Accidental Isomorphisms

Example 1. $B_2 = \mathbb{C}_2 \quad 0 \Rightarrow 0 \quad B_2$

$0 \Leftarrow 0 \quad \mathbb{C}_2$

Consequences:

- $SO_5 \mathbb{C} \cong Sp_4 \mathbb{C}$
- $Spin_5 \mathbb{C} \cong Sp_4 \mathbb{C}$ (there exists $Sp_4 \mathbb{C} \to SO_5 \mathbb{C}$ with kernel of size 2)
- $C^5 \cong \bigwedge^2 C^4 / \langle \alpha \rangle \Rightarrow \bigwedge^2 C^4 / \langle \alpha \rangle$ has an orthogonal form.

- $SO_5 \mathbb{C} \to Sp_4 \mathbb{C}$
- Spin rep $\cong C^4$
- $OFl(1; 5) \cong IFl(2; 4)$

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quadric in $\mathbb{P}^4$

\Rightarrow IFl(2; 4) also quadric

in $\mathbb{P}^4$

Pfaffian of

$\Lambda^2 \mathbb{C}$ restricted
to complement of $\alpha$.
Ex. $A_3 = D_3$

\[\begin{array}{ccc}
0 & \to & 2 \\
\downarrow & & \downarrow \\
1 & \to & 3 \\
& & 3
\end{array}\]

**Consequences:**

- $\text{sl}_4 \mathbb{C} \cong \text{so}_6 \mathbb{C}$
- $\text{SL}_4 \mathbb{C} \cong \text{Spin}_6 \mathbb{C}$ (exists $\text{SL}_4 \mathbb{C} \to \text{so}_6 \mathbb{C}$ with kernel of size 2)
- $\wedge^2 C^4 \cong C^6 \implies \wedge^2 C^4$ has an orthogonal form
- $\text{SL}_4 \mathbb{C} \cong \text{so}_6 \mathbb{C}$

Given $\alpha, \beta \in \wedge^2 C^4$, $\omega(\alpha, \beta) = \text{coeff. of } e_1 e_2 \text{ in } \alpha \wedge \beta$

- $\text{Gr}(2, 4) \cong \text{OFL}(1, 6)$
  - $113$ quadrics in $P^5$

- $P^3 \cong \text{OFL}(3; 6)$; $\text{Gr}(3, 4) \cong \text{OFL}(3'; 6)$
- $\text{Fl}(1, 3; 4) \cong \text{OFL}(3, 3'; 6) \cong \text{OFL}(2; 6)$

Pfaffian
EX.3. $A_n = A_n$ via nontrivial automorphism

\[ 0 \to \cdots \to 0 \]

\[ \begin{array}{c}
0 \\
1 \\
2 \\
\vdots \\
n-1 \\
n \\
\end{array} \]

\[ \bigwedge C^{n+1} = \bigwedge (C^{n+1})^* \]

$SL_{n+1} \cong SL_{n+1}$

$b/c$ Isomorphism

$SL_{n+1} \cong SL_{n+1}$

via $g \to (g^{-1})^T$

$Gr(i, n+1) \cong Gr(n+1-i, n+1)$.

$W \to (C^{n+1}/W)^*$

$D_n$ also has a nontrivial automorphism.

If $n$ is odd, this swaps reps for their duals.

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Classical invariant theory

Schur-Weyl duality $V = f.d. vector space$

$V \otimes \nu$ is rep of $GL(V)$

also rep of $S_n$ symmetric group

$\sigma (V \otimes \cdots \otimes \nu) = V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}$

Define $End(W) = Hom(W, W)$ for any vector space $W$. 


$U = \text{End}(V^\otimes n)$ is a $C$-algebra with composition as multiplication.

Representation structures give

$GL(U) \rightarrow U$, $A =$ linear span of image $C$-subalgebra of $U$

$G_n \rightarrow U$, $B =$ linear span of image $C$-subalg. of $U$

**Def.** Given subalg. $R \subset \text{End}(W)$, let $R^! = \{ x \in \text{End}(W) \mid xR = Rx \ \forall r \in R \}$ commutant

$A \subset B^!$, $B \subset A^!$

**Def.** A f.d. dim $C$-algebra is **semisimple** if it is isomorphic to direct product of matrix algebras.

**Thm (Double commutant theorem).** Let $R \subset \text{End}(W)$ be semisimple subalg. Then:

1. $S := R^!$ is semisimple, and $R = S^!$
2. As $R \times S$-rep, $W \cong \bigoplus_{i \in I} M_i \otimes N_i$ where $M_i =$ simple $R$-reps and $M_i \neq M_j'$ for $i \neq j'$.

$N_i =$ simple $S$-reps and $N_i \neq N_j'$ for $i \neq j'$. 
\textbf{Pf.} \( R \) semi-simple \( \Rightarrow W = \bigoplus_{i \in I} M_i \oplus n_i \)

where \( M_i = \text{simple, distinct} \ R\)-reps \& \( R = \bigoplus_{i \in I} \text{End}(M_i). \) By Schur's lemma, 
\[ S = R^! = \bigoplus_{i \in I} \text{End}(C^{m_i}) =: \text{Semisimple}. \]

\( \Rightarrow \) As \( S\)-rep, \( W = \bigoplus_{i \in I} N_i \oplus \dim M_i \) where \( N_i = C^{m_i}. \n\)

By Schur's lemma again, \( S^! \cong \bigoplus_{i \in I} \text{End}(C^{\dim M_i}). \)

But \( R \subseteq S^! \) and \( \dim R = \dim S^! \Rightarrow R = S^! \square \)

\textbf{Prop.} \( B' = A \)

\textbf{Pf.} \( U = \text{End}(V^\otimes n) = (\text{End}(V))^\otimes n \) as follows:

given \( A_1 \otimes \cdots \otimes A_n, \ A_i \in \text{End}(V), \) get \( V^\otimes n \rightarrow V^\otimes n \) by \( v_1 \otimes \cdots \otimes v_n \rightarrow A(v_1) \otimes \cdots \otimes A(v_n) \n\)

Under this identification, image of \( g \in \text{GL}(V) \) in \( U \) goes to \( g \otimes g \cdots \otimes g \in (\text{End} V)^\otimes n \)
Claim: \( A = \{ \psi \in (\text{End}(V))^n \mid \sigma \psi = \psi \forall \sigma \in \Gamma \} \)

\underline{Proof of claim:} Pick \( f \) linear on symmetric tensors.

\[ \Rightarrow \] degree \( n \) polynomial \( F \) on \( \text{GL}(V) \) via

\[ F(g) = f(g \circ \cdots \circ g), \text{ in fact } F \text{ is defined on } \text{End}(V) \]

\( \text{GL}(V) \subset \text{End}(V) \) is Zariski dense \( \Rightarrow F \) is identically \( 0 \) on \( \text{GL}(V) \) \( \Rightarrow F \equiv 0 \) on \( \text{End}(V) \).

In particular, \( f \) is identically \( 0 \) on \( A \)

\[ \Rightarrow F \equiv 0 \text{ on } \text{GL}(V) \]

\[ \Rightarrow f \equiv 0 \text{ on } \text{End}(V) \]

\[ \Rightarrow f = 0 \text{ on symmetric tensors.} \]

(\( \text{symmetric tensors spanned by } \left\{ x \otimes \cdots \otimes x \mid x \in \text{End}(V) \right\} \))