(1) Let $G$ be a finite group and let $V, W$ be finite-dimensional $G$-representations. Define $\Phi: V^* \otimes W \to \text{Hom}(V, W)$ by $\Phi(\sum_i f_i \otimes w_i) = F$ where $F(v) = \sum_i f_i(v)w_i$. Show that $\Phi$ is well-defined and is a $G$-equivariant isomorphism.

(2) Let $G$ be a finite abelian group and let $V$ be an irreducible representation over an algebraically closed field (of arbitrary characteristic). Use Schur’s lemma to prove that $\dim V = 1$.

(3) Let $G$ be a group. Define $[G, G]$ to be the subgroup of $G$ generated by elements of the form $xyx^{-1}y^{-1}$ where $x, y \in G$.

(a) Show that $[G, G]$ is a normal subgroup and that $G/[G, G]$ is abelian.

(b) Show that $[G, G]$ is in the kernel of any representation $\rho: G \to \text{GL}(V)$ where $\dim(V) = 1$ and deduce that there is a bijection between the 1-dimensional representations of $G$ and of $G/[G, G]$.

(4) Let $X$ be a set with $G$-action and let $V = \mathbb{C}[X]$ be the permutation representation. Let $\chi_1$ be the character of the trivial representation.

(a) Show that $(\chi_V, \chi_1)$ is the number of orbits of $G$ acting on $X$.

(b) For the rest of the problem, assume that $X$ has size at least 2 and that $G$ has 1 orbit on $X$.

The line spanned by $\sum_{x \in X} e_x$ is a subrepresentation, let $U$ be a subrepresentation of $\mathbb{C}[X]$ which is a complement of it. Show that $(\chi_U, \chi_1) = 0$.

(c) Define an action of $G$ on $X \times X$ by $g \cdot (x_1, x_2) = (g \cdot x_1, g \cdot x_2)$. Show that $\chi_{\mathbb{C}[X \times X]} = \chi_U^2$.

(d) Show that $U$ is irreducible if and only if $G$ has exactly 2 orbits on $X \times X$.

(5) Let $F$ be a field, let $G = \text{GL}_2(F)$ be the group of invertible $2 \times 2$ matrices with entries in $F$, and let $X$ be the set of lines, i.e., 1-dimensional subspaces in $F^2$ which has a natural action of $G$. Show that $X \times X$ has exactly 2 orbits. When $F$ is finite, the representation $U$ from above is called the Steinberg representation of $G$.

(6) Let $n > 1$ and let $k$ be a field. Prove that $\{(x_1, \ldots, x_n) \in k^n \mid x_1 + \cdots + x_n = 0\}$ is an irreducible representation of the symmetric group $S_n$ when $k$ has characteristic 0. Show that this remains true if $k$ has characteristic $p > 0$ and $p$ does not divide $n$. What happens when $p$ divides $n$?