the set C in M). Of course the definition of the class $[\langle p, a, b\rangle : p \Vdash^* a \neq b]$ will be by transfinite induction in M as described above.

We prove the other two lemmas by proving them for the sentences on the left of (a)–(e) under the assumption that they are true for the sentences on the right; this is seen to be a valid method of proof as above. The extension lemma is quite trivial; so we consider only the truth lemma.

We first show that if the truth lemma holds for Φ, then

(5.2) $$a \in_G b \ \&\ \vdash_G \Phi \leftrightarrow (\exists p \in G)(\exists q \geqq p)(\langle a, q\rangle \in b \ \&\ p \Vdash^* \Phi).$$

For if the left side holds, there are $q, r \in G$ such that $\langle a, q\rangle \in b$ and $r \Vdash^* \Phi$. Choosing a common extension p of q and r in G by (G3), we have $p \Vdash^* \Phi$ by the extension lemma. Conversely, let the right side of (5.2) hold for p and q. Then $\vdash_G \Phi$ by the truth lemma. Also $q \in G$ by (G2); so $a \in_G b$.

Now we turn to the cases of the truth lemma.

(a) By the definition of $\bar{b}$, $\vdash_G a \in b$ is equivalent to $\exists c(c \in_G b \ \&\ \vdash_G a = c)$. By the hypothesis and (5.2), this is equivalent to

$$\exists c(\exists p \in G)(\exists q \geqq p)(\langle c, q\rangle \in b \ \&\ p \Vdash^* a = c)$$

and hence to $(\exists p \in G)(p \Vdash^* a \in b)$.

(b) Clearly $\vdash_G a \neq b$ if and only if either $\exists c(c \in_G a \ \&\ \vdash_G c \notin b)$ or $\exists c(c \in_G b \ \&\ \vdash_G c \notin a)$. By the hypothesis and (5.2), this is equivalent to

$$\exists c(\exists p \in G)(\exists q \geqq p)(\langle c, q\rangle \in a \ \&\ p \Vdash^* c \notin b)$$
$$\vee\ \exists c(\exists p \in G)(\exists q \geqq p)(\langle c, q\rangle \in b \ \&\ p \Vdash^* c \notin a)$$

and hence to $(\exists p \in G)(p \Vdash^* a \neq b)$.

(c) By hypothesis, $\vdash_G \neg \Phi$ if and only if $\neg(\exists p \in G)(p \Vdash^* \Phi)$. Hence we must prove that exactly one of $(\exists p \in G)(p \Vdash^* \Phi)$ and $(\exists p \in G)(p \Vdash^* \neg \Phi)$ holds. To show that at least one holds, it will suffice by (G4) to show that $D = [p : p \Vdash^* \Phi$ or $p \Vdash^* \neg \Phi]$ is a dense set in M. It is in M by the definability lemma; so we must show every p has an extension in D. But either p has an extension q such that $q \Vdash^* \Phi$ and hence $q \in D$, or $p \Vdash^* \neg \Phi$ and hence p itself is in D.

Now suppose there are $p, q \in G$ such that $p \Vdash^* \Phi$ and $q \Vdash^* \neg \Phi$. By (G2), p and q have a common extension r; and by the extension lemma, $r \Vdash^* \Phi$. This contradicts $q \Vdash^* \neg \Phi$.

We leave the easy proofs of (d) and (e) to the reader.

Now we prove (5.1). Suppose $p \Vdash^* \neg\neg \Phi$. If G is generic and $p \in G$, then $\vdash_G \neg\neg \Phi$ by the truth lemma; so $\vdash_G \Phi$. Thus $p \Vdash \Phi$. Now suppose $\neg(p \Vdash^* \neg\neg \Phi)$. Then $q \Vdash^* \neg \Phi$ for some $q \leqq p$. Choose a generic G such that $q \in G$. By the truth lemma, $\vdash_G \neg \Phi$; and by (G2), $p \in G$. Hence $\neg(p \Vdash \Phi)$.

We have thus proved our lemmas. Substituting $\neg\neg\Phi$ for Φ in (c), we get

$$p \Vdash \neg \Phi \text{ if and only if } (\forall q \leqq p) \neg (q \Vdash \Phi). \tag{5.3}$$

Replacing Φ by $\neg\Phi$ and noting that $p \Vdash \neg\neg\Phi$ is equivalent to $p \Vdash \Phi$,

$$p \Vdash \Phi \text{ if and only if } (\forall q \leqq p) \neg (p \Vdash \neg \Phi). \tag{5.4}$$

HISTORICAL NOTE. Cohen's concept of forcing is essentially our $\Vdash^*$. Part (c) of the definition, which simplified Cohen's definition, is due to Scott. Our concept of forcing was introduced by Feferman [4], who called it *weak forcing*. The three fundamental lemmas are due to Cohen.

6. **Completion of the proof.** We show that $M[G]$ is a model of ZF by showing that it satisfies the conditions of Lemma 2.1. Since $\omega \in M \subset M[G]$, condition (a) holds.

LEMMA 6.1. *Let A be a class in $M(G)$ such that $A \subset \bar{a}$. Then $A \in M[G]$, and A has a name c such that $c \subset \mathrm{Ra}(a) \times C$.*

PROOF. There is a formula $\Phi(x)$ of the forcing language such that

$$\bar{b} \in A \leftrightarrow \vdash_G \Phi(b) \tag{6.1}$$

for all b. Set

$$c = [\langle b, p\rangle : b \in \mathrm{Ra}(a) \;\&\; p \Vdash \Phi(b)].$$

By the definability lemma, c is a class in M. Since $c \subset \mathrm{Ra}(a) \times C$, $c \in M$.

We must prove

$$\bar{b} \in \bar{c} \leftrightarrow \bar{b} \in A.$$

If $\bar{b} \in \bar{c}$, we may suppose, by changing b without changing $\bar{b}$, that $b \in_G c$. Then for some $p \in G$, $\langle b, p\rangle \in c$ and hence $p \Vdash \Phi(b)$. Hence $\vdash_G \Phi(b)$; so $\bar{b} \in A$ by (6.1). Now let $\bar{b} \in A$. Then $\bar{b} \in \bar{a}$; so we may suppose that $b \in_G a$. By (4.1), $b \in \mathrm{Ra}(a)$. By (6.1), $\vdash_G \Phi(b)$; so by the truth lemma, some $p \in G$ forces $\Phi(b)$. Then $\langle b, p\rangle \in c$; so $b \in_G c$; so $\bar{b} \in \bar{c}$.

It follows from Lemma 6.1 that (b) of Lemma 2.1 holds.

LEMMA 6.2. *If $x \subset M[G]$, and every element in x has a name in a, then x is included in a set in $M[G]$.*

PROOF. Let $b = a \times \{1\}$. Any element of x is $\bar{c}$ for some $c \in a$. Then $c \in_G b$ by (G1); so $\bar{c} \in \bar{b}$. Thus $x \subset \bar{b}$.

Now we prove that (c) of Lemma 2.1 holds. Let F be a functional in $M[G]$, and let $\bar{a}$ be a set in $M[G]$ included in the domain of F. There is a formula $\Phi(x, y)$ of the forcing language such that

$$F(\bar{b}) = \bar{c} \leftrightarrow \vdash_G \Phi(b, c)$$

for all b, c. Choose a set d in M with the following property: for each $p \in C$ and each $b \in \mathrm{Ra}(a)$, if there is a c such that $p \Vdash \Phi(b, c)$, then there is such a set in d. By replacing d by its transitive closure, we may suppose that d is transitive.

We now show that every element x of $U([F(\bar{b}):\bar{b} \in \bar{a}])$ has a name in d; in view of Lemma 6.2, this will complete our proof. We have $x \in F(\bar{b})$ with $\bar{b} \in \bar{a}$; and we may suppose that $b \in_G a$. Letting c be a name of $F(\bar{b})$, we have $\vdash_G \Phi(b, c)$; so some $p \in G$ forces $\Phi(b, c)$. Hence for some $c' \in d$, $p \Vdash \Phi(b, c')$. Then $\vdash_G \Phi(b, c')$; so $F(\bar{b}) = \bar{c}'$. Thus $x \in \bar{c}'$; whence $x = \bar{a}'$ with $a' \in_G c'$ and hence $a' \in \mathrm{Ra}(c')$. Since $c' \in d$ and d is transitive, $a' \in d$ as required.

Finally, we prove that (d) holds. Let $\bar{a} \in M[G]$ and let $\bar{b} \in S(\bar{a}) \cap M[G]$. By Lemma 6.1, $\bar{b}$ has a name c such that $c \subset \mathrm{Ra}(a) \times C$ and hence $c \in S^M(\mathrm{Ra}(a) \times C)$. The desired result now follows from Lemma 6.2. Thus $M[G]$ is a model of ZF.

LEMMA 6.3. *If N is a model of* ZF *which includes M and contains G, then there is a functional in N whose restriction to M is K_G.*

PROOF. We define (in N)

$$x \in^* y \leftrightarrow (\exists p \in G)(\langle x, p\rangle \in y), \qquad K^*(y) = [K^*(x) : x \in^* y]$$

(using induction on $\mathrm{rk}(y)$). It is easy to see (using the transitivity of M) that $\in^*$ and K^* agree with $\in_G$ and K_G for arguments in M.

It follows from the lemma that $K_G(a) \in N$ for all $a \in M$, so that $M[G] \subset N$. Thus $M[G]$ is even the smallest model of ZF including M and containing G. Moreover, we can apply Lemma 6.3 to $M[G]$. We obtain a function K in $M[G]$ whose restriction to M is K_G.

Now let $\bar{a} \in M[G]$. Then there is a mapping of an ordinal onto $\mathrm{Ra}(a)$ which is in M and hence in $M[G]$. Composing K with this mapping, we get in $M[G]$ a mapping from an ordinal onto

$$[K(x) : x \in \mathrm{Ra}(a)] = [\bar{b} : b \in \mathrm{Ra}(a)].$$

But by (4.1), this set includes $\bar{a}$. Thus in $M[G]$, the following holds: for every x, there is a mapping from an ordinal onto a set including x. This implies that the axiom of choice holds in $M[G]$. We have thus completed the proof of the fundamental theorem.

HISTORICAL NOTE. Most of the ideas of this section are due to Cohen. The proof that the power set axiom holds in $M[G]$ is essentially due to Solovay; it is simpler than Cohen's proof.

7. **The axiom of constructibility.** To make the best use of the fundamental theorem, we need information about the relation between M and $M[G]$. The following simple result is often useful.

LEMMA 7.1. *M and $M[G]$ have the same ordinals.*

PROOF. Since $M \subset M[G]$, we need only show that every ordinal α in $M[G]$ is in M. A simple induction shows that $\mathrm{rk}(\bar{a}) \leqq \mathrm{rk}(a)$ for all a. Taking a to be a name of α, $\alpha = \mathrm{rk}(\alpha) \leqq \mathrm{rk}(a)$. Since rk is absolute, $\mathrm{rk}(a) \in M$; so $\alpha \in M$ by the transitivity of M.

COROLLARY. *M and $M[G]$ have the same constructible sets.*

Now let M be a countable model of ZFC, and let $C = H(\omega, 2)$. By absoluteness, C is in M. Take G generic over M, and set $F = U(G)$. As seen in §3, F is a mapping from ω to 2. Obviously $F \in M[G]$; we claim $F \notin M$. To see this, let f be a mapping from ω to 2 which is in M. Then $[p:(\exists n \in \omega)(n \in \mathrm{Do}(p) \ \&\ p(n) \neq f(n))]$ is a set in M which is easily seen to be dense. Hence it contains a $p \in G$; and this implies that $F \neq f$.

The set A having F as its characteristic function is thus in $M[G] - M$. By the corollary, A is not constructible in $M[G]$. Hence $M[G]$ is a model of ZFC$'$, where ZFC$'$ is ZFC with the additional axiom: there is a noncontructible subset of ω.

We have shown how to construct a model of ZFC$'$ from a countable model M of ZFC. The existence of M can be seen as follows. We start from any model N of ZFC. Using the Löwenheim-Skolem theorem to extract a countable submodel and then applying the collapsing technique, we obtain a countable model M of ZFC.

Unfortunately, the existence of a (transitive) model N of ZFC cannot be proved in ZFC, even from the assumption that ZFC is consistent. Hence if we wish a finitary proof of the relative consistency of ZFC$'$ to ZFC, we must proceed a little differently. We add to ZFC a constant N and axioms which say that N is transitive and nonempty and that each axiom of ZFC holds in N. The reflection principle **[6]** shows that this is a conservative extension of ZFC. We then define M and $M[G]$ as above. Our proof will then show that each axiom of ZFC$'$ holds in $M[G]$. For a slightly different technique, see **[10]**.

We can get a stronger result by "collapsing" a cardinal in M. Suppose that M satisfies the axiom of constructibility. (We can obtain such a model from a countable model N of ZFC by taking the sets constructible in N.) Take $C = H(\aleph_0, \aleph_1^M)$. Then in $M[G]$, there is a mapping of $\aleph_0$ onto $\aleph_1^M$; so $\aleph_1^M$ is countable in $M[G]$. Now there is a one-one correspondence between $S^M(\omega)$ and $\aleph_1^M$ which is in M and hence in $M[G]$; so $S^M(\omega)$ is countable in $M[G]$. But by the corollary, every subset of ω which is constructible in $M[G]$ is in M and hence in $S^M(\omega)$. Thus in $M[G]$, there are only countably many constructible subsets of ω.

HISTORICAL NOTE. The independence of the axiom of constructibility was proved by Cohen. Cardinal collapsing was introduced by Lévy.

8. **Products.** Let C_1 and C_2 be two notions of forcing in M. We shall put subscripts 1 or 2 on our previous notation to indicate that we are considering C_1 or C_2; e.g. p_1 for an element of C_1, G_2 for a C_2-generic set. We write $M[G_1, G_2]$ for $(M[G_1])[G_2]$.

We define a partial ordering on $C_1 \times C_2$ by

$$\langle p_1, p_2 \rangle \leqq \langle q_1, q_2 \rangle \leftrightarrow p_1 \leqq q_1 \ \&\ p_2 \leqq q_2.$$

Clearly $C_1 \times C_2$ is a notion of forcing in M with largest element $\langle 1_1, 1_2 \rangle$.

PRODUCT THEOREM. *Let C_1 and C_2 be notions of forcing in M. If G_1 is C_1-generic over M and G_2 is C_2-generic over $M[G_1]$, then $G_1 \times G_2$ is $(C_1 \times C_2)$-generic over M, and $M[G_1 \times G_2] = M[G_1, G_2]$. Every set which is $(C_1 \times C_2)$-generic over M is obtained in this way.*

PROOF. It is easy to verify (G1), (G2) and (G3) for $G_1 \times G_2$. Let D be a $(C_1 \times C_2)$-dense set in M. We must show that $(G_1 \times G_2) \cap D \neq 0$, i.e. that $G_2 \cap D_2 \neq 0$, where

$$D_2 = [p_2 : (\exists p_1 \in G_1)(\langle p_1, p_2\rangle \in D)].$$

Since $D_2 \in M[G_1]$, it will suffice to show that D_2 is C_2-dense.

Let $q_2 \in C_2$; we must find $p_2 \leqq q_2$ and $p_1 \in G_1$ such that $\langle p_1, p_2\rangle \in D$. In other words, we must show $G_1 \cap D_1 \neq 0$, where

$$D_1 = [p_1 : (\exists p_2 \leqq q_2)(\langle p_1, p_2\rangle \in D)].$$

Since $D_1 \in M$, it suffices to show D_1 is C_1-dense. Let $q_1 \in C_1$, and choose $\langle p_1, p_2\rangle \leqq \langle q_1, q_2\rangle$ such that $\langle p_1, p_2\rangle \in D$. Then $p_1 \leqq q_1$ and $p_1 \in D_1$.

The equality $M[G_1 \times G_2] = M[G_1, G_2]$ holds because both are the smallest model of ZFC including M and containing G_1 and G_2.

Now let G be $(C_1 \times C_2)$-generic over M, and let G_1 and G_2 be the projections of G on C_1 and C_2 respectively. Clearly $G \subset G_1 \times G_2$. To prove $G = G_1 \times G_2$, let $\langle p_1, p_2\rangle \in G_1 \times G_2$. For some q_1 and q_2, $\langle p_1, q_2\rangle$, $\langle q_1, p_2\rangle \in G$. Hence they have a common extension $\langle r_1, r_2\rangle$ in G. Since $\langle r_1, r_2\rangle \leqq \langle p_1, p_2\rangle$, we have $\langle p_1, p_2\rangle \in G$.

The verification of (G1), (G2), and (G3) for G_1 and G_2 is easy. To verify (G4) for G_1, let D_1 be a C_1-dense set in M. Then $D_1 \times C_2$ is a $(C_1 \times C_2)$-dense set in M. Hence $G \cap (D_1 \times C_2) \neq 0$; so $G_1 \cap D_1 \neq 0$.

To verify (G4) for G_2, let D_2 be a C_2-dense set in $M[G_1]$. Let a be a name of D_2, and let Φ be the sentence of the forcing language which says that $\bar{a}$ is C_2-dense.

We show that

$$D = [\langle p_1, p_2\rangle : p_1 \Vdash \Phi \rightarrow \hat{p}_2 \in a]$$

is dense. Let $\langle q_1, q_2\rangle$ be given, and choose G_1' C_1-generic over M so that $q_1 \in G_1'$. If $K_{G_1'}(a)$ is C_2-dense, choose $p_2 \leqq q_2$ so that $p_2 \in K_{G_1'}(a)$; otherwise, let $p_2 = q_2$. In either case, $\vdash_{G_1'} \Phi \rightarrow \hat{p}_2 \in a$. Hence some $p_1 \in G_1'$ forces $\Phi \rightarrow \hat{p}_2 \in a$; and by (G3) and the extension lemma, we may suppose $p_1 \leqq q_1$. Then $\langle p_1, p_2\rangle \leqq \langle q_1, q_2\rangle$ and $\langle p_1, p_2\rangle \in D$.

Since $D \in M$, it follows that $G \cap D \neq 0$. Let $\langle p_1, p_2\rangle \in G \cap D$. Since $p_1 \in G_1$ and $p_1 \Vdash \Phi \rightarrow \hat{p}_2 \in a$, we have $\vdash_{G_1} \Phi \rightarrow \hat{p}_2 \in a$. But $\vdash_{G_1} \Phi$; so $p_2 \in \bar{a} = D_2$. But also $p_2 \in G_2$; so $G_2 \cap D_2 \neq 0$.

COROLLARY. *Let C_1 and C_2 be notions of forcing in M. Let G_1 be C_1-generic over M and let G_2 be C_2-generic over $M[G_1]$. Then G_1 is C_1-generic over $M[G_2]$, and $M[G_2, G_1] = M[G_1, G_2]$.*

PROOF. By the theorem, $G_1 \times G_2$ is $(C_1 \times C_2)$-generic over M. Applying the obvious isomorphism of $C_1 \times C_2$ and $C_2 \times C_1$, $G_2 \times G_1$ is $(C_2 \times C_1)$-generic

over M. Applying the theorem again, G_1 is C_1-generic over $M[G_2]$. We have $M[G_2, G_1] = M[G_1, G_2]$ because both are the smallest model of ZFC including M and containing G_1 and G_2.

We shall only consider one example of infinite products, the *weak power.* If C is a notion of forcing and I is any set, C^I is the set of all mappings p from I to C such that $[i \mid i \in I \,\&\, p(i) \neq 1]$ is finite. We then define $p \leqq q$ to mean $(\forall\, i \in I)\,(p(i) \leqq q(i))$. Clearly C^I is a notion of forcing whose largest element is the function constantly equal to 1_C. If C and I belong to a model M of ZFC, so does C^I; this is the reason for the finiteness restriction.

If J and K are disjoint subsets of I such that $J \cup K = I$, then C^I is isomorphic to $C^J \times C^K$ in a natural way. In particular, if $i \in I$ and $J = I - \{i\}$, then C^I is naturally isomorphic to $C^J \times C$. In either case, we can apply the product theorem.

9. **The axiom of choice.** By an *automorphism* of a notion of forcing C, we mean an automorphism of the partially ordered set C.

LEMMA 9.1. *Let π be an automorphism of C which is in M. Then $\pi(G)$ is C-generic over M, and $M[G] = M[\pi(G)]$.*

PROOF. Since π maps sets in M into sets in M, the first conclusion is trivial. Since $M[G]$ contains G and π, it contains $\pi(G)$; so $M[\pi(G)] \subset M[G]$ by the fundamental theorem. Substituting $\pi(G)$ for G and π^{-1} for π, we get $M[G] \subset M[\pi(G)]$; so $M[G] = M[\pi(G)]$.

Let $\mathfrak{A}$ be a set of automorphisms of C such that $\mathfrak{A} \in M$. An element a of M is $\mathfrak{A}$-*invariant* if $K_G(a) = K_{\pi(G)}(a)$ for every $\pi \in \mathfrak{A}$. For example, each $\hat{a}$ is invariant, since $K_G(\hat{a}) = K_{\pi(G)}(\hat{a}) = a$. A sentence Φ of the forcing language is $\mathfrak{A}$-*invariant* if every constant in Φ is $\mathfrak{A}$-invariant.

Let $\mathfrak{A}$ be a set of automorphisms of C. We say that C is $\mathfrak{A}$-*homogeneous* if for every $p, q \in C$, there is a $\pi \in \mathfrak{A}$ such that $\pi^{-1}(p)$ and q have a common extension. If $\mathfrak{A}$ is the set of all automorphisms of C, we say *homogeneous* for $\mathfrak{A}$-homogeneous.

If A is infinite, then $H(A, B)$ is homogeneous. For given $p, q \in H(A, B)$, we choose a permutation σ of A such that $\sigma(\mathrm{Do}(p)) \cap \mathrm{Do}(q) = 0$, and define the automorphism π by $\pi(r) = r \circ \sigma$. Then $\pi^{-1}(p) \cup q$ is a common extension of $\pi^{-1}(p)$ and q.

LEMMA 9.2. *Let $\mathfrak{A}$ be a set of automorphisms of C such that $\mathfrak{A} \in M$, and suppose that C is $\mathfrak{A}$-homogeneous in M. Let Φ be a sentence of the forcing language which is $\mathfrak{A}$-invariant. Then $\vdash_G \Phi$ if and only if $1 \Vdash \Phi$.*

PROOF. Since $1 \in G$, $1 \Vdash \Phi$ implies $\vdash_G \Phi$. Now suppose that $\vdash_G \Phi$ but that $\neg(1 \Vdash \Phi)$. By the former, some p forces Φ; and by the latter and (5.4), some q forces $\neg\, \Phi$. Choose $\pi \in \mathfrak{A}$ so that $\pi^{-1}(p)$ and q have a common extension. By the existence theorem and (G2), there is a generic G' containing $\pi^{-1}(p)$ and q. By Lemma 9.1, $\pi(G')$ is generic. Since $p \in \pi(G')$ and $q \in G'$, we have $\vdash_{\pi(G')} \Phi$ and $\vdash_{G'} \neg\, \Phi$. But $M[\pi(G')] = M[G']$ by Lemma 9.1, and the constants in Φ represent the same sets in these two models. Hence $\vdash_{\pi(G')} \Phi$ if and only if $\vdash_{G'} \Phi$, a contradiction.

We assume that the reader is familiar with OD (ordinal-definable) and HOD (hereditarily ordinal-definable) sets.[4] We need a slight generalization.

We say that u is OD *from* $v_1, \dots, v_k$ if there is an α such that $v_1, \dots, v_k \in V(\alpha)$ and u is definable in $\langle V(\alpha), \in, v_1, \dots, v_k\rangle$. We say that u is OD *over* w if for some $v_1, \dots, v_k \in w$, u is OD from $v_1, \dots, v_k, w$. We say that u is HOD *over* w if u is OD over w and every member of u is HOD over w; this is a definition by induction on $\mathrm{rk}(u)$.

The basic results about OD and HOD sets carry over to this situation. Thus every ordinal is OD from any $v_1, \dots, v_k$ and hence OD over any w. If $u_1, \dots, u_m$ are OD from $v_1, \dots, v_k$ (or over w), then $\mu(v_1, \dots, v_k)$ is also (where μ is a term defined in ZFC). The class of sets HOD over w is a model of *ZF* (but not necessarily of the axiom of choice). It is also clear that every member of w is OD over w.

LEMMA 9.3. *Let $\mathfrak{A}$ be a set of automorphisms of C such that $\mathfrak{A} \in M$, and suppose that C is $\mathfrak{A}$-homogeneous. Let u be* OD *from $v_1, \dots, v_k$ in $M[G]$, and suppose that $v_1, \dots, v_k$ have $\mathfrak{A}$-invariant names. Then $u \cap M \in M$.*

PROOF. There is a formula $\Phi(x)$ of the forcing language, containing names only for $v_1, \dots, v_k$ and some $\alpha \in M[G]$, such that

$$\bar{a} \in u \leftrightarrow \vdash_G \Phi(a) \tag{9.1}$$

for all a. Since $\alpha \in M$ by Lemma 7.1, $\hat{\alpha}$ is $\mathfrak{A}$-invariant; so we may suppose that every name in $\Phi(x)$ is $\mathfrak{A}$-invariant. Then putting $\hat{a}$ for a in (9.1) and using Lemma 9.2,

$$a \in u \leftrightarrow \vdash_G \Phi(\hat{a}) \leftrightarrow 1 \Vdash \Phi(\hat{a}).$$

Thus $u \cap M = [a : 1 \Vdash \Phi(\hat{a})]$ is a class in M. But if $\beta = \mathrm{rk}(u)$, then $u \cap M \subset V(\beta) \cap M = V^M(\beta)$; so $u \cap M$ is a set in M.

THEOREM 9.1. *Let C be homogeneous in M. If u is* OD *in $M[G]$, then $u \cap M \in M$; if u is* HOD *in $M[G]$, then $u \in M$.*

PROOF. The first conclusion is a special case of Lemma 9.3. The second conclusion follows easily from the first by induction on $\mathrm{rk}(u)$.

Now let M satisfy the axiom of constructibility, and let $C = H(\omega, 2)$. Using Theorem 9.1, the corollary to Lemma 7.1, and the fact that every constructible set is HOD, we see that in $M[G]$ the constructible sets coincide with the HOD sets. As seen in §7, there is a subset of ω which is nonconstructible in $M[G]$. Since every member of ω is HOD, it follows that this set is not OD in $M[G]$. From this, it follows that there is no OD mapping from an ordinal to $S(\omega)$ in $M[G]$; for if F is OD, then so is every $F(\alpha)$.

Next let $C = H(\omega, 2)^\omega$. Let G_i be the set of ith coordinates of elements in G, and let $H = [G_i : i \in \omega]$. Let N be the set of all sets which are HOD over H in $M[G]$. Then N is a model of ZF. We shall show that N is not a model of the axiom of choice; in fact, that there is no mapping from an ordinal onto $S(\omega)$ in N.

[4] See the article by Myhill and Scott, these Proceedings.

Suppose that there is such a mapping F. Then for some n, F is OD from $G_0, G_1, \ldots, G_{n-1}, H$ (in $M[G]$). Every element of $S(\omega) \cap N$ is $F(\alpha)$ for some α in $M[G]$ and hence is OD from $G_0, G_1, \ldots, G_{n-1}, H$.

We regard C as the product $C' \times C''$, where $C' = H(\omega, 2)^n$ and $C'' = H(\omega, 2)^{\omega-n}$. Let G' and G'' be the projections of G on C' and C'' respectively. By the product theorem, G' is C'-generic over M; G'' is C''-generic over $M' = M[G']$; and $M'[G''] = M[G]$. Applying the product theorem again, G_n is $H(\omega, 2)$-generic over M'. Hence the set A having $U(G_n)$ as characteristic function is a subset of ω not in M'. Since $A \subset \omega \subset M'$, $A \cap M' = A$; so $A \cap M'$ is not in M'.

We complete the proof by using Lemma 9.3 to show that $A \cap M' \in M'$. For each permutation π of $\omega - n$, we define an automorphism π^* of C'' by $(\pi^*(p))_i = p_{\pi(i)}$. Let $\mathfrak{A}$ be the set of all π^* for π in M'. Clearly $\mathfrak{A} \in M'$. Noting that every permutation of $\omega - n$ which moves only finitely many numbers is in M', we see that C'' is $\mathfrak{A}$-homogeneous. Now A is OD from G_n and hence is in N; so A is OD from $G_0, G_1, \ldots, G_{n-1}, H$ in $M[G] = M'[G'']$. Hence we need only show that $G_0, G_1, \ldots, G_{n-1}, H$ have $\mathfrak{A}$-invariant names in M'.

Since $G_0, G_1, \ldots, G_{n-1}$ are in M', they have the $\mathfrak{A}$-invariant names $\hat{G}_0, \hat{G}_1, \ldots, \hat{G}_{n-1}$. Now set

$$a_i = \hat{G}_i \quad \text{if } i < n,$$
$$a_i = [\langle \hat{p}_i, p \rangle : p \in C''] \quad \text{if } i \geqq n,$$
$$a = [a_i : i \in \omega] \times \{1\}.$$

It is easy to see that a_i is a name of G_i and that a is a name of H. In $M'[\pi^*(G'')]$, a_i is a name of G_i if $i < n$ and a name of $\pi^*(G'')_i = G_{\pi^{-1}(i)}$ if $i \geqq n$; so a is a name of H. This shows that a is $\mathfrak{A}$-invariant.

HISTORICAL NOTE. The independence of the axiom of choice is due to Cohen. Models of ZFC in which there is no OD mapping from an ordinal to $S(\omega)$ were first constructed by Feferman [**4**]. Theorem 9.1 is due to Lévy [**7**].

10. **Preserving cardinals.** We now turn to the relation between cardinals in M and cardinals in $M[G]$.

LEMMA 10.1 *Every cardinal in $M[G]$ is a cardinal in M.*

PROOF. Let $\mathfrak{m}$ be a cardinal in $M[G]$. By Lemma 7.1, $\mathfrak{m}$ is an ordinal in M. If $\mathfrak{m}$ is not a cardinal in M, there is a mapping from an ordinal smaller than $\mathfrak{m}$ onto $\mathfrak{m}$ which is in M and hence in $M[G]$. This is impossible.

Since $0, 1, \ldots, \omega$ are absolute, they are cardinals in both M and $M[G]$. On the other hand, uncountable cardinals in M need not be cardinals in $M[G]$, as we saw in the last section.

If α is an ordinal in M, then, since $M \subset M[G]$, we have

$$\text{(10.1)} \qquad \mathrm{cf}^{M[G]}(\alpha) \leqq \mathrm{cf}^{M}(\alpha).$$

The converse holds when $\mathrm{cf}^{M}(\alpha) \leqq \omega$, since being a limit ordinal is absolute. Again, the last section shows that the converse may fail when $\mathrm{cf}(\alpha)$ is uncountable in M.

We shall now obtain some sufficient conditions for the converses to hold.

Let C be a notion of forcing, and let $p, q \in C$. We say p and q are *compatible* if they have a common extension; otherwise, we say p and q are *incompatible*. We say C *satisfies the* $\mathfrak{m}$*-chain condition* if every set of pairwise incompatible elements in C has cardinal $<\mathfrak{m}$.

LEMMA 10.2 *Let* $\mathfrak{m}$ *be a regular cardinal in* M *such that* C *satisfies the* $\mathfrak{m}$*-chain condition in* M. *Then*: (a) *if* $\alpha \in M$ *and* $\mathfrak{m} \leqq \mathrm{cf}^M(\alpha)$, *then* $\mathrm{cf}^M(\alpha) = \mathrm{cf}^{M[G]}(\alpha)$; *every cardinal in* M *which is* $\geqq \mathfrak{m}$ *is a cardinal in* $M[G]$.

PROOF. Let $x(y) = z$ be the formula of set theory which says that x is a function whose value at y is z. We say that γ is a *possible value* of a at β if some p forces $a(\hat{\beta}) = \hat{\gamma}$. We claim that the set of possible values of a at β has cardinal $<\mathfrak{m}$. For each such possible value γ, let p_γ force $a(\hat{\beta}) = \hat{\gamma}$. It will clearly suffice to show that if $\gamma \neq \delta$, then p_γ and p_δ are incompatible. Suppose they had a common extension q. Choose a generic G' such that $q \in G'$. By the extension lemma, q forces $a(\hat{\beta}) = \hat{\gamma}$ and $a(\hat{\beta}) = \hat{\delta}$; so by the truth lemma, $\vdash_{G'} a(\hat{\beta}) = \hat{\gamma}$ and $\vdash_{G'} a(\hat{\beta}) = \hat{\delta}$. Hence $\gamma = \bar{a}(\beta) = \delta$.

Now let α be as in (a). Let $\mathfrak{n} = \mathrm{cf}^{M[G]}(\alpha)$. Then $\mathfrak{n}$ is a cardinal in $M[G]$ and hence in M. Let $\bar{a}$ be a mapping from $\mathfrak{n}$ onto a cofinal subset of α. Let b be the set of possible values of a at ordinals $<\mathfrak{n}$. Clearly $b \in M$. If $\sigma < \mathfrak{n}$ and $\bar{a}(\sigma) = \tau$, then some $p \in G$ forces $a(\hat{\sigma}) = \hat{\tau}$; so τ is a possible value of a at σ. Thus $\mathrm{Ra}(\bar{a}) \subset b$; so b is cofinal in α. Hence $\mathrm{cf}(\alpha) \leqq |b|$ in M.

By the result above, b is, in M, the union of $\mathfrak{n}$ sets, each having cardinal $<\mathfrak{m}$. If $\mathfrak{n} < \mathfrak{m}$, then $|b| < \mathfrak{m}$ in M (since $\mathfrak{m}$ is regular in M). This is impossible since $\mathfrak{m} \leqq \mathrm{cf}(\alpha) \leqq |b|$ in M. Thus $\mathfrak{m} \leqq \mathfrak{n}$; so $|b| \leqq \mathfrak{m} \cdot \mathfrak{n} = \mathfrak{n}$ in M. Thus $\mathrm{cf}(\alpha) \leqq |b| \leqq \mathfrak{n}$ in M, i.e. $\mathrm{cf}^M(\alpha) \leqq \mathrm{cf}^{M[G]}(\alpha)$. Using (10.1), we get equality.

Now let $\mathfrak{n}$ be a cardinal in M such that $\mathfrak{m} \leqq \mathfrak{n}$; we show by induction on $\mathfrak{n}$ that $\mathfrak{n}$ is a cardinal in $M[G]$. If $\mathfrak{n}$ is regular in M, then $\mathrm{cf}^M(\mathfrak{n}) = \mathfrak{n} \geqq \mathfrak{m}$; so $\mathrm{cf}^{M[G]}(\mathfrak{n}) = \mathrm{cf}^M(\mathfrak{n}) = \mathfrak{n}$ by (a); so $\mathfrak{n}$ is a cardinal in $M[G]$. If $\mathfrak{n}$ is singular in M, then $\mathfrak{n}$ is the supremum of the set of cardinals $\mathfrak{p}$ in M such that $\mathfrak{m} \leqq \mathfrak{p} < \mathfrak{n}$. Since these are all cardinals in $M[G]$ and the supremum of a set of cardinals is a cardinal, $\mathfrak{n}$ is a cardinal in $M[G]$.

COROLLARY. *If* C *satisfies the* $\aleph_1$*-chain condition in* M, *then* $\mathrm{cf}^M = \mathrm{cf}^{M[G]}$, *and* M *and* $M[G]$ *have the same cardinals.*

We wish to apply these results to $H_{\mathfrak{m}}(A, B)$. We note first that for each $\mathfrak{p} < \mathfrak{m}$ (including finite $\mathfrak{p}$), there are at most $|A|^{\mathfrak{p}}$ subsets D of A such that $|D| = \mathfrak{p}$, and that for each such D, there are $|B|^{\mathfrak{p}}$ mappings from D to B. Thus

$$|H_{\mathfrak{m}}(A, B)| \leqq \sum_{\mathfrak{p}<\mathfrak{m}} |A|^{\mathfrak{p}} \cdot |B|^{\mathfrak{p}};$$

so

$$|H_{\mathfrak{m}}(A, B)| \leqq (|A| \cdot |B|)^{\underline{\mathfrak{m}}}. \tag{10.2}$$

LEMMA 10.3 *If* $\mathfrak{m}^{\underset{\smile}{\mathfrak{m}}} = \mathfrak{m}$ *and* $|B| \leqq \mathfrak{m}$, *then* $H_{\mathfrak{m}}(A, B)$ *satisfies the* $\mathfrak{m}^+$*-chain condition.*

PROOF Let I be a set of pairwise incompatible elements of $H_{\mathfrak{m}}(A, B)$. We define a subset A_α of A for each α by induction. Let $A_0 = 0$; and for α a limit number, let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Now let A_α be chosen. For each $p \in H_{\mathfrak{m}}(A_\alpha, B)$, choose a $q \in I$ whose restriction to A_α is p, provided that such a q exists. Let $A_{\alpha+1}$ be the union of the domains of these q's and A_α.

We prove by induction that $|A_\alpha| \leqq \mathfrak{m}$ for $\alpha \leqq \mathfrak{m}$. This is trivial if $\alpha = 0$ or α is a limit number. Suppose $|A_\alpha| \leqq \mathfrak{m}$. By (10.2),

$$|H_{\mathfrak{m}}(A_\alpha, B)| \leqq (\mathfrak{m} \cdot |B|)^{\underset{\smile}{\mathfrak{m}}} = \mathfrak{m}^{\underset{\smile}{\mathfrak{m}}} = \mathfrak{m}. \tag{10.3}$$

Then clearly

$$|A_{\alpha+1}| \leqq |A_\alpha| + |H_{\mathfrak{m}}(A_\alpha, B)| \cdot \mathfrak{m} \leqq \mathfrak{m} + \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}.$$

In particular, $|A_{\mathfrak{m}}| \leqq \mathfrak{m}$; so $|H_{\mathfrak{m}}(A_{\mathfrak{m}}, B)| \leqq \mathfrak{m}$ by (10.3).

We complete the proof by showing that $I \subset H_{\mathfrak{m}}(A_{\mathfrak{m}}, B)$. Let $p \in I$. Since $|\mathrm{Do}(p)| < \mathfrak{m}$, there is an $\alpha < \mathfrak{m}$ such that $\mathrm{Do}(p) \cap A_\alpha = \mathrm{Do}(p) \cap A_{\alpha+1}$. Choose $q \in I$ so that p and q have the same restriction to A_α and $\mathrm{Do}(q) \subset A_{\alpha+1}$. If $x \in \mathrm{Do}(p) \cap \mathrm{Do}(q)$, then $x \in A_{\alpha+1}$; so $x \in \mathrm{Do}(p) \cap A_{\alpha+1} \subset A_\alpha$; so $p(x) = q(x)$. It follows that p and q are compatible. Since $p, q \in I$, this implies $p = q$; so $\mathrm{Do}(p) \subset A_{\alpha+1} \subset A_{\mathfrak{m}}$ and hence $p \in H_{\mathfrak{m}}(A_{\mathfrak{m}}, B)$.

For conditions under which $\mathfrak{m}^{\underset{\smile}{\mathfrak{m}}} = \mathfrak{m}$ holds, see §2.

A subset D of C is a *section* if every extension of a condition in D is in D.

LEMMA 10.4. *Let D be a section in M. If $D \cap G' \neq 0$ for every G' which is C-generic over M, then D is dense.*

PROOF. Given p, choose G' generic such that $p \in G'$. Choose $q \in D \cap G'$, and let r be a common extension of p and q. Then $r \leqq p$ and $r \in D$.

A notion of forcing C is $\mathfrak{m}$*-closed* if for each $\alpha < \mathfrak{m}$ and each decreasing sequence $\{p_\beta\}_{\beta<\alpha}$ of conditions in C, there is a $p \in C$ such that $p \leqq p_\beta$ for all $\beta < \alpha$. For example, if $\mathfrak{m}$ is regular, then $H_{\mathfrak{m}}(A; B)$ is $\mathfrak{m}$-closed; for we may take $p = \bigcup_{\beta<\alpha} p_\beta$.

LEMMA 10.5. *If C is $\mathfrak{m}$-closed in M, $\alpha < \mathfrak{m}$, and $\{D_\beta\}_{\beta<\alpha}$ is a sequence in M of dense sections, then $\bigcap_{\beta<\alpha} D_\beta$ is dense.*

PROOF. Given p, we may define inductively in M a decreasing sequence $\{p_\beta\}_{\beta<\alpha}$ such that $p_\beta \in D_\beta$ and $p_\beta \leqq p$. (We must use the hypothesis that C is $\mathfrak{m}$-closed when β is a limit number.) Choosing q so that $q \leqq p_\beta$ for all $\beta < \alpha$, we have $q \leqq p$ and $q \in \bigcap_{\beta<\alpha} D_\beta$.

LEMMA 10.6. *Let C be $\mathfrak{m}$-closed in M, and let $\alpha < \mathfrak{m}$. Then $({}^\alpha a)^M = ({}^\alpha a)^{M[G]}$ for all $a \in M$, and $S^M(\alpha) = S^{M[G]}(\alpha)$.*

PROOF. Clearly $({}^{\alpha}a)^{M} \subset ({}^{\alpha}a)^{M[G]}$. Now let $\bar{b} \in ({}^{\alpha}a)^{M[G]}$. Let $\Phi(x, y)$ be

$$\exists z(z \in \hat{a} \mathbin{\&} \langle z, y\rangle \in b) \rightarrow \langle x, y\rangle \in b;$$

and for each $\beta < \alpha$, let

$$D_{\beta} = [p : \exists c(p \Vdash \Phi(\hat{c}, \hat{\beta}))].$$

Then D_{β} is a section in M. We use Lemma 10.4 to see that it is dense. Let G' be generic. Then for some $c \in a$, $\vdash_{G'} \Phi(\hat{c}, \hat{\beta})$; so by the truth lemma, $D_{\beta} \cap G' \neq 0$.

By Lemma 10.5, $D = \bigcap_{\beta<\alpha} D_{\beta}$ is dense. Hence there is a $q \in G \cap D$. If $q \Vdash \Phi(\hat{c}, \hat{\beta})$,then $c = \bar{b}(\beta)$. Hence $\bar{b}(\beta)$ is the unique c such that $q \Vdash \Phi(\hat{c}, \hat{\beta})$. It readily follows that $\bar{b} \in M$.

The last conclusion follows by taking $a = 2$ and using the correspondence between a set and its characteristic function.

COROLLARY. *Let C be $\mathfrak{m}$-closed in M. Then:* (a) *if $\alpha \in M$ and* $\mathrm{cf}^{M}(\alpha) \leqq \mathfrak{m}$, *then* $\mathrm{cf}^{M}(\alpha) = \mathrm{cf}^{M[G]}(\alpha)$; (b) *every cardinal $\mathfrak{n}$ in M which is $\leqq \mathfrak{m}$ is a cardinal in $M[G]$.*

PROOF. (a) If not, then $\mathrm{cf}^{M[G]}(\alpha) < \mathrm{cf}^{M}(\alpha) \leqq \mathfrak{m}$. Let f be a mapping from $\mathrm{cf}^{M[G]}(\alpha)$ onto a cofinal subset of α in $M[G]$. By the theorem, $f \in M$; so $\mathrm{cf}^{M[G]}(\alpha) \geqq \mathrm{cf}^{M}(\alpha)$, a contradiction. (b) If not, there is a mapping f from an ordinal $<\mathfrak{n}$ onto $\mathfrak{n}$ in $M[G]$. By the theorem, $f \in M$; and this is impossible.

We note that if we prove that M and $M[G]$ have the same cardinals by means of the theorems of this section, then $\mathrm{cf}^{M} = \mathrm{cf}^{M[G]}$. This suggests a problem: can we choose C so that M and $M[G]$ have the same cardinals, but $\mathrm{cf}^{M} \neq \mathrm{cf}^{M[G]}$? Prikry has shown that it is possible if there is a measurable cardinal in M.

HISTORICAL NOTE. The results on $\mathfrak{m}$-closed notions are due to Solovay; the remaining results are due to Cohen.

11. **The continuum hypothesis.**

We first investigate the size of power sets in $M[G]$.

LEMMA 11.1. *Let C satisfy the $\mathfrak{m}$-chain condition in M. Then for every infinite cardinal $\mathfrak{n}$ in M,*

$$|S(\mathfrak{n})|^{M[G]} \leqq ((|C|^{\underset{\smile}{\mathfrak{m}}})^{\mathfrak{n}})^{M}.$$

PROOF. For $a \in M$ and $\alpha < \mathfrak{n}$, let $\phi_a(\alpha) = [p : p \Vdash \hat{\alpha} \in a]$. Then

$$\bar{a} \subset \mathfrak{n} \mathbin{\&} \bar{b} \subset \mathfrak{n} \mathbin{\&} \phi_a = \phi_b \rightarrow \bar{a} = \bar{b}.$$

By symmetry, it suffices to show that $\bar{a} \subset \bar{b}$. Let $\alpha \in \bar{a}$. Then some $p \in G$ is in $\phi_a(\alpha)$ and hence in $\phi_u(\beta)$; so $\vdash_G \hat{\alpha} \in b$; so $\alpha \in \bar{b}$.

It follows that

$$|S(\mathfrak{n})|^{M[G]} \leqq |[\phi_a : a \in M]|^{M}.$$

Now ϕ_a is a mapping from $\mathfrak{n}$ to Q, where Q is the set of all sets $[p : p \Vdash \Phi]$. It will thus suffice to prove $|Q| \leqq |C|^{\underset{\smile}{\mathfrak{m}}}$ in M.

Let $a \in Q$. Using Zorn's lemma, choose a maximal pairwise incompatible subset b of a. Then a can be recovered from b by the equivalence

$$p \in a \leftrightarrow (\forall q \leqq p)(\exists r \in b)(q \text{ and } r \text{ are compatible}).$$

The implication from left to right holds by the extension lemma and the maximality of b. Suppose $p \notin a$. If $a = [p : p \Vdash \Phi]$, then by (5.4) there is a $q \leqq p$ such that $q \Vdash \neg\Phi$. By (5.3) and the extension lemma, no r compatible with q can force Φ; so no such r can be in b.

It follows that $|Q|$ is at most equal to the number of pairwise incompatible subsets b of C. Since $|b| < \mathfrak{m}$ for all such b,

$$|Q| \leqq |S_{\mathfrak{m}}(C)| \leqq |C|^{\underline{\mathfrak{m}}}.$$

Now let $\mathfrak{m}$ and $\mathfrak{n}$ be cardinals in M. Can we choose C so that M and $M[G]$ have the same cardinals and $2^{\mathfrak{m}} = \mathfrak{n}$ in $M[G]$? If we can, then

$$\mathfrak{n}^{\mathfrak{m}} = (2^{\mathfrak{m}})^{\mathfrak{m}} = 2^{\mathfrak{m}} = \mathfrak{n}$$

in $M[G]$. Since $({}^{\mathfrak{m}}\mathfrak{n})^M \subset ({}^{\mathfrak{m}}\mathfrak{n})^{M[G]}$, and M and $M[G]$ have the same cardinals, we have $(\mathfrak{n}^{\mathfrak{m}})^M \leq (\mathfrak{n}^{\mathfrak{m}})^{M[G]}$. Hence $\mathfrak{n}^{\mathfrak{m}} = \mathfrak{n}$ in M.

Let us therefore assume that $\mathfrak{n}^{\mathfrak{m}} = \mathfrak{n}$ in M. To get $2^{\mathfrak{m}} = \mathfrak{n}$, we should introduce $\mathfrak{n}$ subsets of $\mathfrak{m}$. We actually introduce a mapping F from $\mathfrak{n} \times \mathfrak{m}$ to 2, and take $[\beta : F(\alpha, \beta) = 0]$ to be the αth subset of $\mathfrak{m}$.

Let $C = H(\mathfrak{n} \times \mathfrak{m};\, 2)$. Setting $F = \bigcup_{p \in G} p$, F is a mapping from $\mathfrak{m} \times \mathfrak{n}$ to 2 in $M[G]$. Set $A_\alpha = [\beta : \beta < \mathfrak{m} \;\&\; F(\alpha, \beta) = 0]$ for $\alpha < \mathfrak{n}$. Then A_α is a subset of $\mathfrak{m}$ in $M[G]$. Moreover, $\alpha \neq \alpha' \to A_\alpha \neq A_{\alpha'}$; this follows from the fact that

$$[p : \exists\beta(p(\alpha, \beta) \;\&\; p(\alpha', \beta) \text{ are defined and unequal})]$$

is a dense set in M.

By Lemma 10.3 and the Corollary to Lemma 10.2, $\mathrm{cf}^M = \mathrm{cf}^{M[G]}$ and M and $M[G]$ have the same cardinals. In particular, $\mathfrak{m}$ and $\mathfrak{n}$ are cardinals in $M[G]$. Since we have exhibited $\mathfrak{n}$ distinct subset of $\mathfrak{m}$ in $M[G]$, $2^{\mathfrak{m}} \geqq \mathfrak{n}$ in $M[G]$. By Lemma 11.1,

$$(2^{\mathfrak{m}})^{M[G]} \leqq ((|C|^{\aleph_0})^{\mathfrak{m}})^M.$$

Calculating in M, we have by (10.2)

$$|C| \leqq (\mathfrak{m} \cdot \mathfrak{n})^{\underline{\aleph_0}} = \mathfrak{n}$$

(since $\mathfrak{n}^{\mathfrak{m}} = \mathfrak{n}$ implies $\mathfrak{m} < \mathfrak{n}$); so

$$(|C|^{\aleph_0})^{\mathfrak{m}} = |C|^{\mathfrak{m}} \leqq \mathfrak{n}^{\mathfrak{m}} = \mathfrak{n}.$$

Hence $2^{\mathfrak{m}} = \mathfrak{n}$ in $M[G]$. This proves the following theorem.

THEOREM 11.1. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be infinite cardinals in M such that $\mathfrak{n}^{\mathfrak{m}} = \mathfrak{n}$ in M. For a suitable choice of C, $\mathrm{cf}^M = \mathrm{cf}^{M[G]}$; M and $M[G]$ have the same cardinals; and $2^{\mathfrak{m}} = \mathfrak{n}$ in $M[G]$.*

Now suppose that we have defined a constant Γ in ZFC and proved in ZFC that Γ is a cardinal. We would like to show that $2^{\aleph_0} = \Gamma$ is consistent with ZFC. In view of König's theorem, we require that $\mathrm{cf}(\Gamma) > \omega$ is provable in ZFC. Assume this, and take M to satisfy the axiom of constructibility and hence the GCH. From the GCH and $\mathrm{cf}(\Gamma) > \omega$ we can prove $\Gamma^{\aleph_0} = \Gamma$; so $(\Gamma^{\aleph_0})^M = \Gamma^M$. Choosing C as in Theorem 11.1, we have $2^{\aleph_0} = \Gamma^M$ in $M[G]$. We then have the desired result if we can show $\Gamma^M = \Gamma^{M[G]}$. Recalling that M and $M[G]$ have the same cardinals and the same cf function, this holds if Γ is, say, $\aleph_2$ or $\aleph_{\omega_1}$ or the first weakly inaccessible cardinal (provided that there is a weakly inaccessible cardinal in M).

Suppose now that the GCH holds in M. If we make $2^{\mathfrak{m}} = \mathfrak{n}$ in $M[G]$, then $2^{\mathfrak{p}} \geqq \mathfrak{n}$ for $\mathfrak{p} \geqq \mathfrak{m}$; so the GCH may fail in $M[G]$ above $\mathfrak{m}$. We show that we can keep the GCH below $\mathfrak{m}$ if $\mathfrak{m}$ is regular in M.

THEOREM 11.2. *Let the* GCH *hold in M. Let $\mathfrak{m}$ and $\mathfrak{n}$ be infinite cardinals of M such that $\mathfrak{m}$ is regular in M and* $\mathrm{cf}(\mathfrak{n}) > \mathfrak{m}$ *in M. For a suitable choice of C,* $\mathrm{cf}^M = \mathrm{cf}^{M[G]}$*; M and $M[G]$ have the same cardinals;* $2^{\mathfrak{m}} = \mathfrak{n}$ *in $M[G]$; and* $\forall \mathfrak{p}(\mathfrak{p} < \mathfrak{m} \to 2^{\mathfrak{p}} = \mathfrak{p}^+)$ *holds in $M[G]$.*

PROOF. We take $C = H^M_{\mathfrak{m}}(\mathfrak{m} \times \mathfrak{n}, 2)$. The hypotheses show that $\mathfrak{m}^{\underline{\mathfrak{m}}} = \mathfrak{m}$ in M. Hence Lemmas 10.2 and 10.3 and the corollary to Lemma 10.6 show that $\mathrm{cf}^M = \mathrm{cf}^{M[G]}$ and that M and $M[G]$ have the same cardinals. The proof that $2^{\mathfrak{m}} = \mathfrak{n}$ in $M[G]$ is essentially as before (noting that $\mathrm{cf}(\mathfrak{n}) > \mathfrak{m}$ and the GCH imply $\mathfrak{n}^{\mathfrak{m}} = \mathfrak{n}$). If $\mathfrak{p} < \mathfrak{m}$, $S^M(\mathfrak{p}) = S^{M[G]}(\mathfrak{p})$ by Lemma 10.6. Since M and $M[G]$ have the same cardinals and $2^{\mathfrak{p}} = \mathfrak{p}^+$ in M, we see that $2^{\mathfrak{p}} = \mathfrak{p}^+$ in $M[G]$.

It is not known if Theorem 11.2 holds when $\mathfrak{m}$ is singular in M. The simplest unsolved problem is: is it consistent with ZFC to assume that $\forall n(n < \omega \to 2^{\aleph_n} = \aleph_{n+1})$ and $2^{\aleph_\omega} \neq \aleph_{\omega+1}$?

HISTORICAL NOTE. Theorem 11.1 is due to Cohen; Theorem 11.2 is due to Solovay.

12. **Forcing with classes.** So far we have assumed that C is a set in M. Sometimes we can construct forcing models when C is merely a class in M. Since the general situation has not been investigated very thoroughly, we shall consider only a specific problem.

This problem is a generalization of that in the last section. Suppose that H is a mapping from the set of infinite cardinals of M to itself which is a functional in M. We want to choose C so that M and $M[G]$ have the same cardinals, and so that for every infinite cardinal $\mathfrak{m}$ in M, $2^{\mathfrak{m}} = H(\mathfrak{m})$ in $M[G]$.

We must clearly have

(12.1) $$\mathfrak{m} \leqq \mathfrak{n} \to H(\mathfrak{m}) \leqq H(\mathfrak{n}).$$

Moreover

(12.2) $$\mathfrak{m} < \mathrm{cf}(H(\mathfrak{m}))$$

must hold in M. For it will hold in $M[G]$ by König's theorem and hence in M by (10.1).

We shall assume in addition that M satisfies the axiom of constructibility.[5] We will then show that for a suitable C, M and $M[G]$ have the same cardinals, $\mathrm{cf}^M = \mathrm{cf}^{M[G]}$, and $2^{\mathfrak{m}} = H(\mathfrak{m})$ in $M[G]$ for every regular cardinal $\mathfrak{m}$ in M.

Remark. Again not much is known about singular cardinals. Certainly further hypotheses on H must be added to cover this case. For example,

$$(\forall n \in \omega)(2^{\aleph_n} = \aleph_{\omega+1}) \rightarrow 2^{\aleph_\omega} = \aleph_{\omega+1}.$$

For

$$2^{\aleph_\omega} = 2^{\Sigma \aleph_n} = \Pi\, 2^{\aleph_n} = \aleph_{\omega+1}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \aleph_{\omega+1}.$$

We now describe C. We let $\mathfrak{m}$ and $\mathfrak{n}$ vary through *regular* cardinals of M. We set

$$Q_{\mathfrak{m}} = [\langle \mathfrak{n}, \alpha, \beta \rangle : \mathfrak{n} \leqq \mathfrak{m} \;\&\; \alpha < H(\mathfrak{n}) \;\&\; \beta < \mathfrak{n}].$$

We let C be the set of all functions p in M such that $\mathrm{Ra}(p) \subset 2$; $\mathrm{Do}(p) \subset \bigcup_{\mathfrak{m}} Q_{\mathfrak{m}}$; and $|\mathrm{Do}(p) \cap Q_{\mathfrak{m}}| < \mathfrak{m}$ in M for all $\mathfrak{m}$. For $p, q \in C$, $p \leqq q$ means $q \subset p$. Obviously, C is a notion of forcing which is a class in M.

It is natural to require our generic set to meet the dense classes in M. We therefore choose G to be C-generic over M', where M' is the set of all classes in M. Since M' is countable, this can be done by the existence theorem.

We now define $M[G]$ as before. Just as before, we prove that $M[G]$ is countable and transitive and that $M \subset M[G]$. In trying to extend §5, there is a difficulty: the inductive definition of $p \Vdash^* a \neq b$ can no longer be given in M. We therefore proceed differently.

Let

$$C_{\mathfrak{m}} = [p : p \in C \;\&\; \mathrm{Do}(p) \subset Q_{\mathfrak{m}}],$$
$$C^{\mathfrak{m}} = [p : p \in C \;\&\; \mathrm{Do}(p) \cap Q_{\mathfrak{m}} = 0].$$

Both $C_{\mathfrak{m}}$ and $C^{\mathfrak{m}}$ contain 1 and hence are notions of forcing. Moreover $C_{\mathfrak{m}}$ is a set in M; the function which maps $\mathfrak{m}$ into $C_{\mathfrak{m}}$ is a functional in M; and $C^{\mathfrak{m}}$ is a class in M.

For $p \in C$, let $p_{\mathfrak{m}}$ be the restriction of p to $Q_{\mathfrak{m}}$, and let $p^{\mathfrak{m}} = p - p_{\mathfrak{m}}$. It is easy to check that $p \rightarrow \langle p_{\mathfrak{m}}, p^{\mathfrak{m}} \rangle$ is an isomorphism of C and $C_{\mathfrak{m}} \times C^{\mathfrak{m}}$ whose inverse takes $\langle q, r \rangle$ into $q \cup r$. Let $G_{\mathfrak{m}} = [p_{\mathfrak{m}} : p \in G]$, $G^{\mathfrak{m}} = [p^{\mathfrak{m}} : p \in G]$. Then by the product theorem,[6] $G_{\mathfrak{m}}$ is $C_{\mathfrak{m}}$-generic over M and $G^{\mathfrak{m}}$ is $C^{\mathfrak{m}}$-generic over $M[G^{\mathfrak{m}}]$. Moreover, G corresponds to $G_{\mathfrak{m}} \times G^{\mathfrak{m}}$ under the above isomorphism, i.e.

$$G = [p \cup q : p \in G_{\mathfrak{m}} \;\&\; q \in G^{\mathfrak{m}}]. \tag{12.3}$$

[5] Only the GCH and the existence of a definable well-ordering of the universe are used, and the latter can be dispensed with. However, this is immaterial for consistency purposes.

[6] Strictly speaking, the product theorem is not applicable here; but the part of the proof needed carries over without difficulty.

It follows that

$$G \cap C_{\mathfrak{m}} = G_{\mathfrak{m}}. \tag{12.4}$$

Noting that C is the union of the $C_{\mathfrak{m}}$, we may define $\Delta(b)$ in M by induction on $\mathrm{rk}(b)$ as follows: $\Delta(b)$ is the smallest $\mathfrak{m}$ such that $\Delta(a) \leqq \mathfrak{m}$ for all $a \in \mathrm{Ra}(b)$ and $p \in C_{\mathfrak{m}}$ for all $p \in \mathrm{Do}(b)$. We prove by induction on $\mathrm{rk}(b)$ that

$$\Delta(b) \leqq \mathfrak{m} \rightarrow K_G(b) = K_{G_{\mathfrak{m}}}(b). \tag{12.5}$$

If $\langle a, p\rangle \in b$, then $\Delta(a) \leqq \Delta(b) < \mathfrak{m}$, so $K_G(a) = K_{G_{\mathfrak{m}}}(a)$ by induction hypothesis. Also $p \in C_{\mathfrak{m}}$; so $p \in G \leftrightarrow p \in G_{\mathfrak{m}}$ by (12.4). Hence $K_G(b) = K_{G_{\mathfrak{m}}}(b)$.

Now define

$$a^{(\mathfrak{m})} = [\langle b^{(\mathfrak{m})}, p\rangle : \langle b, p\rangle \in a \;\&\; p \in C_{\mathfrak{m}}].$$

Again this is a functional in M. We prove

$$K_G(a^{(\mathfrak{m})}) = K_{G_{\mathfrak{m}}}(a) \tag{12.6}$$

by induction on $\mathrm{rk}(a)$. By the induction hypothesis and (12.4),

$$\begin{aligned} K_G(a^{(\mathfrak{m})}) &= [K_G(b^{(\mathfrak{m})}) : (\exists p \in G)(\langle b, p\rangle \in a \;\&\; p \in C_{\mathfrak{m}})] \\ &= [K_{G_{\mathfrak{m}}}(b) : (\exists p \in G_{\mathfrak{m}})(\langle b, p\rangle \in a)] \\ &= K_{G_{\mathfrak{m}}}(a). \end{aligned}$$

An easy induction shows that

$$\Delta(a^{(\mathfrak{m})}) \leqq \mathfrak{m}. \tag{12.7}$$

From (12.5) and (12.6) we obtain

$$M[G] = \bigcup_{\mathfrak{m}} M[G_{\mathfrak{m}}]. \tag{12.8}$$

Moreover,

$$\mathfrak{m} \leqq \mathfrak{n} \rightarrow M[G_{\mathfrak{m}}] \subset M[G_{\mathfrak{n}}]. \tag{12.9}$$

For $K_{G_{\mathfrak{m}}}(a) = K_G(a^{(\mathfrak{m})}) = K_{G_{\mathfrak{n}}}(a^{(\mathfrak{m})})$ by (12.6), (12.7), and (12.5).

All the $C_{\mathfrak{m}}$ have the same forcing language as C. We define $p \Vdash^*_{\mathfrak{m}} \Phi$ for $C_{\mathfrak{m}}$ as before. We then set $\Delta(a, b) = \max(\Delta(a), \Delta(b))$, and define

$$\begin{aligned} p \Vdash^* a \in b \quad &\text{if} \quad p_{\Delta(a,b)} \Vdash^*_{\Delta(a,b)} a \in b, \\ p \Vdash^* a \neq b \quad &\text{if} \quad p_{\Delta(a,b)} \Vdash^*_{\Delta(a,b)} a \neq b. \end{aligned}$$

The definability and extension lemmas are trivial. To prove the truth lemma for, say, $a \in b$, we have for $\mathfrak{m} = \Delta(a, b)$:

$$\begin{aligned} \vdash_G a \in b &\leftrightarrow \vdash_{G_{\mathfrak{m}}} a \in b \quad \text{by (12.5)} \\ &\leftrightarrow (\exists p \in G_{\mathfrak{m}})(p \Vdash^*_{\mathfrak{m}} a \in b) \\ &\leftrightarrow (\exists p \in G)(p_{\mathfrak{m}} \Vdash^*_{\mathfrak{m}} a \in b) \\ &\leftrightarrow (\exists p \in G)(p \Vdash^* a \in b). \end{aligned}$$

We can now define $p \Vdash^* \Phi$ for nonatomic Φ and prove the three lemmas on forcing as before.

We next observe that $C_{\mathfrak{m}}$ satisfies the $\mathfrak{m}^+$-chain condition (in M). For by Lemma 10.3, $H_{\mathfrak{m}}(Q_{\mathfrak{m}}, 2)$ satisfies the $\mathfrak{m}^+$-chain condition. But $C_{\mathfrak{m}} \subset H_{\mathfrak{m}}(Q_{\mathfrak{m}}, 2)$; and compatible elements in $H_{\mathfrak{m}}(Q_{\mathfrak{m}}, 2)$ are compatible in $C_{\mathfrak{m}}$, since their union is in $C_{\mathfrak{m}}$. This gives the desired result. We also note that $C^{\mathfrak{m}}$ is $\mathfrak{m}^+$-closed in M.

We say that $\{D_\beta\}_{\beta<\mathfrak{m}}$ is a *sequence of classes in* M if the set of pairs $\langle p, \beta\rangle$ such that $p \in D_\beta$ is a class in M. We note that Lemma 10.4 extends to classes in M and that Lemma 10.5, extends to sequences of classes in M.

LEMMA 12.1. *Let $\{D_\beta\}_{\beta<\mathfrak{m}}$ be a sequence of classes in M such that each D_β is a dense section in C. Then there is a $q \in G^{\mathfrak{m}}$ such that*

$$(\forall\alpha < \mathfrak{m})(\exists p \in G_{\mathfrak{m}})(p \cup q \in D_\alpha).$$

PROOF. For $q \in C^{\mathfrak{m}}$, let

$$D_\alpha^q = [p : p \in C_{\mathfrak{m}} \;\&\; p \cup q \in D_\alpha].$$

It will suffice to choose $q \in G^{\mathfrak{m}}$ such that for each $\alpha < \mathfrak{m}$, D_α^q is $C_{\mathfrak{m}}$-dense. Hence setting

$$D'_\alpha = [q : q \in C^{\mathfrak{m}} \;\&\; D_\alpha^q \text{ is } C_{\mathfrak{m}}\text{-dense}],$$

it will suffice to show that $\bigcap_{\alpha<\mathfrak{m}} D'_\alpha$ is dense. In view of the above remarks, we need only show that D'_α is dense.

Let $q' \in C^{\mathfrak{m}}$. Choose r_β inductively in M so that $r_\beta \leqq q'$; $r_\beta \in D_\alpha$; and for all $\gamma < \beta$, $(r_\beta)^{\mathfrak{m}} \leqq (r_\gamma)^{\mathfrak{m}}$ and $(r_\beta)_{\mathfrak{m}}$ and $(r_\gamma)_{\mathfrak{m}}$ are incompatible. If there are many such r_β, we use the definable well-ordering of D_α given by the axiom of constructibility to choose one; if there is no such r_β, then r_β is undefined.

Since $C_{\mathfrak{m}}$ satisfies the $\mathfrak{m}^+$-chain condition in M, we see that the smallest β such that r_β is undefined exists and satisfies $|\beta| \leqq \mathfrak{m}$ in M. Hence there is a $q \in C^{\mathfrak{m}}$ such that $q \leqq (r_\gamma)^{\mathfrak{m}}$ for $\gamma < \beta$ and $q \leqq q'$. We must show that $q \in D'_\alpha$, i.e. that D_α^q is dense.

Let $p \in C_{\mathfrak{m}}$. Then $p \cup q$ has an extension r in D_α. Since r is not a possible value for r_β, $r_{\mathfrak{m}}$ is compatible with some $(r_\gamma)_{\mathfrak{m}}$. Let p' be a common extension of $r_{\mathfrak{m}}$ and $(r_\gamma)_{\mathfrak{m}}$. Then $p' \leqq r_{\mathfrak{m}} \leqq p$; and $p' \cup q \leqq (r_\gamma)_{\mathfrak{m}} \cup (r_\gamma)^{\mathfrak{m}} = r_\gamma$, so $p' \cup q \in D_\alpha$ and hence $p' \in D_\alpha^q$.

LEMMA 12.2. *Let F be a functional in $M[G]$ such that $\mathfrak{m}$ is included in the domain of F. Then the restriction f of F to $\mathfrak{m}$ is in $M[G]$. If $[F(\alpha) : \alpha < \mathfrak{m}]$ is included in a set in $M[G_{\mathfrak{m}}]$, then f is in $M[G_{\mathfrak{m}}]$.*

PROOF. Let $\Phi(x, y)$ be a formula of the forcing language such that

$$F(\bar{a}) = \bar{b} \leftrightarrow \vdash_G \Phi(a, b). \tag{12.10}$$

For each $\alpha < \mathfrak{m}$, let

$$D_\alpha = [p : \exists b(p \Vdash \exists x\Phi(\hat{\alpha}, x) \to \Phi(\hat{\alpha}, b))].$$

Then $\{D_\alpha\}$ is a sequence of classes in M. Using Lemma 10.4, we see that D_α is a dense section. Hence by Lemma 12.1, there is a $q \in G^{\mathfrak{m}}$ such that

$$(12.11) \qquad (\forall \alpha < \mathfrak{m})(\exists p \in G_{\mathfrak{m}})(p \cup q \in D_\alpha).$$

For $\alpha < \mathfrak{m}$ and $p \in C_{\mathfrak{m}}$, let $A_{\alpha,p}$ be the set of b such that

$$(12.12) \qquad p \cup q \Vdash \exists x \Phi(\hat{\alpha}, x) \to \Phi(\hat{\alpha}, b).$$

Let g be a function in M whose domain is the set of $\langle \alpha, p\rangle$ in $\mathfrak{m} \times C_{\mathfrak{m}}$ such that $A_{\alpha,p} \neq 0$, and such that $g(\langle \alpha, p\rangle) \in A_{\alpha,p}$ for each such $\langle \alpha, p\rangle$. We show that

$$(12.13) \qquad f = [\langle K_G(b), \alpha\rangle : (\exists p \in G_{\mathfrak{m}})(b = g(\langle \alpha, p\rangle))].$$

By (12.11), there is for each $\alpha < \mathfrak{m}$ a $p \in G_{\mathfrak{m}}$ such that $A_{\alpha,p} \neq 0$, and hence such that $g(\langle \alpha, p\rangle)$ is defined. Hence we must show that if $p \in G_{\mathfrak{m}}$ and $b = g(\langle \alpha, p\rangle)$, then $K_G(b) = f(\alpha)$. Since (12.12) holds and since $p \cup q \in G$ by (12.3), $\vdash_G \exists x \Phi(\hat{\alpha}, x) \to \Phi(\hat{\alpha}, b)$. From this and (12.10), $F(\alpha) = K_G(b)$; so $f(\alpha) = K_G(b)$.

Choose $\mathfrak{n}$ so that $\mathfrak{m} \leqq \mathfrak{n}$ and $\Delta(b) \leqq \mathfrak{n}$ for $b \in \mathrm{Ra}(g)$. By (12.4) and (12.5), (12.13) becomes

$$f = [\langle K_{G_{\mathfrak{n}}}(b), \alpha\rangle : (\exists p \in G_{\mathfrak{n}})(b = g(\langle \alpha, p\rangle))].$$

Since $G_{\mathfrak{n}} \in M[G_{\mathfrak{n}}]$ and there is a functional in $M[G_{\mathfrak{n}}]$ coinciding with $K_{G_{\mathfrak{n}}}$ on arguments in M, it follows that $f \in M[G_{\mathfrak{n}}]$. Hence $f \in M[G]$.

Now suppose that $[F(\alpha) : \alpha < \mathfrak{m}] \subset K_{G_{\mathfrak{m}}}(a)$; we want to show that we may take $\mathfrak{n} = \mathfrak{m}$. We modify D_α to be the set of p such that

$$(\exists b \in \mathrm{Ra}(a^{(\mathfrak{m})}))[p \Vdash \exists x(x \in a^{(\mathfrak{m})} \ \&\ \Phi(\hat{\alpha}, x)) \to \Phi(\hat{\alpha}, b)].$$

In proving this is dense, we have to note that every member of $K_G(a^{(\mathfrak{m})})$ has a name in $\mathrm{Ra}(a^{(\mathfrak{m})})$ by (4.1). The fact that $\vdash_G \exists x(x \in a^{(\mathfrak{m})} \ \&\ \Phi(\alpha, x)) \to \Phi(\alpha, b)$ yields $F(\alpha) = K_G(b)$ now uses (12.6). We can now suppose that every b in $\mathrm{Ra}(g)$ is in $\mathrm{Ra}(a^{(\mathfrak{m})})$ and hence is of the form $b'^{(\mathfrak{m})}$. But $\Delta(b'^{(\mathfrak{m})}) \leqq \mathfrak{m}$; so we may indeed take $\mathfrak{n} = \mathfrak{m}$.

LEMMA 12.3. *If $\bar{a} \in M[G]$, and $\bar{a} \neq 0$, then there is an $\mathfrak{m}$ such that $\bar{a} \in M[G_{\mathfrak{m}}]$ and a mapping f of $\mathfrak{m}$ onto $\bar{a}$ which is in $M[G_{\mathfrak{m}}]$.*

PROOF. Choose $\mathfrak{n}$ by 12.8 so that $\bar{a} \in M[G_{\mathfrak{n}}]$. Choose $\mathfrak{m}$ so that $\mathfrak{n} \leqq \mathfrak{m}$ and $|\bar{a}| \leqq \mathfrak{m}$ in $M[G_{\mathfrak{n}}]$. Then there is an f in $M[G_{\mathfrak{n}}]$ mapping $\mathfrak{m}$ onto $\bar{a}$; and $\bar{a}, f \in M[G_{\mathfrak{m}}]$ by (12.9).

Now we verify that $M[G]$ satisfies the conditions of Lemma 2.1. Since $\omega \in M \subset M[G]$, (a) holds. Now let $\bar{a} \in M[G]$, $\bar{a} \neq 0$; and choose $\mathfrak{m}$ and f as in Lemma 12.3. Suppose A is a class in $M[G]$ such that $A \subset \bar{a}$ and $A \neq 0$. Since $f \in M[G]$, it is easy to define a functional F in $M[G]$ with domain $\mathfrak{m}$ and range A. By Lemma 12.2, $F \in M[G_{\mathfrak{m}}]$; so $A \in M[G_{\mathfrak{m}}] \subset M[G]$. This proves (b). It also shows that every subset of $\bar{a}$ in $M[G]$ is in $M[G_{\mathfrak{m}}]$ and hence in the power set of $\bar{a}$ in $M[G_{\mathfrak{m}}]$. This proves (d).

Now let F be a functional in $M[G]$, and let $\bar{a}$ be a nonempty subset of the domain of F. Again let $\mathfrak{m}$ and f be as in Lemma 12.3, and let F_1 be the functional defined by $F_1(\alpha) = F(f(\alpha))$. Then $F_1 \in M[G]$ by Lemma 12.2. Hence for some $\mathfrak{n}$, $F_1 \in M[G_\mathfrak{n}]$; so $U(\mathrm{Ra}(F_1)) \in M[G_\mathfrak{n}] \subset M[G]$. But $U(\mathrm{Ra}(F_1)) = U([F(x): x \in \bar{a}])$; so we have proved (c).

By Lemma 2.1, $M[G]$ is a model of ZF. Using Lemma 12.3, we see that $M[G]$ is a model of ZFC.

Now we show that $\mathrm{cf}^M = \mathrm{cf}^{M[G]}$. If not, we would have $\aleph_0 \leqq {}^{M[G]}(\alpha) < \mathrm{cf}^M(\alpha)$ for some α. Set $\mathfrak{m} = \mathrm{cf}^{M[G]}(\alpha)$. This is an infinite cardinal in M; and since $\mathfrak{m} = \mathrm{cf}^{M[G]}(\mathfrak{m}) \leqq \mathrm{cf}^M(\mathfrak{m}) \leqq \mathfrak{m}$, it is regular in M. Let f be a mapping of $\mathfrak{m}$ onto a cofinal subset of α such that $f \in M[G]$. By Lemma 12.2, $f \in M[G_\mathfrak{m}]$. Hence $\mathrm{cf}^{M[G_\mathfrak{m}]}(\alpha) \leqq \mathfrak{m}$. On the other hand, $\mathfrak{m} < \mathrm{cf}^M(\alpha)$ implies $\mathfrak{m}^+ \leqq \mathrm{cf}^M(\alpha)$; so $\mathrm{cf}^M(\alpha) = \mathrm{cf}^{M[G_\mathfrak{m}]}(\alpha)$ by Lemma 10.2. Thus $\mathfrak{m} < \mathrm{cf}^M(\alpha) = \mathrm{cf}^{M[G_\mathfrak{m}]}(\alpha) \leqq \mathfrak{m}$, a contradiction.

From $\mathrm{cf}^M = \mathrm{cf}^{M[G]}$ it follows that M and $M[G]$ have the same cardinals; the proof is like the proof of Lemma 10.2(b).

Any nonempty subset of $\mathfrak{m}$ in $M[G]$ is the image of $\mathfrak{m}$ under a functional in $M[G]$ and hence is in $M[G_\mathfrak{m}]$ by Lemma 12.2. Thus $S^{M[G]}(\mathfrak{m}) = S^{M[G_\mathfrak{m}]}(\mathfrak{m})$; so to prove $2^\mathfrak{m} = H(\mathfrak{m})$ in $M[G]$, it suffices to prove it in $M[G_\mathfrak{m}]$. Setting $F = U(G_\mathfrak{m})$ and

$$A_\alpha = [\beta : \beta < \mathfrak{m} \,\&\, F(\mathfrak{m}, \alpha, \beta) = 0]$$

for $\alpha < H(\mathfrak{m})$, we prove as before that the A_α are $H(\mathfrak{m})$ distinct subsets of $\mathfrak{m}$. Making use of (12.1), we have in M,

$$|Q_\mathfrak{m}| \leqq \mathfrak{m} \cdot H(\mathfrak{m}) \cdot \mathfrak{m} = H(\mathfrak{m})$$

(since (12.2) gives $\mathfrak{m} < H(\mathfrak{m})$). Since $C_\mathfrak{m} \subset H_\mathfrak{m}(Q_\mathfrak{m}, 2)$, we get

$$|C_\mathfrak{m}| \leqq H(\mathfrak{m})^{\underset{\smile}{\mathfrak{m}}} = H(\mathfrak{m})$$

by (10.2) and (12.2). Then by Lemma 11.1,

$$|S(\mathfrak{m})|^{M[G_\mathfrak{m}]} \leqq (H(\mathfrak{m})^{\mathfrak{m}\cdot\mathfrak{m}})^M = H(\mathfrak{m}).$$

Thus $2^\mathfrak{m} = H(\mathfrak{m})$ in $M[G_\mathfrak{m}]$.

HISTORICAL NOTE. The results of this section are due to Easton [3].

BIBLIOGRAPHY

1. P. J. Cohen, *The independence of the continuum hypothesis, Parts* I, II, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1143–1148; **51** (1964), 105–110.

2. ———, *Set theory and the continuum hypothesis*, Benjamin, New York, 1966.

3. W. Easton, *Powers of regular cardinals*, Thesis, Princeton University, 1964.

4. S. Feferman, *Some applications of the notion of forcing and generic sets*, Fund. Math. **56** (1965), 325–345.

5. K. Gödel, *The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory*, Ann. of Math. Studies No. 3, Princeton University Press, Princeton, N.J., 1940.

6. A. Lévy, *Axiom schemata of strong infinity in axiomatic set theory*, Pacific J. Math. **10** (1960), 223–238.

7. ———, Definability in axiomatic set theory. I, (Proc. 1964 Internat. Congr.) *Logic, Methodology, and Philosophy of Science*, North-Holland, Amsterdam, 1966.

8. A. Mostowski, *An undecidable arithmetical statement*, Fund. Math. **36** (1949), 143–164.

9. J. C. Shepherdson, *Inner models for set theory, Part* I, J. Symbolic Logic **16** (1951), 161–190.

10. J. R. Shoenfield, *Mathematical logic*, Addison-Wesley, Reading, Mass., 1967.

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