Thm (Stable branching rule $GL \to Sp$) let $\lambda$ be a partition w/ $\ell(\lambda) \leq n$. Then we have $Sp_{2n}$-rep isom:

$$S_\lambda \mathbb{C}^{2n} \cong \bigoplus_{\mu} \left( S_{[\mu]} \mathbb{C}^{2n} \right)^{\otimes m_{\lambda,\mu}}$$

where $m_{\lambda,\mu} = \sum_{\nu} c_{\mu,\nu}^{\lambda}$

**Ex 1.** \(\lambda = \mu\), \(m_{\lambda,\mu} = 1\) since only \(C_{\lambda,\mu}^\mu = 1\) contributes to sum

\(\Rightarrow S_{\lambda} \mathbb{C}^{2n}\) appears w/ mult. 1 in \(S_{\lambda} \mathbb{C}^{2n}\)

In fact, h.w. vector for $GL_{2n} \mathbb{C}$ in $S_{\lambda} \mathbb{C}^{2n}$ is a h.w. vector for $Sp_{2n} \mathbb{C}$ of weight $\lambda$

**Ex 2.** \(\lambda = (1^d)\). \(C_{(1^d),(1^d)}^{(1^d)} > 0\) implies that

\(\mu = (1^e) (2^f)^T = (1^f), f\) even. \& \(e + f = d\)

In that case, \(C_{(1^e),(1^f)} = 1\) by Pieri rule.
Ex 3. Unstable example:

\[ \lambda = (1^{2n}) \quad S(1^{2n}) \cong \mathbb{C}^{2n} = \bigwedge^2 \mathbb{C}^{2n} \cong \mathbb{C}^{2n} \oplus S_{(1^{a-1})} \mathbb{C}^{2n} \oplus \cdots \]

\[ g \in \text{Sp}_{2n} \mathbb{C} \Rightarrow \det g = 1, \text{ hence } \bigwedge^2 \mathbb{C}^{2n} = \text{trivial} = \mathbb{C}_{\lambda_0} \mathbb{C}^{2n} \]

Characters of the symplectic group

\[ \rho : \text{Sp}_{2n} \mathbb{C} \rightarrow \text{GL}(\mathbb{C}) \text{ rep. } \Rightarrow (\text{char } \rho)(x_1, \ldots, x_n) = \text{trace } \rho \left( \begin{array}{ccc} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \end{array} \right) \]

Can see that char \( \rho \) is symmetric in \( x_i \)'s.

but more is true.

Let \( H_n = \text{permutations } \sigma \text{ of } \{x_1, \ldots, x_n, x_n^{-1}, x_1^{-1}\} \)

s.t. \( \sigma(x_i) = \frac{1}{\sigma(x_i)} \). \( G_n \subset H_n \text{ as permutations of } \{x_1, \ldots, x_n\} \)

\[ |H_n| = 2^n n! \quad \text{hyperoctahedral group} \]

Abstractly, \( H_n \cong G_n \times (\mathbb{Z}/2)^n \)
Lemma (Sharp) \((x, \ldots, z_1)\) is invariant under \(H_n\).

Proof. \(H_n\) is generated by \(G_n\) and \(t_1, \ldots, t_n\), where 
\[
t_i(x_j) = \begin{cases} x_i & \text{if } i = j, \\ x_j & \text{if } i \neq j. \end{cases}
\]
so suffices to show invariance under these

\(\sigma \in G_n\): let \(M(\sigma)\) be an \(n\times n\) permutation matrix

\[
(M(\sigma) \quad 0) \quad 0 \quad I'M(\sigma)'I' \quad \in \quad Sp_{2n} \mathbb{C}
\]

Conjugation by \(S\) sends \(x_i \mapsto x_{\sigma(i)}\).

\(t_i\): define \(g_i \in Sp_{2n} \mathbb{C}\) via:

- \(e_j \mapsto e_j\) for \(j \neq i\)
- \(e_i \mapsto e_{-i}\)
- \(e_{-i} \mapsto -e_i\)

\(w(e_i, e_{-i}) = 1\)
\(w(g_i e_{-i}, g_i e_{-i}) = w(e_{-i}, -e_i) = 1\)

Conjugation by \(g_i\) sends \(x_i \mapsto x_i^{-1}\), \(x_j \mapsto x_j^{-1}\) \(x_i^{-1} \mapsto x_i\) \(\quad \square\)

Corollary. Every \(Sp_{2n}\)-rep is self-dual (isom. to its dual).

Proof. If \(f\) is char. of rep, its dual has character

\[
\tilde{f}(x_1^{-1}, \ldots, x_n^{-1}) = f(x_1, \ldots, x_n) \quad \square
\]
Define \( \Lambda_{sp(2n)} = \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm] \mathfrak{H}_n \) \( \mathfrak{H}_n \)-invariant elements

\((\text{char } p)\) \( \mathfrak{H}_n \)-reps. \( \rho \) of \( sp(2n) \).

The irreducible characters form basis for \( \Lambda_{sp(2n)} \).

let \( \sigma_{\lambda, \gamma}(x_1, \ldots, x_n) = \text{char } \sigma_{\lambda, \gamma}(\mathbb{C}^{2n}) \).

Given \( \sigma \in \mathfrak{H}_n \subset G_{2n} \), let \( \text{sgn}(\sigma) \) be its sign as an element of \( G_{2n} \).

Thm (Weyl character formula) \( \rho = (n, n-1, \ldots, 2, 1) \)

\( \sigma_{\lambda, \gamma}(x_1, \ldots, x_n) = \det \left( x_i^{\lambda_j + n - j + 1} - x_i^{-(\lambda_j + n - j + 1)} \right)_{i,j=1}^n \left( \prod_{i=1}^n (x_i - x_i^{-1}) \prod_{1 \leq i < j \leq n} (x_i + x_i^{-1} - x_j - x_j^{-1}) \right) \)

\( = \sum_{\sigma \in \mathfrak{H}_n} \text{sgn}(\sigma) \sigma(x^\rho) \)

\( \sum_{\sigma \in \mathfrak{H}_n} (\text{sgn } \sigma) \sigma(x^\rho) \)
The (Symplectic Cauchy identity)

\[ \prod_{1 \leq i < j \leq n} \left( 1 - y_i y_j t^2 \right) \prod_{i,j=1}^n (1 - x_i y_j t) (1 - x_i^{-1} y_j t) = \sum_{x \in \chi} \sum_{x \in \chi} (x, \ldots, x_n) s(x, y_1, \ldots, y_n) t^{\chi_1} \]

\[ \chi_1 \geq \ldots \geq \chi_n \geq 0 \in \mathbb{Z}_n \]

Proof. From multiple action, we have

\[ C[x] \cong \bigoplus s_{\chi \in \chi} C^{2n} \otimes S_{\lambda} C^n \]

\[ \text{char}(C[x]) = \sum_{\chi \in \chi} s_{\chi \in \chi} (x, \ldots, x_n) s(x, y_1, \ldots, y_n) t^{\chi_1} \]

Also, we have

\[ C[x] \otimes \text{Sym}(\mathbb{C}^n) \cong \mathbb{C}[U] \]

\[ \text{char}(C[x]) \cdot \prod_{1 \leq i < j \leq n} (1 - y_i y_j t^2)^{-1} = \prod_{i,j=1}^n (1 - x_i y_j t)^{-1} \cdot (1 - x_i^{-1} y_j t)^{-1} \]

Combine both. □
From stable branching rule, we have formula like:

\[ S_\lambda(x_1, \ldots, x_n) = SC_\lambda(x_1, \ldots, x_n) + \sum_{\mu \vdash \lambda, \mu \neq \lambda} m_{\lambda \mu} S_\mu(x_1, \ldots, x_n) \]

Order partitions of size \( \leq |\lambda| \) in a way that refines size. \( \Rightarrow \) matrix M whose elements are \( m_{\lambda \mu} \). It is upper triangular w/ 1's on the diagonal \( \Rightarrow \) invertible. Let \( N = (n_{\lambda \mu} = M^{-1}) \).

\[ \Rightarrow S_{\lambda\mu}(x_1, \ldots, x_n) = \sum_{\mu} n_{\lambda \mu} S_\mu(x_1, \ldots, x_n) \]

\( \sum_{\mu} \Rightarrow n_{\lambda \mu} \neq 0 \Rightarrow |\mu| \leq |\lambda| \).

Also: \( m_{\lambda \mu} \) is independent of \( n \) if \( n \geq c(\lambda) \).

\( \Rightarrow \) \( n_{\lambda \mu} \) is independent of \( n \) if \( n \geq c(\lambda) \).

**IDEA:** Define \( sc_\lambda \in \Lambda \) - symmetric functions via \( sc_\lambda = \sum_{\mu} n_{\lambda \mu} \sum_{\mu} \).

**symplectic Schur functions** via \( sc_\lambda \in \sum_{\mu} \sum_{\mu} \).

**Schur function**
By upper-triangularity, \( \{ S_{\lambda} \} \) is basis for \( \Lambda \).

Can define specialization maps

\[
\pi_{\text{Sp}(2n)} : \Lambda \longrightarrow \Lambda_{\text{Sp}(2n)}
\]

\[
S_{\lambda} \longmapsto s_{\lambda}(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1})
\]

If \( n \geq \ell(\lambda) \), then \( \pi_{\text{Sp}(2n)}(S_{\lambda}) = S_{\lambda}(x_1, \ldots, x_n) \)

But note: \( \pi_{\text{Sp}(2n)}(S_{\lambda}) \) makes sense even if \( n < \ell(\lambda) \)

**Next time:** study tensor products of symplectic reps